MA3514 — Assignment No. 3

1. Consider the linear boundary value problem

$$y'' + p(x)y' + q(x)y = r(x) \quad \text{for} \quad a < x < b$$
$$y(a) = \alpha y(b) + \beta, \quad y'(b) = \gamma,$$

where a, b, α, β and γ are given constants, p, q and r are given functions. Show that if we solve three initial value problems (one for the original differential equation, two for the homogeneous differential equation with zero right hand), then the BVP can be solved.

2. Assume that the following linear boundary value problem:

$$u'' + q(x)\,u = 0, \quad 0 < x < L$$
$$u'(0) = \alpha\, u(0), \quad u'(L) = \beta\, u(L) + \gamma,$$

where L, α, β and γ are given constants and q is a given function, has a unique solution which is non-zero for all $x \in [0, L]$. Derive an IVP for $Q = u'/u$. Show that if the IVP for Q is solved, we can find $u(L)$. Derive another IVP for $Y(x) = u(x)/u(0)$. Show that if Q and Y are solved together, we can find both $u(L)$ and $u(0)$.

3. The following two-point boundary value problem

$$u'' + [1 + x(u')^2]u = 0, \quad 0 < x < 1,$$
$$u'(0) = 0, \quad u'(1) = u(1)$$

is nonlinear but has a trivial solution $u(x) = 0$ for all x. Use a shooting method (based on the secant method as the nonlinear equation solver and any Runge-Kutta method as the ODE IVP solver), find a non-zero solution. Submit your MATLAB programs and a plot of the non-zero solution.

4. Use the finite element method to solve the following boundary value problem

$$y'' + (2 - x)y = 1, \quad \text{for} \quad 0 < x < 1$$
$$y(0) = 1, \quad y(1) = 0$$

based on equally spaced grid points and the grid size $h = 1/3$.

5. Let $\{\phi_j\}$ be the piecewise linear functions used in the finite element method and

$$a_{kj} = \int_a^b [-\phi_k' \phi_j' + p(x)\phi_k \phi_j' + q(x)\phi_k \phi_j] dx.$$

When p and q are constants, verify the formulae:

$$a_{kk} = -\frac{1}{h} - \frac{1}{H} + \frac{q}{3}(h + H)$$

where $h = x_k - x_{k-1}$ and $H = x_{k+1} - x_k$.