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A New Stochastic Duration Based on the Vasicek and CIR Term Structure Theories

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1. INTRODUCTION

Macaulay's duration, being the most easily understandable measure of exposure to interest rate risk, is widely used by practitioners for the purpose of bond immunization.[1] In its original form, the model unrealistically assumes a flat yield curve and purely parallel shifts in the yield curve; but modifications made in Bierwag and Kaufman (1977), Bierwag (1977) and Khang (1979), now allow for non-flat term structures as well as linear and decreasing-in-term shifts. However, as pointed out by Ingersoll, Skelton and Weil (hereafter, ISW) (1978) and Cox, Ingersoll and Ross (hereafter, CIR) (1979), a more fundamental problem with the model is that even in its generalized form it cannot be an equilibrium model, because it violates the no-arbitrage condition. For these reasons, they suggest a theoretically sounder measure of yield curve risk that is based on the CIR one-factor term structure model. Boyle (1977) discusses a similar duration measure that is based on the Vasicek model. However, ISW (1978) and CIR (1979) did not empirically test the so-called stochastic duration (hereafter, ISW/CIR's

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duration), although they show numerically that the empirically observed mean-reverting property of short-term interest rate dynamics is consistent with dampened fluctuations of long yields, which are captured by their stochastic duration but missed badly by the traditional duration measure.

Unfortunately, using coupon bond data, empirical tests of stochastic duration measures based on these (or even more sophisticated) theoretical models have not demonstrated any actual superiority to the simple Macaulay duration.[2] One apparent reason is that both the Vasicek and CIR models imply near-constant zero-coupon yields at the longest maturities; thus, they fail to capture the movement in the long end of yield curves so badly that even the ad hoc Macaulay duration still turns out to be superior. The practical advantages of Macaulay's duration seem to stem from two aspects: (a) internal rates of return reflect averages of zero-coupon rates, which means that they can capture a lot of yield-curve information; and (b) internal rates of return are bond- and hence maturity-specific, so that they capture the average zero-coupon rates that are relevant for that particular bond. In contrast, one-factor models focus on one specific yield only, the instantaneous rate, which would be justified only if, as predicted by the theoretical term structure models, all zero-coupon yields were intimately linked to this short-term rate – and this is manifestly not the case.

Given the poor immunization performance caused by the misspecification of the one-factor equilibrium term structure models, this paper proposes an alternative stochastic duration measure within the one-factor framework. The proposed stochastic duration uses the change in a longer zero-coupon yield rather than the instantaneous rate as a proxy for the relevant risk source of the unexpected changes in interest rates. Empirically, a longer yield carries more useful information on the movement of the term structure than the instantaneous interest rate and, thus, compensates for much of the information loss due to specification errors in the theoretical models. More precisely, the risk-factor proxy we use in the price sensitivity (or duration measure) of a coupon bond with time to maturity τ is specified as the change in a $w\tau$-period zero-coupon yield, where w is a number below unity and, in fact, close to zero. Of course, these risk factors differ across bonds with different maturities;

thus, in the logic of the one-factor models these $w\tau$-yields (factors) should all have different standard deviations. To make the modified duration work effectively, however, one has to assume that the differences of these factors are trivial. Although such an assumption is not literally compatible with the underlying bond pricing model, it is not *a priori* a very unreasonable assumption if all these $w\tau$-yields (factors) still fall in the short end of the maturity spectrum and are, therefore, very similar across bonds. Nevertheless, the optimal choice of w is largely empirically determined.

The advantages of the proposed approach are two-fold. First, like the bond's theoretical sensitivity to the instantaneous rate and in contrast to the standard Macaulay duration, the proposed interest-exposure measure preserves much of the theoretical and empirical tractability that one-factor term structure models enjoy and can be directly obtained from the models that a financial firm may already be using for the pricing of bonds. Second, the performance of the proposed measure of interest risk is quite comparable to Macaulay's duration and definitely superior to the performance of a bond's theoretical sensitivity to the instantaneous rate. This claim is substantiated by the results of a bond-immunization performance test using data on Belgian default-free, non-callable bonds. The test shows unequivocally that the new durations from both the Vasicek and CIR versions are superior to the original ISW/CIR duration and, for values of w between 2.5% and 5%, often outperform Macaulay's duration. The success of the modified stochastic duration becomes possible because, as proven in Appendix A, the new stochastic duration magnifies the original ISW/CIR duration, and does so more for long bonds than for short bonds.

Thus, one contribution of this paper to the literature is that it provides a stochastic duration measure preserving considerable theoretical and empirical tractability, which outperforms Macaulay's duration in reasonable cases and beats definitely the theoretical measures of term structure risk. Most importantly, the proposed approach sheds light on how practitioners can apply term structure models which are bound to suffer specification errors. A second contribution is that the paper adds one more dimension in testing the Vasicek and CIR one-factor term structure models. While much work has been done in estimating the competing models' parameters, little empirical work has been provided as to how well these models fare in bond immunization.[3]

At first sight, immunizing M bonds with M factors may look similar to using an M-factor, APT-style model (Ross, 1976). However, there is a fundamental difference: in our application, the M yields are mathematically linked by a term structure theory while no functional structure is imposed among the APT factors. One implication is that, in this paper, the sensitivity of each bond to its own factor is not constant, but varies over time in a way that is determined by the underlying bond pricing model. In contrast, empirical work on APT bond models assumes constant exposures to pre-specified factors. Because of this fundamental difference in modeling, our use of the zero-coupon yields with only short maturities does not necessarily contradict, for instance, the finding by Elton, Gruber and Michaely (1990) that the four-year spot rate best proxies for the risk source of a one-factor model of the APT type.

