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Options on the Minimum or the Maximum of Two Average Prices

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Abstract. This paper studies options on the minimum/maximum of two average prices. We provide a closed-form pricing formula for the option with geometric averaging starting at any time before maturity. We show overwhelming numerical evidence that the variance reduction technique with the help of the above closed-form solution dramatically improves the accuracy of the simulated price of an option with arithmetic averaging. The proposed options are found widely applicable in risk management and in the design of incentive contracts. The paper also discusses some parity relationships within the family of average-rate options and provides the upper and lower bounds for the proposed options with arithmetic averaging.

Keywords: option, average-rate, rainbow, risk management, incentive contract.

JEL classification: G13.

Introduction

Corporate hedging has been increasingly important in the environment of today's volatile financial markets. The ever-increasing demand for various hedging instruments, which are suitable for more complex applications, has stimulated the development of more and more sophisticated derivatives. Among these often called "exotic derivatives," average-rate options are most successful. Average-rate options, or Asian options, have become widely used by corporations that consider hedging against the unwanted average price movement during an extended period of time. For example, domestic firms, having outstanding foreign account payables due on the last working day of each week and wishing to hedge away the foreign exchange risk for the next 26 weeks, can obtain the most suitable protection by using a 26-week Asian option, with an arithmetic average of weekly foreign account payables, which depends on the arithmetic average of the exchange spot rates. Thus, the Asian option, whose use is more cost-effective than the use of a portfolio of ordinary options, which are supposed to match individual cash flows, provides a one-shot solution to the risk management that aims to cap an aggregate-level cost in domestic currency over a finite future period.

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In many situations, a firm dealing with foreign account payables may enjoy the foreign suppliers' flexible invoicing policy that allows the firm to choose between different foreign currencies. For example, to promote sales, an Italian exporter would like to make it easy on a US importer by allowing the US firm to decide at the start of a designated period to pay either British Pounds (GBP) or Italian Lira (ITL) for the period, since the Italian firm is likely to have business with British firms and hence would need GBP anyway. Either ITL or GBP can turn out to be favorable to the US firm but the US firm does not know it at the moment it chooses the invoicing currency for payment. Assume that equal-amount account payables are made on a frequent basis and that the risk-management-conscientious US firm would like to cap the average cost in US dollars (USD) for the period. The issue boils down to a desire of the US firm to choose between ITL and GBP without any regret later on. This desire can not be met satisfactorily by using an ordinary Asian option because two average prices rather than a single one enter the desired payoff function. Thus, an option on the maximum of two average prices is needed.

The option on the maximum of two average prices obviously has an advantage over an ordinary Asian option. This can be understood as follows. If either the average ITL account payables (A_1) or the average GBP account payables (A_2), which are already translated into USD (foreign exchange/forward rates are assumed to be quoted in USD/FC), turn out to be larger than a controlled value or the strike, K, during the concerned period, the proposed option expires in the money anyway. Hence, the US firm ends up with a hedged cost at or, luckily, less than K, depending on which foreign currency it has chosen. In contrast, this is not generally true by using a standard Asian option. If the US firm has agreed to pay a foreign currency whose future spot rates result in the minimum of A_1 and A_2, the Asian option on the invoicing currency allows the firm to cap the cost only at K, while the proposed option would allow the firm to incur a hedged cost less than K by $|A_1 - A_2|$. As a matter of fact, it can be easily shown that the option on the maximum of two average prices always has equal or higher payoffs than an ordinary Asian option.[1]

The use of the options this paper investigates is not limited to multinational firms' hedging of transaction exposures. It is shown in this paper that firms, when considering at least two kinds of commodities in the production and when wishing to hedge away price uncertainties of the commodities involved, will find the options on the minimum or the maximum of two average prices to be convenient hedging instruments. Financial contracts, which are based on the average prices of commodities, have the merit of anti-manipulation because commodity prices, such as oil prices, are usually thought to be prone to manipulation by big market participants as a result of the high level of concentration in the industry.[2]

Another application that goes beyond risk management shows that the option on the minimum of two average prices appropriately enters the payoff function of incentive contracts for executive compensation. A manager can be given, instead of the standard stock option, a compensation package that includes the option on the minimum of the average stock price of the firm the manager runs and the average stock price of a close competitor or the industrial representative. This package is truly merit-based and should be fair to both managers and shareholders, because there is little incentive for managers to boost the stock prices temporarily for their own good. On the other hand, the average feature smoothes the randomness or the "noise" inherent in the stock prices so that the managers can be evaluated more fundamentally.

In view of the potential wide applications of the options on the minimum or the maximum of two average prices, it is somewhat surprising that, to the best of our knowledge, such options have never been investigated.

This paper analyzes and values the European-style options on the minimum or the maximum of two average prices. In particular, we provide a closed-form pricing formula for the option with the geometric averaging, which may start at any time before maturity. Thus, our work is closely related to the two well-known contributions in derivatives research, that is, average-rate options by Kemna and Vorst (1990), and options on the minimum or the maximum of two risky assets by Stulz (1982). Both kinds of options are so popular that they have become "classic" examples of non-standard options discussed in textbooks.

Kemna and Vorst (1990) investigate average-rate options with both arithmetic and geometric averaging for the underlying asset price that obeys the geometric Brownian motion and come up with an analytical pricing model for geometric average-rate options. Although the direct application of an option with geometric averaging makes only limited economic sense, the closed-form solution of the option with geometric averaging is indispensable in the variance reduction technique employed in the Monte Carlo simulation approach. In pricing path-dependant options without analytical expressions such as in the Asian option with arithmetic averaging, the simulation approach is often used.[3] The simulation approach has the advantage that it can provide standard errors for the estimates and is traditionally used as a benchmark approach in the horse-race with different techniques.

Stulz (1982) provides a closed-form pricing formula for (European-style) options on the minimum or the maximum of two risky assets. He illustrates many applications ranging from valuation of foreign currency debt and option-bonds to risk-sharing contracts and growth opportunities involving mutually exclusive investments. As a matter of fact, options on the maximum or the minimum, labeled as Rainbow options, are often used in global asset allocation. A Rainbow option allows the holder to choose between two indexes such as S&P500 and Nikkei225 (two colors in this case), or among many indexes (see e.g., Johnson, 1987). Rainbow options are also called outperformance (or relative performance) options.[4]

The options we propose in this paper non-trivially combine average-rate options with rainbow ones; that is, both the average-rate options and rainbow ones in the existing literature are special cases of our hybrid options. This is important for two reasons. First, our options can be used to meet many kinds of firms' financial management demands that cannot be satisfied by derivative assets already known in the literature. In light of the MM theory and market efficiency hypothesis, one way for financial managers to add value to their firms is to introduce innovative financial products that meet as yet unsatisfied demands and render the market more complete (see Chapter 17 of the textbook by Brealey and Myers, 1996). Thus, by adding to such financial innovations, our paper serves a useful purpose to both the supplier and the user of the products. Second, creating an option that encompasses existing ones is unusual and is more likely to occur when academics are involved: in the street, most efforts are directed towards producing increasingly specific derivative assets. Thus, the generality of our options would spawn wider and more interesting applications than the existing literature can offer.

The paper is organized as follows. Section 1 analyzes and values the proposed options. We start with presenting a closed-form pricing formula for the option with geometric averaging and then we demonstrate how the closed-form solution is implemented in the variance reduction technique to obtain an accurately simulated price of the option with arithmetic averaging. Section 2 discuses some parity relationships and provides the upper and lower bounds for the options with arithmetic averaging. Section 3 illustrates some typical applications of options on the minimum or the maximum of two average asset prices. Section 4 concludes.

