# MA3514 — Assignment No. 1

1. Consider the following system of equations,

$$\begin{aligned} x' &= -xy + 2y - \cos t \\ y' &= x^2 + ty, \end{aligned}$$

for $t > 0$. The initial conditions are $x(0) = -1$, $y(0) = 2$. Find the numerical solution at $t = 0.1$, with step size $h = 0.1$ based on the modified Euler's method.

2. Consider the following system

$$\begin{cases} x' = -y, & x(0) = 1 \\ y' = x, & y(0) = 0 \end{cases}$$

which describes a unit circle (why). For each of the three numerical methods below, if ($x_n$, $y_n$) is the numerical solution at $t_n = nh$, where $h$ is the time step, find $\lim_{n\to\infty}(x_n^2 + y_n^2)$.

(a) Euler's method, (b) midpoint method, (c) backward Euler's method, (d) implicit midpoint method.

3. For the differential equation $y' = f(t, y)$, the local truncation error of the general 2nd order Runge-Kutta method (with parameter $\alpha$) is

$$T_{j+1} = \frac{h^3}{4}\left[\frac{2-3\alpha}{3}y'''(t_j) + \alpha y''(t_j)\frac{\partial f(t_j, y(t_j))}{\partial y}\right] + O(h^4).$$

Verify this result for the special case of $\alpha = 1$.

4. The following is another third order Runge-Kutta method for $y' = f(t, y)$:

$$\begin{aligned} k_1 &= f(t_j, y_j) \\ k_2 &= f(t_j + \frac{h}{2}, y_j + \frac{h}{2}k_1) \\ k_3 &= f(t_j + \frac{3}{4}h, y_j + \frac{3}{4}hk_2) \\ y_{j+1} &= y_j + \frac{h}{9}(2k_1 + 3k_2 + 4k_3). \end{aligned}$$

Describe an embedded Runge-Kutta method based on the above third order method and a related second order Runge-Kutta method. Use this method to solve the Lorenz equation with the initial conditions given on page 26 of the lecture notes and from $t = 0$ to $t = 40$. Choose your own error tolerance parameter. Show figures for (a) $y_1$ vs. $y_2$, (b) $y_1$ vs. $y_3$, (c) $y_2$ vs. $y_3$.

5. Consider the following linear system of ordinary differential equations

$$\begin{aligned} \frac{dy}{dt} &= iAy, \quad t > 0 \\ y(0) &= y_0 \end{aligned}$$

where $y$ is a complex vector function of $t$ (a column vector of length $n$), $A$ is a real symmetric $n \times n$ matrix and $i = \sqrt{-1}$ is the imaginary number. For numerical computation, we use $t_j = jh$, where $h$ is the step size and denote the numerical approximation of $y(t_j)$ by $y_j$.

(a) Apply the trapezoid method to this system, write down the formula for $y_{j+1}$ in terms of $y_j$.

(b) Show that the numerical solutions obtained from the trapezoid method satisfy

$$y_{j+1}^H y_{j+1} = y_j^H y_j$$

where $a^H = \overline{a}^T$ is the transpose of the complex conjugate of $a$.