The remainder of this paper is organized as follows. Section 2 gives a brief review of the Vasicek and CIR one-factor term structure models and proposes the new stochastic duration measure in both the Vasicek and CIR versions. Section 3 describes the data and the estimates of model parameters. Section 4 compares immunization performance among different duration measures. Performance evaluation is based on a comparison, across models, of root mean square errors (RMSE's) of daily time series of residual returns of individual bonds. The residual return of a bond is defined as the difference between the holding period return of the bond and that of a duration-and-value matched portfolio formed from the rest of the bonds in the sample. Section 5 concludes the paper.

2. TERM STRUCTURE MODEL AND STOCHASTIC DURATION

For the sake of clarity and continuity, the Vasicek and CIR term structure models are briefly presented before introducing the new duration measure. In a one-state-variable model, the price at t, $P(r, t, T)$, of a zero bond maturing at T, is treated as a contingent claim on the instantaneous interest rate, $r(t)$, which usually follows a mean reverting stochastic process:

$$dr = \kappa[m - r(t)] + \delta(r,\ t)dz. \tag{1}$$

Mean reversion means that $r(t)$ is pulled back toward its long term mean, m, at rate of κ. The volatility, $\delta(r,\ t)$, is constant, σ, for the Vasicek model and $\sigma\sqrt{r(t)}$ for the CIR model. In (1), dz is a Wiener process, and κ and m are positive constants.

Since dz is the only risk source of both the contingent claim and the state variable, $r(t)$, a locally risk-free hedge can be established and hence there exists an intertemporal no-arbitrage condition, known as the fundamental PDE,

$$\frac{\partial P}{\partial t} + \mu(r,\ t)\frac{\partial P}{\partial r} + \frac{\delta(r,\ t)}{2}\frac{\partial^2 P}{\partial r^2} - rP = 0. \tag{2}$$

In the Vasicek model, $\mu(r,\ t)$ equals $\kappa[m - r(t)] - q\sigma$ with q being specified as a constant price of risk of changes in the instantaneous rate, $r(t)$, while in the CIR model, $\mu(r,\ t)$ equals $\kappa[m - r(t)] - q(r,\ t)\sigma(r,\ t) = \kappa m - (\kappa + \lambda)r(t)$ with the risk premium, $\lambda r(r,\ t) = q(r,\ t)\sigma(r,\ t)$, being endogenously determined. The other parameters are the same as in (1).

Given boundary conditions of a zero-coupon bond, both the Vasicek and CIR models lead to closed-form bond pricing formulas. Because of the constant volatility specification in (1), the Vasicek zero-coupon bond pricing model takes a simpler form:

$$P(r,\ \tau) = A(\tau)e^{-r(t)B(\tau)}, \tag{3}$$

with

$$A(\tau) = e^{\phi(1-\kappa\tau-e^{-\kappa\tau})-\phi_2(1-e^{-\kappa\tau})^2}, \tag{4}$$

and

$$B(\tau) = \frac{1}{\kappa}(1 - e^{-\kappa\tau}), \tag{5}$$

where $\phi_1 = \dfrac{\kappa m - q\sigma}{\kappa^2} - \dfrac{1}{2}\dfrac{\sigma^2}{\kappa^3}$; $\phi_2 = \dfrac{1}{4}\dfrac{\sigma^2}{\kappa^3}$; and τ is time-to-maturity, i.e., $\tau = T - t$.

The CIR model also conforms with the general form in (3). However, both $A(\tau)$ and $B(\tau)$, which are solely determined by time to maturity, become less neat, namely:

$$A(\tau) = \left[\frac{\theta_1 e^{\theta_2 \tau}}{\theta_2(e^{\theta_1 \tau} - 1) + \theta_1}\right]^{\theta_3}, \tag{6}$$

and

$$B(\tau) = \frac{e^{\theta_1 \tau} - 1}{\theta_2(e^{\theta_1 \tau} - 1) + \theta_1}, \tag{7}$$

where $\theta_1 = \sqrt{(\kappa + \lambda)^2 + 2\sigma^2}$; $\theta_2 = \dfrac{\kappa + \lambda + \theta_1}{2}$; and $\theta_3 = \dfrac{2\kappa m}{\sigma^2}$.

In cross-sectional estimation, the unobservable state variable, $r(t)$, can be estimated as an implied instantaneous rate. So, there are four parameters in each of the models. Since a zero coupon bond with a maturity of more than one year is not always available in all countries, estimation is usually done using coupon bond data. In fact, the price of a coupon bond at t, $P(r, c, t, T)$, with coupon rate, c, maturing at T, is simply a portfolio of zero coupon bonds, whose prices are determined directly by the zero-coupon bond pricing models in (3), (4) and (5) or (3), (6) and (7). These models can be easily expressed in the form of the continuously compounded zero-coupon yield, namely:

$$R(r, \tau) = -\frac{1}{\tau} \ln P(r, \tau). \tag{8}$$

With the above estimated term structures, bond price sensitivity to interest rate changes can be expressed in a more elegant fashion. Duration in general is a measure of bond return risk caused by unanticipated changes in interest rates. In the traditional Macaulay duration measure, the unanticipated changes are assumed to come from the parallel shifts in a flat term structure of interest rates; that is, all yields are assumed to have the same variance and to be perfectly correlated. Theoretically as well as empirically, the assumption of pure parallel shifts and flat term structures is untenable. In contrast, the one-factor Vasicek and CIR framework identifies a specific risk source of the unanticipated changes in interest rates because the whole term structure movement is assumed to hinge on the instantaneous rate, $r(t)$. Still, it would be naive to believe that the unexpected changes in interest rates can be satisfactorily explained by just the instantaneous rate. To solve this problem, one could adopt two- or multiple-factor models; however, these are often hard to estimate. Alternatively, one could look for a single risk-factor that (a) is better at capturing twists and shifts of the term structure of interest rates than the instantaneous interest rate, and (b) is actually observable. The latter alternative is the route adopted in this paper.