1. Pricing Options on the Minimum or the Maximum of Two Average Prices

In this section, we solve the pricing of options on the minimum or the maximum of two average prices. We first provide a closed-form pricing formula for a call option on the minimum of two geometric averages (subsection 1.1) and then use this analytical solution in a variance reduction technique to obtain accurately simulated option prices when arithmetic averages are considered (subsection 1.2). Average-rate options with arithmetic averaging have no closed-form pricing formulas if the underlying variables are assumed to follow a geometric Wiener process because the arithmetic average of a so-assumed variable does not remain in the family of the Ito process.

1.1. A Model of the Call Option on the Minimum of Two Geometric Averages

We price a European-style call option on the minimum of two Geometric averages. The maximum case can be easily derived from the parity relationship discussed in Section 3. Using the case of the minimum as the attacking point directly follows the tradition of the literature (e.g., Stulz, 1982).

Assume a continuous-time framework to start with, and assume that the price of asset i, denoted by $S_i(t)$, obeys the following Geometric Wiener process:

$$dS_i(t) = \mu_i S_i(t)dt + \sigma_i S_i(t)dW_i(t), \qquad i = 1, 2, \tag{1}$$

where $W_i(t)$ is a Wiener process and μ_i and σ_i are constant. The correlation coefficient between $W_1(t)$ and $W_2(t)$ is ρ.

For $0 \leq s \leq t$, we introduce a new variable

$$I_i = \int_0^t \ln S_i(s)ds, \qquad i = 1, 2, \tag{2}$$

and hence the geometric mean over $[0, t]$ is simply $G_i(t) = e^{I_i/t}$.[5]

The payoff of a European-style call option on the minimum of two geometric averages with maturity T is $\text{Max}\{\text{Min}[G_1(T), G_2(T)] - K, 0\}$. Let the option price at $t(0 \leq t \leq T)$ be $V(S_1, S_2, I_1, I_2, K, t)$. The option value is path-dependant on $I_1(t)$ and $I_2(t)$, but it is fundamentally driven by the two original underlying variables in (1), which are traded assets. Assume that the instantaneous rate of interest r is constant over the remaining life of the option. Using the riskless hedging argument and applying Ito's Lemma, we can derive the PDE as follows (see Black and Scholes, 1973, and Merton, 1973):

$$\frac{\partial V}{\partial t} + \ln S_1 \frac{\partial V}{\partial I_1} + \ln S_2 \frac{\partial V}{\partial I_2} + \frac{1}{2}\sigma_1^2 S_1^2 \frac{\partial^2 V}{\partial S_1^2} + \sigma_1\sigma_2\rho S_1 S_2 \frac{\partial^2 V}{\partial S_1 \partial S_2} + \frac{1}{2}\sigma_2^2 S_2^2 \frac{\partial^2 V}{\partial S_2^2} + r S_1 \frac{\partial V}{\partial S_1} + r S_2 \frac{\partial V}{\partial S_2} - rV = 0. \tag{3}$$

Given the boundary conditions of the option:

$$V(S_1, S_2, I_1, I_2, T) = \mathrm{Max}\{\mathrm{Min}[G_1(T), G_2(T)] - K, 0\}, \tag{4}$$

$$V(0, S_2, I_1, I_2, t) = 0, \tag{5}$$

and

$$V(S_1, 0, I_1, I_2, t) = 0, \tag{6}$$

the solution to (3)–(6) is given as follows (see Appendix A for the derivation):

$$\begin{aligned} V(S_1, S_2, I_1, I_2, K, t) &= S_1^* N_2\left(\frac{\ln\frac{S_1^*}{K} + (r + \frac{1}{2}\sigma_1^{*2})(T-t)}{\sigma_1^*\sqrt{T-t}}, \frac{\ln\frac{S_2^*}{S_1^*} - \frac{1}{2}\sigma^{*2}(T-t)}{\sigma^*\sqrt{T-t}}, \frac{\rho\sigma_2^* - \sigma_1^*}{\sigma^*}\right) \\ &+ S_2^* N_2\left(\frac{\ln\frac{S_2^*}{K} + (r + \frac{1}{2}\sigma_2^{*2})(T-t)}{\sigma_2^*\sqrt{T-t}}, \frac{\ln\frac{S_1^*}{S_2^*} - \frac{1}{2}\sigma^{*2}(T-t)}{\sigma^*\sqrt{T-t}}, \frac{\rho\sigma_1^* - \sigma_2^*}{\sigma^*}\right) \\ &- Ke^{-r(T-t)} N_2\left(\frac{\ln\frac{S_1^*}{K} + (r - \frac{1}{2}\sigma_1^{*2})(T-t)}{\sigma_1^*\sqrt{T-t}}, \frac{\ln\frac{S_2^*}{K} + (r - \frac{1}{2}\sigma_2^{*2})(T-t)}{\sigma_2^*\sqrt{T-t}}, \rho\right) \end{aligned} \tag{7}$$

where

$$S_i^* = S_i^{(T-t)/T} \exp\left\{\frac{I_i}{T} + (\mu_i^* - r)(T-t)\right\}, \qquad i = 1, 2, \tag{8}$$

$$\mu_i^* = (r - \frac{1}{2}\sigma_i^2)\frac{T-t}{2T} + \frac{1}{6}\sigma_i^2 \frac{(T-t)^2}{T^2}, \qquad i = 1, 2, \tag{9}$$

$$\sigma_i^* = \frac{\sigma_i}{\sqrt{3}} \frac{T-t}{T}, \qquad i = 1, 2, \tag{10}$$

$$\sigma^* = \frac{\sigma}{\sqrt{3}} \frac{T-t}{T}, \qquad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}, \tag{11}$$

and N_2 (α, β, θ) is the bivariate cumulative standard normal distribution with upper limits of integration α and β, and correlation coefficient θ.

The closed-form formula in (7)–(11) gives the price of a European-style call option on the minimum of two geometric averages. Next, we discuss the pricing of a similar option with arithmetic averaging, which most applications call for.

1.2. Pricing of the Option on the Minimum of Two Arithmetic Averages

The pricing of a path-dependant European-style option can always be implemented using the popular Monte Carlo simulation. The simulation approach sometimes becomes the last resort especially when there is no closed-form formula available, such as the pricing of the option with arithmetic averaging. One important issue in implementing the simulation method is the accuracy of the calculated option price. To achieve high accuracy of the simulation results within a feasible number of simulation runs, the implementation of a variance reduction technique is necessary (see e.g., Boyle, 1977, and Hull and White, 1987).[6]

The geometric average can serve not only as a lower bound for the arithmetic average but also as a control variable in the variance reduction technique. Thus, the closed-form formula in (7)–(11) with geometric averaging becomes indispensable in the simulation approach to the pricing of options with arithmetic averaging because the formula plays an integrated part in the variance reduction.

Without loss of generality, we focus on the price at the inception ($t = 0$, denoted by $AV(S_1(0), S_2(0), 0, T)$, of a European-style call option on the minimum of two arithmetic averages with maturity T in the simulation that follows. The arithmetic counterpart of (2) is defined as,

$$I_i^A = \int_0^t S_i(s)ds, \quad i = 1, 2, \tag{12}$$

and hence the arithmetic mean over $[0, t]$ is simply $A_i(t) = I_i^A/t$.

