MA3514 — Assignment No. 2

1. Find constant a_0, a_1, b_0, b_1 and b_2, such that the local truncation error of the following method

$$y_{j+1} + a_0 y_j + a_1 y_{j-1} = h[b_0 f_{j+1} + b_1 f_j + b_2 f_{j-1}]$$

has an order as high as possible. Analyze the zero-stability of this method.

2. Derive a 3-step method for ODE $y' = f(t, y)$ based on $P_3'(t_j) = f(t_j, y_j)$, where P_3 is a polynomial interpolating

$$(t_{j+1}, y_{j+1}), \quad (t_j, y_j), \quad (t_{j-1}, y_{j-1}), \quad (t_{j-2}, y_{j-2}).$$

Find the local truncation error and the order of this method. Analyze the zero stability of this method.

3. The Korteweg-de Vries (KdV) equation (1895)

$$u_t + u u_x + \delta^2 u_{xxx} = 0,$$

where δ is a constant, models water waves in a shallow canal. The KdV equation has soliton solutions which were first observed by J. Scott Russell on the Edinburgh-Glasgow canal in 1834. Many mathematical properties of the KdV equation were found after the initial numerical study of Zabusky and Kruskal in 1965. They solved the KdV equation for $\delta = 0.022$ and $0 < x \leq 2$ assuming periodic boundary condition: $u(x+2, t) = u(x, t)$ for all x, with the initial condition: $u(x, 0) = \cos(\pi x)$, and for $0 < t \leq 3.6/\pi$. We will repeat this calculation with our own method. Discretizing x by $x_k = 2k/n$ for $n = 200$ and $1 \leq k \leq n$, the KdV equation is approximated by the following system of ODEs:

$$\frac{du_k}{dt} + u_k \frac{u_{k+1} - u_{k-1}}{2d} + \frac{\delta^2}{2d^3}(u_{k+2} - 2u_{k+1} + 2u_{k-1} - u_{k-2}) = 0,$$

for $1 \leq k \leq n$, where $d = 2/n$ and $u_k \approx u(x_k, t)$. Due to the periodic boundary condition, we need to set $u_0 = u_n$, $u_{-1} = u_{n-1}$, $u_{n+1} = u_1$ and $u_{n+2} = u_2$ for $k = 1$, 2, $n - 1$ and n. Solve this system of equations with the 3-step Adams-Bashforth method (together with a third order Runge-Kutta method for the first a few steps) using the time step $h = 1/(m\pi)$ for $m = 600$. Plot the initial condition and the solutions at $t = 1/\pi$ and $t = 3.6/\pi$ in one figure. Submit the programs and the figure.