Suppose, initially, that the unexpected changes in interest rates can be tracked by one state variable, the instantaneous interest rate. Even though one-week (and, in many markets, overnight) interest rates are observable, the instantaneous rate itself is not. However, one can always construct an intermediary and observable factor, F, that, in turn, is driven by the instantaneous rate. Then a general duration measure takes the following form:

$$\begin{aligned}\text{Duration} &= \frac{-1}{P(r,\ c,\ t,\ T)} \frac{dP(r,\ c,\ t,\ T)}{dF} \\ &= \frac{-1}{P(r,\ c,\ t,\ T)} \sum_{l=1}^{N} CF(\tau_l) \frac{dP(r,\ \tau_l)}{dF}, \qquad (9)\end{aligned}$$

where $P(r,\ c,\ t,\ T) = \sum_{l=1}^{N} CF(\tau_l)P(r,\ \tau_l)$ is the price of a bond with N coupons (plus face value at T) to be due and $CF(\tau_l)$ is the l-th component cash flow with time to due date, $\tau_l(\tau_l \leq \tau = T - t;\ \forall l)$. As the intermediary risk factor, F, we propose one of the zero-coupon yields in (8), and we assume it is able to capture, with the help of a term structure model, the unanticipated changes in all interest rates.

Thus, one can easily get around the problem of unobservability of the instantaneous rate. More problematic is the fundamental assumption that one single factor, such as $F(r(t))$ or $r(t)$ itself, drives all bond prices. In reality, even the best-chosen single factor can only capture the systematic risk for all bonds, that is, the risk caused by the 'average' component of shifts in the term structure. But the term structure movements often cause more unanticipated price changes for one group of bonds than for another, as illustrated by, for instance, Elton, Gruber and Michaely (1990). In other words, one factor does not seem to be able to take care, at the same time, of 'average' (systematic) risk as well as maturity-specific risk. Nor can we reasonably assume, as we often do in models for equity markets, that the maturity-specific risk can be satisfactorily diversified away: by definition, that specific risk equally affects all bonds with similar terms to maturity. To remedy this, we let the risk factor be partly determined by the time-to-maturity of the bond. More precisely, the zero-coupon yield that we choose as our intermediary factor is bond-specific, in the sense that we select the yield that corresponds to a fixed fraction, w, of the time-to-maturity of the coupon bond in question, τ.[4] Using (3), (8) and defining $F = R(r, w\tau)$, we find:

$$\text{Duration} = \frac{-1}{P(r, c, t, T)} \sum_{l=1}^{N} CF(\tau_l) \frac{dP(r, \tau_l)}{dR(r, w\tau)} \tag{10}$$

$$= \frac{-1}{P(r, c, t, T)} \sum_{l=1}^{N} CF(\tau_l) \frac{-B(\tau_l)P(r, \tau_l)\,dr}{\frac{B(w\tau)}{w\tau}\,dr} \tag{11}$$

$$= \sum_{l=1}^{N} \frac{CF(\tau_l)P(r, \tau_l)}{P(r, c, t, T)} \left[B(\tau_l) \frac{w\tau}{B(w\tau)} \right]. \tag{12}$$

Equation (10) specifies the risk-factor proxy to be the change in a zero-coupon yield for time to maturity $w\tau$. The time to maturity of the yield factor is tied to the maturity of the bond in question. As a result, the factor for a long bond is a longer zero-coupon yield while the factor for a short bond is a shorter one. For example, if we set w at 0.05, the factor for a 10-year bond is the six-month interest rate, while for a 5-year bond the factor is the three-month rate. Equation (11) says that both zero-coupon bond prices, $P(r, \tau_l)$, and the risk factor, $R(r, w\tau)$, for the specific coupon bond in turn depend on the instantaneous rate, $r(t)$, with $B(\tau)$ being defined in (3); and equation (12) simply results from rearranging (11). Notice that the first fraction after the summation operator is the relative present value weights of component cash flows of the coupon bond. The term in the square brackets is the price sensitivity of a zero-coupon bond with a time-to-maturity of τ_l for the l-th coupon to be due. It is worth mentioning that, unlike Macaulay's duration, the dimension of the price sensitivity in (12) is not a number of units of time.[5] Since the change in the $w\tau$-period yield is taken as the risk factor instead of the change in the instantaneous interest rate, the price sensitivity is modified by $\frac{w\tau}{B(w\tau)}$. It will be argued below that such an adjustment is appropriate.

It follows that the general form of duration in (12) can be specified according to the term structure model either in (3), (4) and (5) or in (3), (6) and (7). That is:

$$D_{Vasicek} = \sum_{l=1}^{N} \frac{CF(\tau_l)P(r,\ \tau_l)}{P(r,\ c,\ t,\ T)} \left[\frac{1}{\kappa}(1 - e^{-\kappa\tau_l}) \times \frac{\kappa(w\tau)}{1 - e^{-\kappa(w\tau)}}\right], \quad (13)$$

and

$$D_{CIR} = \sum_{l=1}^{N} \frac{CF(\tau_l)P(r,\ \tau_l)}{P(r,\ c,\ t,\ T)}$$

$$\left\{\frac{e^{\theta_1\tau_l} - 1}{\theta_2(e^{\theta_1\tau_l} - 1) + \theta_1} \times \frac{\{\theta_2[e^{\theta_1(w\tau)} - 1] + \theta_1\}w\tau}{e^{\theta_1(w\tau)} - 1}\right\} \quad (14)$$