To implement the simulation, we take the discrete approximation of A_i defined as follows:

$$A_i(T) = \sum_{j=0}^{n} \frac{S_i(T_j)}{n+1}, \quad i = 1, 2, \tag{13}$$

where $T_j = j(T/n)$ with $T_0 = 0$, $T_n = T$, and $j = 1, \ldots, n$. For comparison with Kemna and Vorst (1990), we choose $T = 4$ months or $1/3$ year and $n = 88$ in our later calculation. In other words, our sampling interval is deemed as one trading day.

Following the risk-neutral valuation argument by Cox and Ross (1976), the price of a European-style call option on the minimum of two arithmetic averages can generally be expressed as follows:

$$AV(S_1(0), S_2(0), 0, T) = e^{-rT} E_Q \left[\text{Max}\{\text{Min}[A_1(T), A_2(T)] - K, 0\}\right], \tag{14}$$

where E_Q is the expectation under a risk-neutral probability measure (see e.g., Harrison and Kreps, 1979, and Harrison and Pliska, 1981).

Let $R_i(T_j) = \ln(S_i(T_j)/S_i(T_{j-1}))$, $(i = 1, 2)$. Under the risk-neutral probability measure, we can replace the drift coefficient, μ_i, in (1) by the instantaneous riskless rate r, and hence, $\{R_1(T_j), R_2(T_j)\}$ is bivariate-normally distributed with means $(r - \frac{1}{2}\sigma_i^2)T/n$, variances $\sigma_i^2 T/n$, $(i = 1, 2)$, and a correlation coefficient ρ. Thus, the random twin sequence $\{S_1(T_1), S_2(T_1)\}, \ldots, \{S_1(T_n), S_2(T_n)\}$ can be generated by the following processes:

$$\ln S_1(T_j) = \ln S_1(T_{j-1}) + \left(r - \frac{1}{2}\sigma_1^2\right)\frac{T}{n} + \sigma_1\sqrt{\frac{T}{n}}x_j \tag{15}$$

$$\ln S_2(T_j) = \ln S_2(T_{j-1}) + \left(r - \frac{1}{2}\sigma_2^2\right)\frac{T}{n} + \sigma_2\sqrt{\frac{T}{n}}y_j \tag{16}$$

where $\{x_j, y_j\}$ is governed by a standard bivariate normal distribution with a correlation coefficient ρ. As a result, $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$ consist of a two-dimensional sequence of independent drawings from the standard bivariate normal distribution.

We implement a total of 10,000 simulation runs. For every run, a realization of a two-dimensional sequence can be obtained and a single simulated option price can be calculated as follows,

$$Z(T) = e^{-rT}\,\mathrm{Max}\{\mathrm{Min}[A_1(T), A_2(T)] - K, 0\}. \tag{17}$$

The simulation estimate of the option price is simply the expected value of $Z(T)$, namely, the mean of $Z(T)$ over 10,000 runs.

The simulation results are reported in Table 1. The choice of the current asset prices of 40 USD, the time-to-maturity of four months, and other various numerical inputs mainly follows the literature (e.g., Cox and Rubinstein, 1985, and Kemna and Vorst, 1990). The column under AV reports the simulated option prices with various inputs. The column under $Std(AV)$ shows the standard error of simulated AV under each set of inputs. For example, in Table 1, Panel A, with the following set of inputs, $r = 0.03$, $\rho = 0.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, $K = 40$ USD, and $T = 4$ months or 1/3 year, the estimated price at the inception of the call option on the minimum of two arithmetic averages is 0.66847 USD with a standard error of 0.01303 USD. If a firm wishes to hedge an average risk exposure of four million USD in the above situation, the standard error of the average hedging cost to the firm will stand at 1,303 USD.

Of course, derivative houses would like to provide prices of their derivative products that are as fair as possible in a competitive market. Fortunately, a more accurate simulation estimate can be achieved by using the variance reduction technique. In order to implement the variance technique to AV, there should be available a control variable, namely a random variable, $W(T)$, which is driven by the same random twin sequence $\{S_1(T_1), S_2(T_1)\}, \ldots, \{S_1(T_n), S_2(T_n)\}$ as for $Z(T)$ in (17) and is a close approximation of $Z(T)$ but has a closed-form expression for its expected value, $E[W(T)]$.

We choose the following random variable as the control variable:

$$W(T) = e^{-rT}\,\mathrm{Max}\{\mathrm{Min}[G_1(T), G_2(T)] - K, 0\} \tag{18}$$

Table 1. Valuation results of options on the minimum of two average prices.[a]

Panel A: Valuation at $r = 0.03$							
σ_1	σ_2	K	AV	$Std(AV)$	AV^*	$Std(AV^*)$	GV^*
$\rho = -0.3$							
		35	3.12351	0.02134	3.14199	0.00050	3.09182
0.2	0.3	40	0.25436	0.00691	0.26012	0.00026	0.25054
		45	0.00081	0.00035	0.00083	0.00004	0.00070
		35	2.90434	0.02295	2.89729	0.00073	2.83643
0.2	0.4	40	0.26721	0.00737	0.27617	0.00038	0.26432
		45	0.00183	0.00049	0.00170	0.00006	0.00145
		35	2.67154	0.02548	2.64503	0.00079	2.57226
0.3	0.4	40	0.33972	0.00968	0.34043	0.00051	0.32273
		45	0.01085	0.00161	0.00991	0.00023	0.00825
$\rho = 0.1$							
		35	3.47097	0.02397	3.47741	0.00049	3.42712
0.2	0.3	40	0.45019	0.01018	0.44149	0.00035	0.42697
		45	0.00540	0.00091	0.00667	0.00010	0.00591
		35	3.29790	0.02625	3.25902	0.00074	3.19653
0.2	0.4	40	0.46524	0.01078	0.46961	0.00043	0.45283
		45	0.01054	0.00147	0.01077	0.00016	0.00953
		35	3.15056	0.03010	3.11394	0.00083	3.03727
0.3	0.4	40	0.60500	0.01432	0.59423	0.00066	0.56737
		45	0.04288	0.00345	0.04425	0.00040	0.03869
$\rho = 0.5$							
		35	3.88484	0.02696	3.88982	0.00051	3.83803
0.2	0.3	40	0.66847	0.01303	0.66665	0.00041	0.64746
		45	0.02294	0.00235	0.02283	0.00022	0.02033
		35	3.67025	0.02919	3.68789	0.00077	3.62390
0.2	0.4	40	0.71828	0.01434	0.70582	0.00052	0.68324
		45	0.03092	0.00256	0.03101	0.00029	0.02751
		35	3.70914	0.03388	3.68633	0.00093	3.60252
0.3	0.4	40	0.91422	0.01897	0.91706	0.00082	0.87999
		45	0.11302	0.00655	0.11377	0.00063	0.10176

[a] The table presents the prices of options on the minimum of two average prices. The options prices with geometric averaging, denoted by GV^*, are determined by the formula in (7)–(11) while the option prices with arithmetic averaging are calculated using the Monte Carlo simulation with (AV^*) and without (AV) a variance reduciton technique. The two underlying asset prices are chosen at $S_1 = S_2 = 40$. The time to maturity is 4 months or $1/3$ year and n = 88. Valuation results at three different values of the interest rate r: 3%, 5%, and 7%, and standard deviations for simulation resutls AV and AV^* are reported in Panels A, B, and C, respectively. In each panel, various values of other input parameters, σ_1, σ_2, ρ, and K, are used. The number of simulation runs is 10,000.

where

$$G_i(T) = \left[\prod_{j=0}^{n} S_i(T_j)\right]^{\frac{1}{n+1}}, \quad i = 1, 2. \tag{19}$$

It is easy to notice that $E[W(T)]$ is the expected price of a call option on the minimum of two geometric averages and its closed-form expression is already given by (7)–(11).