where all parameters were defined before. The first part of the bracketed terms in (13), $\frac{1}{\kappa}(1 - e^{-\kappa\tau_l}) = B(\tau)$ (for the Vasicek model from (5)), and in (14), $\frac{e^{\theta_1\tau_l} - 1}{\theta_2(e^{\theta_1\tau_l} - 1) + \theta_1} = B(\tau)$ (for the CIR model, from (7)), represent the price sensitivity using the single instantaneous rate as the risk factor. This part becomes constant as $\tau_l \to \infty$. It is well known that short-term interest rates fluctuate much more than long rates. Thus, prices of a long *zero-coupon* bond tend to be much less sensitive to changes in short-term interest rates than those of a short one, as reasonably described by both the Vasicek and CIR models. The second part, $\frac{\kappa(w\tau)}{1 - e^{-\kappa(w\tau)}}$ in (13) and $\frac{\{\theta_2[e^{1(w\tau)} - 1]+_1\}w\tau}{e^{\theta_1(w\tau)} - 1}$ in (14), help to recoup part of the maturity-specific risk lost due to specification errors. This part is greater than unity and approaches unity as w shrinks to zero (see proofs in the Appendix). Therefore, the use of the $w\tau$-period yield as a risk factor for a specific coupon bond does not simply modify but magnify, more for long bonds than for short bonds, the price sensitivity with little loss of both theoretical and empirical tractability. At the same time, the original ISW/CIR stochastic duration measure ($w = 0$) is nested in the new duration measure ($w > 0$).

According to a one-factor equilibrium model, of course, any given bond has a unique theoretical price variability. Thus, if we re-specify the factor in a way that the bond's duration increases, there would be an offsetting drop in the variability of the factor – which brings us back to square one. To get out of this circle, an ad hoc assumption is needed that is similar to the hypothesis of parallel shifts of internal yields in Macaulay's duration. We assume that the $w\tau(i)$- and $w\tau(j)$-yields of two different bonds i and j, have the same variability. This obviously violates the spirit of the theoretical models. However, if w is small, say, 0.05 or 0.025, then all bond-specific factors are really short-term interest rates, which, although they are of slightly different maturities, still do not appear to have overly different volatilities in practice. Our second defense of the ad hoc assumption is eminently pragmatic. If, in practice, this approach does better than either the theoretically correct approach (which fails to capture term structure movements) or the generalized Macaulay model (that has a totally unstructured approach to the term structure), then the assumption of equal volatilities is not a bad one after all. To verify whether the assumption holds well, one can immunize bonds by going short in a portfolio with a matched duration measure. Thus, we test whether that immunization strategy outperforms strategies based on the theoretically correct approach or on the purely ad-hoc Macaulay duration.

The potential gains with the $w\tau$-period yield for a specific coupon bond in capturing the relevant risk come from two aspects. First, any (finite) positive value of w means that we are choosing, as the factor, a zero-coupon yield with a finite maturity rather than the instantaneous interest rate. Thus, the risk factor carries some yield-curve information that would otherwise have been missed out due to specification errors in the model. Second, $w\tau$ is tied to the time-to-maturity of the (coupon) bond in question. This is important because it is known that, the longer the time-to-maturity of the bond, the less the instantaneous rate tends to matter, and hence the more information on longer yields is needed. The only remaining issue is the choice of w for the $w\tau$-period yield. For simplicity, unlike τ, we prefer w not to be bond-specific. And w should be small if the equal variability for yields $R(r, w\tau(i))$ and $R(r, w\tau(j))$ is able to hold reasonably.

Apart from these *a priori* considerations, we let the choice of w be settled as a purely empirical matter.

3. DATA AND TERM STRUCTURE PARAMETERS

Data are obtained from the data service of the *Financieel Economische Tijd* (a major financial newspaper in Belgium). The data consist of prices of OLOs (Obligations Linéaires/Lineaire Obligaties, a class of non-callable Belgian government bonds first introduced around 1990) and short-term discount bonds constructed from Brussels interbank offer rates in Belgian Franc (BIBORs). There are 351 daily cross sections, after deleting non-trading and thin-trading days from March 27, 1991 through September 16, 1992. Parameter estimates of the Vasicek and CIR term structure models are directly taken from Sercu and Wu (1997), who provide (time-varying) daily cross-sectional estimations.

Over time, the number of traded OLOs increases from six to eleven because of introduction of new issues. At the beginning of the sample period, times to maturity of OLOs range from three to 12 years while near the end, from 1.5 to 15 years. The OLO prices are last-trade quotes from the continuous on-screen trading in the CATS (Computer Aided Trading System). Bond prices (invoice or trade prices) were computed from closing quotes (flat prices) plus accrued interest according to the 360-day year rule that holds in the bond market. The one-week settlement rule means that the invoice prices are actually one-week forward prices. This effect is corrected for by computing the implied spot prices from the forward prices. Our use of BIBORs rather than T-bill yields is dictated by the fact that, because of poor liquidity, the yields on the T-bills were often higher than comparable BIBORs by at least ten basis points. Therefore, we prefer BIBORs to fill in the gap in the short end of the full yield curve. The bid-ask spread is 12.5 basis points usually, and we used the midpoint rates. Five BIBORs, namely, 1-, 2-, 3-, 6- and 12-month, are available throughout the whole sample period.

As shown in Figure 1 in Sercu and Wu (1997), estimated zero-coupon-yield curves for both the Vasicek and CIR models are sharply humped around five months, and the zero-coupon-yield curves tend to shift downward over the sample period indicating a bull market. With time to maturity in units of days, the mean cross-sectional estimates of the parameters are $\phi_1 = 0.0240$, $\phi_2 = 0.0048$, $\kappa = 0.0101$ and $r = 8.76\%$ per annum (Vasicek) and $\theta_1 = 0.0103$, $\theta_2 = 0.0079$, $\theta_3 = 0.2061$, and $r = 8.90\%$ per annum (CIR). The mean RMSEs of cross-sectional regressions are 13.5 (Vasicek) and 12.5 basis points (CIR), respectively.