We run the simulation to obtain the estimated value of $E[Z(T) - W(T)]$. Because $Z(T)$ and $W(T)$ are closely related random variables, the estimation errors of both $Z(T)$

Table 1. Continued.

Panel B: Valuation at $r = 0.05$							
σ_1	σ_2	K	AV	$Std(AV)$	AV^*	$Std(AV^*)$	GV^*
$\rho = -0.3$							
		35	3.26151	0.02159	3.23298	0.00048	3.18359
0.2	0.3	40	0.28985	0.00742	0.28772	0.00028	0.27727
		45	0.00089	0.00031	0.00101	0.00004	0.00088
		35	3.02007	0.02329	2.98099	0.00072	2.92010
0.2	0.4	40	0.31224	0.00812	0.30280	0.00037	0.28987
		45	0.00189	0.00052	0.00205	0.00007	0.00177
		35	2.73642	0.02549	2.72267	0.00079	2.64906
0.3	0.4	40	0.36288	0.00985	0.36650	0.00057	0.34661
		45	0.01075	0.00148	0.01083	0.00017	0.00940
$\rho = 0.1$							
		35	3.55305	0.02406	3.56832	0.00048	3.51805
0.2	0.3	40	0.47338	0.01051	0.47607	0.00035	0.46157
		45	0.00757	0.00124	0.00799	0.00014	0.00694
		35	3.30745	0.02617	3.34172	0.00074	3.27945
0.2	0.4	40	0.50033	0.01121	0.50438	0.00047	0.48620
		45	0.01204	0.00166	0.01252	0.00018	0.01099
		35	3.18207	0.02985	3.19158	0.00085	3.11419
0.3	0.4	40	0.61742	0.01446	0.62704	0.00066	0.59889
		45	0.05121	0.00389	0.04851	0.00042	0.04241
$\rho = 0.5$							
		35	3.92185	0.02664	3.98098	0.00052	3.92831
0.2	0.3	40	0.69035	0.01327	0.70942	0.00039	0.69016
		45	0.02227	0.00222	0.02571	0.00022	0.02308
		35	3.80992	0.02937	3.77139	0.00077	3.70627
0.2	0.4	40	0.74090	0.01440	0.74791	0.00052	0.72466
		45	0.03309	0.00276	0.03462	0.00028	0.03096
		35	3.73540	0.03432	3.76491	0.00093	3.68027
0.3	0.4	40	0.93123	0.01887	0.95800	0.00081	0.91931
		45	0.12339	0.00669	0.12194	0.00061	0.10937

and $W(T)$ that are bound to occur during the simulation should be very similar in a well-controlled simulation test. As a result, $E[Z(T)-W(T)]$ incurs very small estimation errors. To obtain the call option price, we take the sum of the simulated result, $E[Z(T) - W(T)]$, and the analytical value, $E[W(T)]$, from (7)–(11). It is worth mentioning that there is an inevitable small bias between the (continuous-time) analytical value and the simulated value of $E[W(T)]$ due to discrete sampling. Nevertheless, such a bias is much offset by a similar bias for $E[Z(T)]$ in simulated $E[Z(T) - W(T)]$. Thus, the estimated $E[Z(T)]$ using the control-variate approach is, strictly speaking, of continuous-time type and has reduced variance since it bears the same small estimation errors as $E[Z(T) - W(T)]$ does.

In Table 1, the column under AV^* reports the simulation estimates of $E[Z(T)]$ for various inputs, using the variance reduction technique described above, and the column under $Std(AV^*)$ shows the standard error of AV* under each set of inputs. The evidence of substantial variance reduction is overwhelming. For example, in Table 1, Panel A, with the same set of inputs, $r = 0.03$, $\rho = 0.5$, $\sigma_1 = 0.2$, $\sigma_2 = 0.3$, and $K = 40$ USD, the estimated

Table 1. Continued.

Panel C: Valuation at $r = 0.07$							
σ_1	σ_2	K	AV	$Std(AV)$	AV^*	$Std(AV^*)$	GV^*
$\rho = -0.3$							
		35	3.30846	0.02152	3.32477	0.00048	3.27525
0.2	0.3	40	0.29849	0.00746	0.31696	0.00028	0.30598
		45	0.00117	0.00036	0.00128	0.00004	0.00110
		35	3.06885	0.02330	3.06476	0.00074	3.00382
0.2	0.4	40	0.33594	0.00839	0.33108	0.00039	0.31713
		45	0.00224	0.00055	0.00256	0.00010	0.00215
		35	2.81670	0.02561	2.80055	0.00079	2.72628
0.3	0.4	40	0.40477	0.01060	0.39329	0.00056	0.37174
		45	0.01062	0.00146	0.01236	0.00019	0.01070
$\rho = 0.1$							
		35	3.64576	0.02415	3.65972	0.00050	3.60883
0.2	0.3	40	0.50545	0.01080	0.51358	0.00034	0.49802
		45	0.00683	0.00098	0.00899	0.00011	0.00812
		35	3.46978	0.02651	3.42370	0.00074	3.36237
0.2	0.4	40	0.54945	0.01195	0.54069	0.00047	0.52116
		45	0.01424	0.00161	0.01438	0.00017	0.01264
		35	3.28295	0.03022	3.26986	0.00086	3.19141
0.3	0.4	40	0.65495	0.01510	0.66019	0.00068	0.63159
		45	0.04716	0.00359	0.05204	0.00039	0.04643
$\rho = 0.5$							
		35	4.07721	0.02755	4.07119	0.00054	4.01835
0.2	0.3	40	0.73767	0.01376	0.75547	0.00041	0.73460
		45	0.02827	0.00238	0.02936	0.00024	0.02615
		35	3.88041	0.02954	3.85420	0.00076	3.78855
0.2	0.4	40	0.79688	0.01494	0.79283	0.00053	0.76756
		45	0.04059	0.00312	0.03921	0.00030	0.03478
		35	3.84091	0.03468	3.84267	0.00093	3.75815
0.3	0.4	40	0.97967	0.01959	0.99959	0.00083	0.95972
		45	0.12300	0.00665	0.13073	0.00066	0.11743

price of the call option on the minimum of two arithmetic averages is 0.66665 USD with a standard error of 0.00041 USD. For an average risk exposure of four million USD, the standard error of the average hedging cost drops to only a paltry 41 USD from 1,303 USD for the estimated price of the same option without using the variance reduction technique.

2. Properties of Options with Two Average Prices

In the last section, we focused only on the pricing of a call option on the minimum of two average prices. The foreign account payables hedging problem illustrated in the introduction requires, however, a call option on the maximum of two average prices. In this section, we show that the call option on the maximum of two average prices can be derived from the option on the minimum of two average prices via a parity relationship. In fact, we discuss several useful parity relationships from which some variants of the options on the minimum or the maximum can be easily derived (subsection 2.1). We also provide the upper and lower bounds for the call option on the minimum of two arithmetic average prices (subsection 2.2).

For ease of discussion in this section, we employ a new notation system as follows. Let $CMA(S_1, S_2, A_1, A_2, K, t)$ be a call option on the minimum of two average prices with the terminal payoff $CMA(T) = \text{Max}\{\text{Min}[A_1(T), A_2(T)] - K, 0\}$, which we use as a benchmark option and whose pricing has been already solved in Section 2. Although arithmetic averaging is presented, the discussion on parity relationships holds for geometric averaging as well.

2.1. Some Useful Parity Relationships

We discuss here some parity relationships among related average-rate European-style options.[7] It is sufficient to show that the parity relationships hold at maturity.

2.1.1. Pricing of a Call Option on the Maximum of Two Average Prices

Let $CXA(S_1, S_2, A_1, A_2, K, t)$ be the price of a call option on the maximum of two average prices with the terminal payoff of $CXA(T) = \text{Max}\{\text{Max}[A_1(T), A_2(T)] - K, 0\}$. It is easy to verify that the following parity relationship holds:

$$CXA(T) = \text{Max}\{A_1(T) - K, 0\} + \text{Max}\{A_2(T) - K, 0\} - CMA(T). \tag{20}$$

Thus, a call option on the maximum of two average prices can be synthetically created by taking long positions in two corresponding ordinary average-rate options and a short position in a call option on the minimum of two average prices, $CMA(S_1, S_2, A_1, A_2, K, t)$.

From parity relationship (20), we find that either value of the options on the minimum or the maximum of two average asset prices is smaller than the sum of the values of two corresponding standard average-rate options. This cost-effectiveness justifies the preference of the proposed option over two corresponding standard average-rate options.

2.1.2. Pricing of an Option on the Best of Two Averages and a Discount Bond

Let $BA(S_1, S_2, A_1, A_2, K, t)$ be the price of an option on the best of two average prices and a discount bond, K, with the terminal payoff $BA(T) = \text{Max}\{A_1(T), A_2(T), K\}$. Clearly, we have

$$BA(T) = K + CXA(T). \tag{21}$$

Thus, the price of an option on the best of two average prices and a discount bond is determined by the value of the discount bond, $Ke^{-r(T-t)}$, and the price of a call option on the maximum of two average prices, $CXA(S_1, S_2, A_1, A_2, K, t)$.

2.1.3. *Pricing of an Average-Rate Exchange Option*

Let $XA(S_1, S_2, A_1, A_2, t)$ be the price of an average-rate exchange option with the terminal payoff $XA(T) = \text{Max}\{A_2(T) - A_1(T), 0\}$. This option is the average-rate version of an exchange option by Margrabe (1978) and is analyzed by Boyle (1993), who provides a closed-form formula, which is a special case of our formula in (7)–(11). As a matter of fact, the payoff of the average-rate exchange option can be decomposed into

$$XA(T) = \text{Max}\{A_2(T), 0\} - \text{Max}\{\text{Min}[A_1(T), A_2(T)], 0\}. \qquad (22)$$

Thus, the price of an exchange average-rate option is determined by the price of an ordinary average-rate option with a zero strike minus the price of a call option on the minimum of two average prices with a zero strike, namely $CMA(S_1, S_2, A_1, A_2, 0, t)$.

2.1.4. *Pricing of a Put Option on the Minimum of Two Average Prices*

Let $PMA(S_1, S_2, A_1, A_2, K, t)$ be the price of a put option on the minimum of two average prices with the terminal payoff of $PMA(T) = \text{Max}\{K - \text{Min}[A_1(T), A_2(T)], 0\}$. Then, we have

$$PMA(T) = K - \text{Max}\{\text{Min}[A_1(T), A_2(T)], 0\} + CMA(T). \qquad (23)$$

Thus, a put option on the minimum of two average prices can be synthetically created by a portfolio of a long position in a discount bond with face value K, a short position in a call option on the minimum of two average prices with a zero strike, which has a value of $CMA(S_1, S_2, A_1, A_2, 0, t)$, and a long position in a call option on the minimum of two average prices with exercise price K, which has a value of $CMA(S_1, S_2, A_1, A_2, K, t)$.

2.1.5. *Pricing of a Put Option on the Maximum of Two Average Prices*

Let $PXA(S_1, S_2, A_1, K, t)$ be the price of a put option on the maximum of two average prices with the terminal payoff of $PXA(T) = \text{Max}\{K - \text{Max}[A_1(T), A_2(T)], 0\}$. Then, we have

$$PXA(T) = K - \text{Max}\{\text{Max}[A_1(T), A_2(T)], 0\} + CXA(T). \qquad (24)$$

Thus, a put option on the maximum of two average prices can be synthetically created by a portfolio of a long position in a discount bond with face value K, a short position in a call option on the maximum of two average prices with a zero strike, whose value is determined by $CXA(S_1, S_2, A_1, A_2, 0, t)$, and a long position in a call option on the maximum of two average prices with exercise price K, whose value is determined by $CXA(S_1, S_2, A_1, A_2, K, t)$.

2.2. Upper and Lower Bounds for the Option with Two Arithmetic Averages

This subsection provides the upper and lower bounds for the option on the minimum or the maximum of two average arithmetic average prices. Our goal is to find tight upper and lower bounds that can be easily determined by the values of traditional options or options with closed-form pricing formulae. We first show such upper and lower bounds for the option on the minimum of two arithmetic average prices. It is easy to verify that the following inequality holds:

$$CMA(T) \leq \text{Min}[\text{Max}\{A_1(T) - K, 0\}, \text{Max}\{A_2(T) - K, 0\}]. \tag{25}$$

Thus, the price of the option on the minimum of two arithmetic average prices is no greater than the minimum value of the two corresponding standard arithmetic average-rate options. As a matter of fact, similar to (25), the geometric counterpart defines an upper bound for the concerned option with geometric averaging as well.

Furthermore, the lower bound for the value of an option with arithmetic averaging can be found using a similar option with geometric averaging. Let $CMG(S_1, S_2, G_1, G_2, K, t)$ be the price of a European-style option on the minimum of two geometric average prices with the terminal payoff of $CMG(T) = \text{Max}\{\text{Min}[G_1(T), G_2(T)] - K, 0\}$, where G_1 and G_2 are defined as a continuous-time version of (19). Then, we have the following relation:

$$CMG(T) \leq CMA(T). \tag{26}$$

Thus, the lower bound of the price of an European-style option on the minimum of two arithmetic average prices is the price of its geometric-average counterpart. This is from the fact that a geometric average is no larger than its arithmetic counterpart.

Table 1, column GV^* reports values of the option on the minimum of two geometric average prices with various input parameters according to the closed-form formula in (7)–(11). It is worth mentioning that (26) is verified by the numerical results that all the values in the GV^* column are smaller than the comparable values in column AV^*, which are the estimated values of the option on the minimum of two arithmetic average prices.

The upper and lower bounds for an option on the maximum of two arithmetic average prices are given as follows:

$$\begin{aligned} &\text{Max}[\text{Max}\{A_1(T) - K, 0\}, \text{Max}\{A_2(T) - K, 0\}] \\ &\quad \leq CXA(T) \\ &\quad \leq \text{Max}\{A_1(T) - K, 0\} + \text{Max}\{A_2(T) - K, 0\} - CMG(T), \end{aligned} \tag{27}$$

where the first inequality can be easily understood from the definition of $CXA(T)$ while the second inequality is from parity relationship (20) using the fact of (26).

3. Applications

The options on the maximum or the minimum of two average prices that we have discussed so far not only have applications in risk management (subsections 3.1–3.3) but also enter in the payoff function of an incentive contract of a management compensation plan (subsection 3.4).

3.1. Hedging under a Flexible Two-FX Invoicing Policy

As discussed in the introduction, the flexible invoicing policy a foreign exporter offers can benefit a domestic importer if the option on the maximum of two average prices is available. Let S_1 and S_2 be the two foreign exchange rates (quoted as USD/FC) that the foreign exporter is willing to receive, and let c_1 and c_2 be the positive constant foreign account payables in S_1 and S_2, respectively, in each period for an extended period T. The US importer is concerned about the average cost in USD during the period and wishes to cap it at K without any regret later about which foreign currency it has chosen in the first place.