4. COMPARISON OF IMMUNIZATION PERFORMANCE

Table 1 reports, for each asset, the time-series averages of the Macaulay, Vasicek and CIR duration measures during the sample period, as well as the modified stochastic durations for various

Table 1

Duration Measures

Maturity			*w = 0%*		*w = 5%*		*w = 10%*	
		Macaulay	*Vasicek*	*CIR*	*Vasicek*	*CIR*	*Vasicek*	*CIR*
Bibor-1m		0.082	0.071	0.077	0.071	0.077	0.072	0.077
Bibor-2m		0.167	0.123	0.143	0.125	0.144	0.127	0.145
Bibor-3m		0.249	0.161	0.195	0.165	0.197	0.168	0.199
Bibor-6m		0.496	0.222	0.290	0.233	0.295	0.244	0.302
Bibor-12m		1.000	0.256	0.343	0.281	0.357	0.308	0.374
Bond05	28 Feb. 94	1.390	0.257	0.344	0.295	0.366	0.336	0.394
Bond02	5 Apr. 96	3.071	0.259	0.347	0.357	0.410	0.475	0.503
Bond08	29 Aug. 97	4.194	0.255	0.341	0.406	0.444	0.597	0.609
Bond04	1 Jan. 98	4.203	0.261	0.350	0.404	0.447	0.584	0.600
Bond11	30 Jul. 98	4.785	0.260	0.350	0.435	0.471	0.657	0.668
Bond01	1 Jun. 99	5.294	0.260	0.349	0.463	0.494	0.725	0.732
Bond03	1 Aug. 00	5.836	0.260	0.350	0.505	0.531	0.829	0.832
Bond07	27 Jun. 01	6.347	0.260	0.349	0.538	0.560	0.909	0.910
Bond10	25 Jun. 02	6.842	0.260	0.349	0.577	0.594	1.000	1.000
Bond06	1 Mar. 03	6.989	0.259	0.347	0.603	0.616	1.064	1.063
Bond09	1 Oct. 07	8.281	0.245	0.327	0.748	0.749	1.416	1.410

Notes:
The average duration measures over the sample period (from March 27, 1991, or the first issue date, through September 16, 1992) are reported. Macaulay's duration is in units of time (years) but the stochastic duration measures have no dimension. For the Vasicek and the CIR duration measures, w is the fraction of the time to maturity(τ) of each bond, and the change in the $w\tau$-period yield (annualized) is used as a risk factor proxy for the unexpected changes in interest-rates. Bonds are ascendingly tabulated from top to bottom by maturity.

finite values of w. The original ISW/CIR stochastic duration in both the Vasicek and CIR versions, which takes the instantaneous rate as the risk factor, is tabulated in the columns under $w = 0\%$. To interpret the numbers, Macaulay's duration of five years means that a 1% change in the continuously compounded internal yield leads to a 5% change in bond prices.[6] Likewise, a stochastic duration of, say, 0.3 indicates that a 1% change in the $w\tau$-zero-coupon yield leads to a 0.3% change in the prices of a bond with time to maturity, τ.

From the table we see that when time-to-maturity is very short, the Macaulay, Vasicek and CIR duration measures are close.[7] The stochastic duration measures initially increase with the bond's maturity (but more slowly than their Macaulay counterpart), then remain almost constant for a wide range of coupon bonds in the middle, and decrease for the long bonds, a pattern that reflects the models' prediction that very long-term yields are constant. When the Macaulay duration reaches the value of 4.785, the Vasicek and CIR duration measures peak at 0.26 and 0.35 respectively.

Different duration definitions take different proxies for the unobservable risk sources, so one cannot draw any conclusions from the different levels of the durations. Specifically, if the instantaneous interest rate has a sufficiently high variability and reasonably succeeds in capturing movements in the entire term structure (via a bond pricing model), then a low ISW/CIR duration can still explain a substantial part of bond price changes. Thus, even though in Table 1 the price sensitivities of long (coupon) bonds according to the original ISW/CIR duration ($w = 0\%$) are much lower than those according to the Macaulay counterpart and the difference between the two tend to increase with τ, one cannot conclude from this that the Macaulay duration would overstate the true price sensitivity of long bonds or that the stochastic duration measure is likely to understate it.[8] Even more importantly, it is not sufficient that a factor has the ability to predict the *variance* of bond prices; the factor's predicted bond price should also be highly correlated with the actual bond price. Thus, what counts is not the durations' individual magnitudes, but their cross-sectional patterns across bonds, the variability of the chosen source of risk, and the factor's ability to capture overall movements of the term structure.

In practice, the variability of the short-term interest rate turns out to be too low to explain even the variance of price changes of longer-term coupon bonds. Improvement may be possible with a magnified stochastic duration measure, that is, a duration measure with $w > 0$. If the zero-coupon yields for maturity $w\tau$ are taken as the relevant risk factors, the Vasicek and CIR duration measures become larger, as expected, and tend, reasonably, to increase with time-of-maturity (see columns under $w = 5\%$ and $w = 10\%$, respectively). This confirms the price-sensitivity-intensifying effect of taking zero-coupon yields longer than the instantaneous rate as the risk sources. However, no conclusion on the effectiveness of such a modification can be drawn until it is verified that bond portfolios with equal modified durations also have highly correlated price changes. Thus, we need to evaluate the immunization performance.

Instead of immunizing a single liability for one specific date, we immunize individual coupon bonds using each of the alternative duration measure. The procedure is as follows. On each trading day, t, we obtain, from previous trading day, duration information about three assets: a duration for a specific coupon bond, m, the duration for the equal-weight short maturity portfolio consisting of five zero-coupon bonds constructed from BIBORs, Sd, and the duration for the equal-weight long maturity portfolio consisting of all available OLOs excluding the one to be matched, Ld. Since the specification of duration (either a stochastic one or Macaulay's) has not been made here, what follows is very general. The equal-weight matched portfolio is determined by finding out X and Y (weights on the short portfolio and long portfolio, respectively) such that $m = X\,Sd + Y\,Ld$ (duration matching) and $1 = X + Y$ (value matching). Thus, one can calculate the matched portfolio effective return as $mr = X\,Sr$ (short portfolio return)$+Y\,Lr$(long portfolio return), and hence the abnormal (residual) return using this matched portfolio as benchmark. Of course, using different duration measures will generate different matched portfolios (X and Y) and hence result in different hedging performances.

Besides the residual return for the 1-day horizon, five other residual returns for the 2-, 3-, ..., and the 6-day horizon are calculated and measured by j-day averages.[9] The motivation for looking at a longer holding period is that a stochastic duration measure may not be able to show its potential within a short horizon. The reason is that bond prices tend to fluctuate around their fundamental values, which are implied by the estimated term structures using cross-sectional bond prices.[10] Therefore, within a short horizon, such a temporary departure will introduce noise into the performance of the stochastic duration measures that largely rely on the estimated term structures.

Average RMSE's for different holding periods and for different values of w (only relevant for the Vasicek and CIR duration measures) are reported in Table 2. The conclusions are as follows. Comparing across methods we see that for all horizons, the Macaulay duration measure resoundingly beats the original ISW/CIR duration in both the Vasicek and CIR versions ($w = 0\%$). For example, at the one-day horizon, Macaulay immunization has a RMSE of less than 6.1 basis points for the 1-day horizon, while the

Table 2

Bond Immunization Performance Comparison

w (%)	*Duration Measure*	*Horizon (Trading Days)*					
		1	2	3	4	5	6
	Macaulay	6.08	4.27	3.79	3.73	3.32	3.08
0.0	Vasicek	8.57	6.26	5.27	4.75	4.19	3.80
	CIR	8.58	6.26	5.28	4.75	4.20	3.80
2.5	Vasicek	6.50	4.56	3.89	3.70	3.32	3.08
	CIR	6.91	4.83	4.05	3.76	3.33	3.04
5.0	Vasicek	6.50	4.57	3.95	3.82	3.44	3.22
	CIR	6.42	4.46	3.80	3.63	3.24	2.99
7.5	Vasicek	6.75	4.76	4.15	4.01	3.60	3.38
	CIR	6.55	4.57	3.94	3.79	3.38	3.14
10.0	Vasicek	6.93	4.91	4.30	4.15	3.72	3.48
	CIR	6.75	4.73	4.11	3.97	3.52	3.28

Notes:
The performance is gauged by the average of RMSE's of the daily time-series of residual returns (in basis points) over all OLO bonds. The residual (abnormal) return is the difference between a holding period effective return of individual bonds and the benchmark return of the duration-and-value matched portfolio. The residual return of the j-day horizon at a cross section is defined as the cumulative daily residual returns up to the j-day horizon divided by j days. The average RMSE's no greater than the Macaulay counterparts are marked with a single underline. w is the fraction of the time-to-maturity (τ) of a bond, and the bond-specific $w\tau$-period yields (annualized) are used to proxy for the relevant risk sources of the unexpected changes in interest-rates.

ISW/CIR immunization in both the Vasicek and CIR versions has a RMSE close to 8.6 basis. However, the performance of stochastic duration measures does improve across the board when w is set greater than zero, confirming that longer zero-coupon yields better capture the unexpected changes in interest rates than the instantaneous interest rate. Moreover, in some cases (marked with a single underline) the performance of the modified stochastic duration measures is comparable to, or better than, the performance of Macaulay's duration. For example, this is true for the modified duration of the Vasicek model when $w = 2.5\%$ for the 4-, 5- and 6-day horizons, and for the CIR model when $w = 2.5\%$ for the 6-day horizon and when $w = 5\%$ for the 4-, 5- and 6-day horizons. Tables 3 and 4 further show more detailed immunization performance results by asset, which are in general consistent with the results in Table 2. These results confirm our early conjecture that the new duration may only work effectively if w is smallish.

In a nutshell, a stochastic duration becomes effective if one considers changes in longer zero-coupon yields rather than the single instantaneous rate, as the relevant risk sources of the unexpected changes in interest rates. The proposed stochastic duration measure undoubtedly beats the original ISW/CIR duration. The bond immunization performance race also shows that the new duration outperforms the popular Macaulay duration in some cases when the value of w is in a range between 2.5% and 5%. The optimum value of w tends to be small, at least in this particular sample. It is likely that, in different samples, the optimal value for w would be different. However, even in different databases one would not expect substantially larger values for the optimal w. The reason is that, if w is very large, say, 0.5, the factors would become very different across bonds, which would violate the assumption that they are all driven by the same single factor and have comparable volatilities.