The desire of the US firm can be best met by its holding an option with the terminal payoff $\text{Max}\{\text{Max}[c_1 B_1, c_2 B_2] - K, 0\}$, where B_1 and B_2 are simply two arithmetic averages of the spot rates S_1 and S_2 over the concerned period, respectively. Define $A_1 = c_1 B_1$, and $A_2 = c_2 B_2$. The price of the option is given by $CXA(S_1, S_2, A_1, A_2, K, t)$.

3.2. Hedging Production Costs with Stochastic Prices of Two Substitutable Inputs

Some production facilities are designed to be able to take either of two kinds of substitutable raw materials as inputs. For example, an oil refinery can use two kinds of different crude oils. If the firm orders periodic shipments of one kind of raw material for an extended period and wishes to hedge the average production cost for the period without worrying about what kind of raw material would turn out to be cost-effective, it is most suitable for the firm to hold an option on the maximum of the two average prices.

Let S_1 and S_2 be the prices of the two kinds of substitutable raw materials the firm can use as inputs, and let h_1 and h_2 be the comparable, constant quantities of the two kinds of raw materials in each shipment for an extended period T. The desire of the firm to be able to cap the average cost at K without any regret later about which raw materiel has been chosen in the first place can be best met by the option with the terminal payoff $\text{Max}\{\text{Max}[h_1 D_1, h_2 D_2] - K, 0\}$, where D_1 and D_2 are simply arithmetic averages of the prices S_1 and S_2 over the concerned period, respectively. Define $A_1 = h_1 D_1$, and $A_2 = h_2 D_2$. The price of the option is given by $CXA(S_1, S_2, A_1, A_2, K, t)$.

3.3. Hedging Profit Markups with Stochastic Input and Output Prices

An oil refinery faces the hedging problem with price uncertainties not only in inputs like crude oil but also in outputs such as heating oils and jet fuels that the refinery produces. Risk management using a single derivative that takes into account all stochastic prices involved in production is desired because of the cost-effectiveness of hedging.

If a firm wants to hedge away unwanted price movements of the finished products as well as those of the raw material, two ordinary average-rate options are traditionally needed. However, the average exchange option that consists of the option on the minimum of two average prices proposed in this paper can provide a one-shot solution to this hedging problem.

Let S_1 and S_2 be the price of input and the price of the output of a comparable unit, respectively, and let π be the average profit markup (a positive percentage) for each production cycle the firm wishes to secure for an extended period T. Thus, the firm's desire can be best met by holding an option with the terminal payoff $\text{Max}\{(1+\pi)H_1 - H_2, 0\}$, where H_1 and H_2 are simply arithmetic averages of the input and output prices S_1 and S_2 over the concerned period, respectively. Define $A_1 = (1+\pi)H_1$, and $A_2 = H_2$. This is the average-rate exchange option and its price is given by $XA(S_1, S_2, A_1, A_2, t)$.

3.4. Incentive Contracts for Executive Compensation

The incentive contract discussed in Stulz (1982) can be motivated by using average underlying prices. For one thing, average prices are anti-manipulative. On such an incentive contract, consequently, managers find it difficult to boost their firm's stock prices temporarily just because they want to exercise the options that are traditionally included in their compensation package. Moreover, the average feature smoothes out the randomness and the "noise" inherent in the stock prices so that the managers can be evaluated more fundamentally.

Let S_1 and S_2 be the stock prices of the managers' firm and the competing firm or the industrial representative, and let L_1 and L_2 be the average prices, respectively, for an evaluation period T. The compensation plan involves a fixed amount of reward and a variable incentive-related reward. Following Stulz (1982), the latter can be designed to have the following payoff

$$\text{Min}[\text{Max}\{\gamma L_1 - \delta L_2, 0\}, \text{Max}\{\gamma L_1 - K, 0\}],$$

where γ and δ are positive constants for desired adjustments, and K is a fixed performance benchmark.

The managers would not be entitled to an extra reward if $\gamma L_1 < \delta L_2$ (worse than competitors) or $\gamma L_1 < K$ (below the mark). Otherwise, the extra reward depends on how they have outperformed the competitors and the pre-agreed fixed benchmark. As argued earlier, the incentive contract so designed is truly merit-based and should be fair to both managers and shareholders.

The payoff of the incentive contract can be decomposed into

$$\text{Max}\{\gamma L_1 - K, 0\} - \text{Max}\{\text{Min}\{\gamma L_1, \delta L_2\} - K, 0\}.$$

It follows that the value of the incentive contract is the value of a portfolio of a long position in an ordinary average-rate option and a short position in an option on the minimum of two average prices.

4. Concluding Remarks

This paper presents a new variety of financial derivatives: options on the minimum or the maximum of two average asset prices. In particular, we provide a closed-form pricing formula for the options with the geometric averaging, which may start at any time before maturity. We then price the option with arithmetic averaging in conventional simulations. The overwhelming numerical evidence demonstrated in the paper confirms that the variance reduction technique with the help of the above closed-form formula dramatically improves the accuracy of the simulated price. Thus, substantial savings in computation time can be achieved.

One way for financial managers to add value to their firms is to use financial innovations which are able, for example, to mitigate financial distress costs. Thus, risk management is closely liked to financial innovations and has become increasingly important to modern corporations. The proposed options that non-trivially bridge the Asian options and the Rainbow options obviously provide great value-added potentials. In particular, the options not only have wide applications in risk management but also appropriately enter the payoff function of incentive contracts for management compensation, a problem in the theory of corporate finance.

Appendix A: Derivation of the Pricing Formula in (7)–(11)

To start with, we apply the following transformation

$$y_i = \frac{I_i}{T} + \frac{T-t}{T}\ln S_i, \qquad i = 1, 2, \tag{28}$$

$$V(S_1, S_2, I_1, I_2, t) = U(y_1, y_2, t), \tag{29}$$

to equation (3) and (4) to yield

$$\begin{aligned} \frac{\partial U}{\partial t} &+ \frac{1}{2}\frac{(T-t)^2}{T^2}\left(\sigma_1^2\frac{\partial^2 U}{\partial y_1^2} + 2\sigma_1\sigma_2\rho\frac{\partial^2 U}{\partial y_1\partial y_2} + \sigma_2^2\frac{\partial^2 U}{\partial y_2^2}\right) \\ &+ \left(r - \frac{1}{2}\sigma_1^2\right)\frac{T-t}{T}\frac{\partial U}{\partial y_1} + \left(r - \frac{1}{2}\sigma_2^2\right)\frac{T-t}{T}\frac{\partial U}{\partial y_2} - rU = 0, \end{aligned} \tag{30}$$

$$U(y_1, y_2, T) = \max\left[\min(e^{y_1}, e^{y_2}) - K, 0\right]. \tag{31}$$

This transformation is also employed by Wilmott, Howison and Dewynne (1995) in pricing a standard Asian option.