5. CONCLUSION

In this paper, we argue that changes in zero-coupon yields, which are slightly longer than the instantaneous rate, can be a better proxy for the relevant risk sources of unexpected changes in

Table 3

Performance of Macaulay's and ISW/CIR's Duration Measures by Asset

		Horizon (Trading Days)					
Asset	*Obs*	*1*	*2*	*3*	*4*	*5*	*6*
Panel A: Macaulay's Duration							
Bond05	299	4.72	3.28	2.70	2.30	2.01	1.89
Bond02	321	5.51	3.80	3.08	2.74	2.32	2.14
Bond08	218	4.59	3.26	2.69	2.65	2.38	2.24
Bond04	314	4.62	3.24	2.73	2.39	2.17	1.94
Bond11	30	4.40	3.92	3.85	4.10	3.82	3.25
Bond01	313	7.25	4.79	4.06	3.34	2.81	2.37
Bond03	323	6.24	3.88	3.18	2.87	2.68	2.54
Bond07	281	6.02	4.15	3.86	4.52	3.18	2.83
Bond10	50	6.71	4.18	3.76	4.75	4.45	4.27
Bond06	318	8.39	5.40	4.50	3.94	3.55	3.33
Bond09	117	8.42	7.03	7.25	7.47	7.15	7.07
Panel B: Vasicek Duration (w = 0%)							
Bond05	299	13.46	10.95	9.57	8.39	7.42	6.77
Bond02	321	7.92	6.06	5.05	4.92	4.25	3.86
Bond08	218	5.65	4.59	4.14	4.14	3.72	3.52
Bond04	314	5.19	3.85	3.36	3.01	2.69	2.36
Bond11	30	6.04	5.11	4.61	4.89	4.50	3.83
Bond01	313	7.30	4.75	3.97	3.31	2.72	2.22
Bond03	323	7.10	4.46	3.55	3.03	2.63	2.42
Bond07	281	7.82	5.08	4.17	3.72	3.13	2.72
Bond10	50	10.45	7.57	5.77	5.14	4.67	4.38
Bond06	318	10.03	6.88	5.65	4.86	4.39	3.95
Bond09	117	13.31	9.57	8.18	6.81	6.03	5.73
Panel C: CIR Duration (w = 0%)							
Bond05	299	13.44	10.94	9.58	8.40	7.44	6.80
Bond02	321	7.96	6.06	5.02	4.92	4.25	3.84
Bond08	218	5.59	4.55	4.11	4.11	3.69	3.51
Bond04	314	5.23	3.87	3.35	2.99	2.69	2.36
Bond11	30	6.03	5.10	4.60	4.89	4.49	3.83
Bond01	313	7.34	4.79	4.01	3.36	2.77	2.27
Bond03	323	7.17	4.52	3.62	3.12	2.69	2.47
Bond07	281	7.85	5.10	4.20	3.74	3.13	2.72
Bond10	50	10.46	7.57	5.78	5.14	4.67	4.38
Bond06	318	9.98	6.84	5.63	4.85	4.38	3.93
Bond09	117	13.29	9.55	8.14	6.77	5.99	5.69

Notes:
RMSE's no greater than the Macaulay counterparts are marked with a single underline. See also the detailed explanatory notes in Table 2.

Table 4

Performance of the New Stochastic Duration Measures (w=2.5%) by Asset

		Horizon (Trading Days)					
Asset	*Obs*	*1*	*2*	*3*	*4*	*5*	*6*
Panel A: Vasicek Duration (w = 2.5%)							
Bond05	299	7.02	5.60	4.94	4.31	3.86	3.57
Bond02	321	5.78	4.17	3.42	3.20	2.68	2.42
Bond08	218	4.64	3.40	2.92	2.91	2.58	2.42
Bond04	314	4.71	3.30	2.87	2.50	2.22	1.98
Bond11	30	4.54	3.90	3.68	3.84	3.53	2.99
Bond01	313	7.16	4.60	3.83	3.18	2.64	2.19
Bond03	323	6.40	3.96	3.18	2.80	2.53	2.38
Bond07	281	6.50	4.17	3.59	3.63	2.80	2.49
Bond10	50	8.01	5.29	3.80	4.08	3.91	3.79
Bond06	318	8.27	5.15	4.11	3.50	3.18	2.95
Bond09	117	8.48	6.63	6.44	6.69	6.62	6.72
Panel B: CIR Duration (w = 2.5%)							
Bond05	299	8.91	7.11	6.17	5.33	4.64	4.18
Bond02	321	6.31	4.61	3.75	3.61	3.00	2.69
Bond08	218	4.82	3.69	3.28	3.34	3.02	2.88
Bond04	314	4.72	3.40	2.95	2.60	2.33	2.06
Bond11	30	4.73	4.11	3.81	4.05	3.72	3.15
Bond01	313	7.19	4.62	3.85	3.21	2.68	2.22
Bond03	323	6.49	4.03	3.25	2.94	2.70	2.57
Bond07	281	7.00	4.40	3.64	3.49	2.74	2.38
Bond10	50	8.70	5.96	4.37	4.28	4.02	3.84
Bond06	318	8.77	5.45	4.30	3.57	3.20	2.85
Bond09	117	8.35	5.79	5.24	4.88	4.59	4.64

Notes:
RMSE's no greater than the Macaulay counterparts are marked with a single underline. See also the detailed explanatory notes in Table 2.

interest rates. We prove that, using such a zero-coupon yield as a risk factor for a specific coupon bond, the stochastic duration derived from the Vasicek and CIR models is larger, increasing with bond maturity, than the original ISW/CIR duration. The immunization performance test shows that the proposed stochastic duration definitely beats ISW/CIR's duration and can in some cases outperform Macaulay's duration.

APPENDIX

First, using L'Hopital's rule, it is trivial to prove that the modified part of the price sensitivity, $\frac{\kappa(w\tau)}{1-e^{-\kappa(w\tau)}}$, (Vasicek) or $\frac{\{\theta_2[e^{\theta_1(w\tau)}-1]+\theta_1\}w\tau}{e^{\theta_1(w\tau)}-1}$ (CIR) approaches unity when w goes to zero.