Next, we apply another transformation

$$z_i = y_i - \int_T^t \left(r - \frac{1}{2}\sigma_i^2\right)\frac{T-t}{T}dt = y_i + \left(r - \frac{1}{2}\sigma_i^2\right)\frac{(T-t)^2}{2T}, i = 1, 2, \tag{32}$$

$$U(y_1, y_2, t) = e^{-r(T-t)}W(z_1, z_2, t) \tag{33}$$

to equations (30) and (31) to obtain

$$\frac{\partial W}{\partial t} + \frac{1}{2}\frac{(T-t)^2}{T^2}\left(\sigma_1^2\frac{\partial^2 W}{\partial z_1^2} + 2\sigma_1\sigma_2\rho\frac{\partial^2 W}{\partial z_1\partial z_2} + \sigma_2^2\frac{\partial^2 W}{\partial z_2^2}\right) = 0, \tag{34}$$

$$W(z_1, z_2, T) = \max\left[\min(e^{z_1}, e^{z_2}) - K, 0\right]. \tag{35}$$

Then, we transform the time variable as follows:

$$\tau = -\int_T^t \frac{(T-s)^2}{T^2} ds = \frac{(T-t)^3}{3T^2}. \tag{36}$$

It follows that equations (34) and (35) above become

$$\frac{\partial W}{\partial \tau} = \frac{1}{2}\sigma_1^2 \frac{\partial^2 W}{\partial z_1^2} + \sigma_1 \sigma_2 \rho \frac{\partial^2 W}{\partial z_1 \partial z_2} + \frac{1}{2}\sigma_2^2 \frac{\partial^2 W}{\partial z_2^2}, \tag{37}$$

$$W(z_1, z_2, 0) = \max\left[\min(e^{z_1}, e^{z_2}) - K, 0\right], \tag{38}$$

where equation (37), after an appropriate change of variables, becomes a classical two-variable heat equation (see, e.g., Logan, 1998). Further, we realize that the following transformation

$$x_i = e^{z_i - (r - \frac{1}{2}\sigma_i^2)\tau}, \tag{39}$$

$$W(z_1, z_2, \tau) = F(x_1, x_2, \tau) e^{r\tau} \tag{40}$$

can transform equations (37) and (38) to the PDE in Stulz (1982):

$$\begin{aligned} \frac{\partial F}{\partial \tau} &= \frac{1}{2}\sigma_1^2 x_1^2 \frac{\partial^2 F}{\partial x_1^2} + \rho \sigma_1 \sigma_2 x_1 x_2 \frac{\partial^2 F}{\partial x_1 \partial x_2} \\ &\quad + \frac{1}{2}\sigma_2^2 x_2^2 \frac{\partial^2 F}{\partial x_2^2} + r x_1 \frac{\partial F}{\partial x_1} + r x_2 \frac{\partial F}{\partial x_2} - rF, \end{aligned} \tag{41}$$

$$F(x_1, x_2, 0) = \max\left[\min(x_1, x_2) - K, 0\right]. \tag{42}$$

And the solution to equations (41) and (42), which was first derived by Stulz (1982) and has been verified by us using Green's function approach, takes the following form:

$$\begin{aligned} F(x_1, x_2, \tau) &= x_1 N_2 \left(\frac{\ln \frac{x_1}{K} + \left(r + \frac{1}{2}\sigma_1^2\right)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln \frac{x_2}{x_1} - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \frac{\rho\sigma_2 - \sigma_1}{\sigma} \right) \\ &\quad + x_2 N_2 \left(\frac{\ln \frac{x_2}{K} + \left(r + \frac{1}{2}\sigma_2^2\right)\tau}{\sigma_2\sqrt{\tau}}, \frac{\ln \frac{x_1}{x_2} - \frac{1}{2}\sigma^2\tau}{\sigma\sqrt{\tau}}, \frac{\rho\sigma_1 - \sigma_2}{\sigma} \right) \\ &\quad - K e^{-r\tau} N_2 \left(\frac{\ln \frac{x_1}{K} + \left(r - \frac{1}{2}\sigma_1^2\right)\tau}{\sigma_1\sqrt{\tau}}, \frac{\ln \frac{x_2}{K} + (r - \frac{1}{2}\sigma_2^2)\tau}{\sigma_2\sqrt{\tau}}, \rho \right), \end{aligned} \tag{43}$$

where $N_2(\alpha, \beta, \theta)$ is the bivariate cumulative standard normal distribution with upper limits of integration α and β, correlation coefficient θ, and

$$\sigma^2 = \sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2. \tag{44}$$

Thus, the solution to equations (37) and (38) is given by

$$\begin{aligned} W(z_1, z_2, \tau) \;=\; & e^{z_1+\frac{1}{2}\sigma_1^2\tau} N_2 \left(\frac{\ln \frac{e^{z_1}}{K} + \sigma_1^2\tau}{\sigma_1\sqrt{\tau}}, \frac{z_2 - z_1 + \frac{1}{2}(\sigma_2^2 - \sigma_1^2 - \sigma^2)\tau}{\sigma\sqrt{\tau}}, \frac{\rho\sigma_2 - \sigma_1}{\sigma} \right) \\ & + e^{z_2+\frac{1}{2}\sigma_2^2\tau} N_2 \left(\frac{\ln \frac{e^{z_2}}{K} + \sigma_2^2\tau}{\sigma_2\sqrt{\tau}}, \frac{z_1 - z_2 + \frac{1}{2}(\sigma_1^2 - \sigma_2^2 - \sigma^2)\tau}{\sigma\sqrt{\tau}}, \frac{\rho\sigma_1 - \sigma_2}{\sigma} \right) \\ & - K N_2 \left(\frac{\ln \frac{e^{z_1}}{K}}{\sigma_1\sqrt{\tau}}, \frac{\ln \frac{e^{z_2}}{K}}{\sigma_2\sqrt{\tau}}, \rho \right). \end{aligned} \tag{45}$$

It follows that the solution to our original problem (3)–(6) becomes

$$V(S_1, S_2, I_1, I_2, t) = e^{-r(T-t)} W(z_1, z_2, \tau), \tag{46}$$

where function $W(z_1, z_2, \tau)$ is given by (45), with z_i, $i = 1, 2$, defined by (32) using (28) and τ defined by (36) respectively.

Alternatively, the solutin to our problem (3)–(6) can be written as

$$V(S_1, S_2, I_1, I_2, t) = e^{r\tau - r(T-t)} F(x_1, x_2, \tau), \tag{47}$$

where the function $F(x_1, x_2, \tau)$ is given by (43), with

$$x_i = \exp\left(\frac{I_i}{T} + \frac{T-t}{T} \ln S_i + \left(r - \frac{1}{2}\sigma_i^2 \right) \frac{(T-t)^2}{2T} - \left(r - \frac{1}{2}\sigma_i^2 \right) \tau \right), \quad i = 1, 2. \tag{48}$$

By doing some algebra, the solution to our problem (3)–(6) can be further written as (7)–(11) in the paper.

Appendiz B: Pricing the Forward-start-averaging Option with the Minimum or the Maximum of Two Geometric Average Prices

For a forward-start-averaging option on the minimum or the maximum of two geometeric average prices, we use time notations as follows: 0 = start of the option; t = option valuation date; T_0 = start of the averaging; and T = maturity of the option or the end of the averaging. We assume $0 \le t \le T_o < T$, with the forward-start-averaging taken over $[T_o, T]$. Note that for $0 \le T_0 \le t \le T$, we can invoke the “plain vanilla” pricing formula of (7)–(11).

We already know from (7)–(11) that the option price at $t = T_0$ as follows:

$$\begin{aligned} &V(S_1, S_2, K, T_0) \\ &= S_1^* N_2\left(\frac{\ln\frac{S_1^*}{K} + \left(r + \frac{1}{2}\sigma_1^{*2}\right)(T-T_0)}{\sigma_1^*\sqrt{T-T_0}}, \frac{\ln\frac{S_2^*}{S_1^*} - \frac{1}{2}\sigma^{*2}(T-T_0)}{\sigma^*\sqrt{T-T_0}}, \frac{\rho\sigma_2^* - \sigma_1^*}{\sigma^*}\right) \\ &\quad + S_2^* N_2\left(\frac{\ln\frac{S_2^*}{K} + \left(r + \frac{1}{2}\sigma_2^{*2}\right)(T-T_0)}{\sigma_2^*\sqrt{T-T_0}}, \frac{\ln\frac{S_1^*}{S_2^*} - \frac{1}{2}\sigma^{*2}(T-T_0)}{\sigma^*\sqrt{T-T_0}}, \frac{\rho\sigma_1^* - \sigma_2^*}{\sigma^*}\right) \\ &\quad - Ke^{-r(T-T_0)} N_2\left(\frac{\ln\frac{S_1^*}{K} + \left(r - \frac{1}{2}\sigma_1^{*2}\right)(T-T_0)}{\sigma_1^*\sqrt{T-T_0}}, \frac{\ln\frac{S_2^*}{K} + \left(r - \frac{1}{2}\sigma_2^{*2}\right)(T-T_0)}{\sigma_2^*\sqrt{T-T_0}}, \rho\right) \end{aligned} \tag{49}$$

where

$$S_i^* = S_i e^{(\mu_i^* - r)(T-T_0)}, \qquad i = 1, 2, \tag{50}$$

$$\mu_i^* = \frac{1}{2}\left(r - \frac{1}{2}\sigma_i^2\right) + \frac{1}{6}\sigma_i^2, \qquad i = 1, 2, \tag{51}$$

$$\sigma_i^* = \frac{\sigma_i}{\sqrt{3}}, \qquad i = 1, 2, \tag{52}$$

$$\sigma^* = \frac{\sigma}{\sqrt{3}}, \quad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2} \tag{53}$$

Thus, the option price at $t < T_0$ is simply the t-time value of a derivative with a terminal value at T_0 determined by (49)–(53), i.e.,

$$f(S_1, S_2) = V(S_1, S_2, T_0). \tag{54}$$

It follows that the option price can be obtained by solving the following integral:

$$\begin{aligned} &V(S_1, S_2, t) \\ &= e^{-rl}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(S_1 e^{\sigma_1 y_1\sqrt{l} + (r - \frac{1}{2}\sigma_1^2)l}, S_2 e^{\sigma_2 y_2\sqrt{l} + (r - \frac{1}{2}\sigma_2^2)l}) n_2(y_1, y_2, \rho) dy_1\, dy_2, \end{aligned} \tag{55}$$

where $l = T_0 - t$ and n_2 is the bivariate normally distributed density function. Through some tedious algebra, we have the forward-start-averaging option formula:

$$
\begin{aligned}
&V(S_1, S_2, K, t)\\
&= S_1^* N_2\left(\frac{\ln\frac{S_1^*}{K} + (r + \frac{1}{2}\sigma_1^{*2})(T-t)}{\sigma_1^*\sqrt{T-t}}, \frac{\ln\frac{S_2^*}{S_1^*} - \frac{1}{2}\sigma^{*2}(T-t)}{\sigma^*\sqrt{T-t}}, \frac{\rho\sigma_2^* - \sigma_1^*}{\sigma^*}\right)\\
&\quad + S_2^* N_2\left(\frac{\ln\frac{S_2^*}{K} + (r + \frac{1}{2}\sigma_2^{*2})(T-t)}{\sigma_2^*\sqrt{T-t}}, \frac{\ln\frac{S_1^*}{S_2^*} - \frac{1}{2}\sigma^{*2}(T-t)}{\sigma^*\sqrt{T-t}}, \frac{\rho\sigma_1^* - \sigma_2^*}{\sigma^*}\right)\\
&\quad - Ke^{-r(T-t)} N_2\left(\frac{\ln\frac{S_1^*}{K} + (r - \frac{1}{2}\sigma_1^{*2})(T-t)}{\sigma_1^*\sqrt{T-t}}, \frac{\ln\frac{S_2^*}{K} + (r - \frac{1}{2}\sigma_2^{*2})(T-t)}{\sigma_2^*\sqrt{T-t}}, \rho\right) \qquad (56)
\end{aligned}
$$

where

$$S_i^* = S_i e^{(\mu_i^* - r)(T-T_0)}, \qquad i = 1, 2, \qquad (57)$$

$$\mu_i^* = \frac{1}{2}\left(r - \frac{1}{2}\sigma_i^2\right) + \frac{1}{6}\sigma_i^2, \qquad i = 1, 2, \qquad (58)$$

$$\sigma_i^* = \sigma_i\sqrt{1 - \frac{2}{3}\frac{T-T_0}{T-t}}, \qquad i = 1, 2, \qquad (59)$$

$$\sigma^* = \sigma\sqrt{1 - \frac{2}{3}\frac{T-T_0}{T-t}}, \qquad \sigma = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho\sigma_1\sigma_2}. \qquad (60)$$

Note that (56) takes the same form as (7), but (57)–(60) are different from their counterparts (8)–(11).

Notes

1. Most exotic options are promoted as more cost-effective than ordinary options. The option on the maximum of two average prices is more expensive than a standard Asian option but is cheaper than the portfolio of two corresponding Asian options, as shown later in the paper.
2. Commodity (such as oil) bonds come into being as a good application of the Asian option. A commodity bond allows the bond holder to redeem a fixed face value plus the difference, if positive, between a variable related to an average of the commodity prices and the face value (e.g., the Oranje Nassau bond, which is detailed in Kemna and Vorst, 1990, and Bouaziz, Briys, and Crouhy, 1994). The average feature commodity bonds embed purges much of the influence of the high volatility of commodity prices due to temporary manipulation. Such bonds have two particular advantages. First, the issuing firm can enjoy built-in hedging by tying its debt financing cost to its future revenues from producing the commodity. Second, those investors who are not able or not allowed to trade commodity options can have a more sophisticated bet on commodity prices. Naturally, such commodity bonds can be easily modified to the case of two commodities the firm produces.
3. See e.g., Kemna and Vorst (1990), Haykov (1993), and Corwin, Boyle, and Tan (1996). Extensive research has focused on other computationally efficient techniques such as the analytic approximations that yield closed-form expressions (see e.g., Turnbull and Wakeman, 1991, Levy, 1992, Geman and Yor, 1992, Vorst, 1992, 1996, Bouaziz, Briys and Crouhy, 1994, and Milevsky and Posner, 1998).

4. Since the options this paper develops have characteristics of both Asian options and Rainbow options, we can appropriately call them Asian Rainbow options. In general, the use of Asian Rainbow options is legitimately justified as long as Stulz's Rainbow options require a modification with average prices. The Asian Rainbow options are shown to generate many important and interesting applications.
5. For the ease of presentation, we focus on the "plain vanilla" case, namely the averaging is taken over the whole life of a new option, i.e., $[0, T]$. For completeness while saving space, we deal with the forward-start-averaging, which starts in the middle of the life of a new option, T_0 $(0 \leq T_o \leq T)$, in Appendix B. In fact, as Kemna and Vorst (1990) have already discussed regarding the forward-start-averaging setting with an ordinary Asian option, the value of the proposed option at t before $[T_0, T]$ can be determined in two steps. First, the option with averaging taken over $[T_0, T]$ is evaluated at T_0 as the "plain vanilla" case. Second, the value of the derivative with the terminal value at T_0 that is set equal to the calculated value of the option with two average prices in the first step can be routinely determined.
6. In this paper we use the variance-reduction technique which is first employed by Boyle (1977) for estimating option prices. For innovations in variance-reduction techniques with applications in path-dependent options, see e.g., Glasserman, Heidelberger, and Shahabuddin (1999) which combines the importance sampling with the stratified sampling.
7. Stulz (1982) has a similar discussion on his options on the minimum or the maximum of two risky assets. Ritchken (1996) presents additional parity relationships related to Stulz's options.

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