Second, to prove that the modified part is greater than unity, let us first look at the case of the Vasicek duration measure. Let $f(x) = x - 1 + e^{-x}$, $(x = \kappa w\tau \geq 0)$, then, the problem is to prove:

$$f(x) = x - 1 + e^{-x} \geq 0. \tag{A.1}$$

We have $f'(x) = 1 - e^{-x}$ and $f''(x) = e^{-x}$ Because $f''(x) = e^{-x} > 0$, there exists a global minimum for $x \geq 0$. For $f'(x) = 1 - e^{-x} = 0$, there is only one solution, $x = 0$ So, $f(x)$ has a minimal value at $x = 0$ Therefore, (A.1) holds.

Next, let us look at the case of the CIR duration measure. Let $f(x) = [\theta_2(e^{\theta_1 x} - 1) + \theta_1]x - (e^{\theta_1 x} - 1)$, $(x = w\tau \geq 0)$, likewise, the problem becomes to prove:

$$f(x) = [\theta_2(e^{\theta_1 x} - 1) + \theta_1]x - (e^{\theta_1 x} - 1) \geq 0. \tag{A.2}$$

And we have:

$$\begin{aligned} f'(x) &= \theta_1\theta_2 x e^{\theta_1 x} + [\theta_2(e^{\theta_1 x} - 1) + \theta_1] - \theta_1 e^{\theta_1 x} \\ &= [\theta_1\theta_2 x - (\theta_1 - \theta_2)]e^{\theta_1 x} + (\theta_1 - \theta_2) \end{aligned} \tag{A.3}$$

and

$$\begin{aligned} f''(x) &= \theta_1\theta_2 e^{\theta_1 x} + \theta_1[\theta_1\theta_2 x - (\theta_1 - \theta_2)]e^{\theta_1 x} \\ &= e^{\theta_1 x}[\theta_1(2\theta_2 - \theta_1) + x\theta_1^2\theta_2]. \end{aligned} \tag{A.4}$$

Note that, when the current interest rate is above $\frac{\kappa m}{\kappa + \lambda}$, the term structure is falling; and when the rate is below it, the term structure is humped or rising. See Cox, Ingersoll and Ross (1985, p. 394, following equation (26)). It follows that $\kappa + \lambda > 0$ because $\kappa m > 0$ and an interest rate is positive, and hence that $\theta_2 > 0$. See the definition of the parameter θ_2 below equation (7), and also notice that $2\theta_2 - \theta_1 = \kappa + \lambda > 0$. Therefore,

$f''(x) > 0$ (from A.4), and hence there exists a global minimum for $x \geq 0$ For $f'(x) = 0$ there is only one solution, $x = 0$, (from A.3). So, $f(x)$ has a minimal value at $x = 0$ Therefore, (A.2) holds.

Note that in both cases, we have $f''(x) > 0$. Thus, $f(x)$ is an increasingly monotonic function in x and hence in τ, indicating that the price sensitivity of long bonds is more magnified than that of short bonds.

NOTES

1 Fisher and Weil (1971). See also Bierwag, Kaufman and Khang (1978) for a review of applications of Macaulay's Duration.
2 See also Ingersoll (1983), Nelson and Schaefer (1983) and Brennan and Schwartz (1983). There is an exception. According to Haugen (1993), the unpublished work of Lau (1983) did show that the CIR duration measure is comparable to Macaulay's duration in immunizing a single liability using two monthly rebalanced highest-yield bonds from each side of the duration of the liability.
3 See Brown and Dybvig (1985), Pearson and Sun (1994), De Munnik and Schotman (1994), and Sercu and Wu (1997), among others.
4 If there are M coupon bonds with different maturities, we will have M factors, $\tau(k)$, $k = 1, \ldots,$ M, just like M internal yields in the case of Macaulay's duration. However, the M $w\tau$-yields are fundamentally different from the M internal yields because the relation among the formers is governed by a bond pricing model.
5 The stochastic duration measure in (10) stands for the price sensitivity and is consistent with that in Ingersoll, Skelton and Weil (1978). However, without the concern on duration matching, Cox, Ingersoll and Ross (1979) further propose to convert, non-linearly, the dimensionless stochastic duration measure into one in units of time. It is straightforward to make duration matching with the measure in (10) because price sensitivities should be additive. Therefore, this paper uses the dimensionless stochastic duration measure.
6 The modified version of Macaulay's duration (Macaulay's duration divided by one plus the internal yield) is used.
7 For a very short τ, the dimensionless zero bond price sensitivity, $\tau dR(\tau)/dr$, approaches τ. Therefore, as long as the unit of interest rates is consistent (annualized), Macaulay's duration and stochastic duration measures are comparable at the very short end.
8 With numerical examples of duration measures, Cox, Ingersoll and Ross (1979) point out that, for coupon bonds, the stochastic duration measure converted into units of years peaks at 10 years, and they contend that this is realistic compared to the remote peak at 50 years with the Macaulay counterpart. However, which duration measure better captures the risk in bond returns requires an empirical check on immunization performance. And the remainder of the paper will carry out such a task.

9 If the RMSE of the cumulative return is wanted rather than the average, it suffices to multiply by the number of days. Whether one uses averages or sums, the RMSEs for different horizons obviously cannot be compared across horizons; however, either RMSE can always be compared across immunization methods. Note also that the j-day cumulative return starting from date t substantially overlaps with the j-day cumulative return starting from date $t+1$ etc., which provides one more reason not to compare the RMSEs across horizons. However, there is no reason to believe the overlap in the observations would be in favor of a particular duration measure; that is, it is unlikely to undermine the cross-sectional comparison.

10 There are always bond pricing model residuals. Nevertheless, Sercu and Wu (1997) find that these residuals tend to revert to the mean over time.

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