MA3514 — Assignment No. 4

1. For the heat equation $u_t = au_{xx}$ where a is a positive constant, show that the following method

$$\frac{1}{2\Delta t}(u_j^{k+1} - u_j^{k-1}) = \frac{a}{(\Delta x)^2}(u_{j+1}^k - 2u_j^k + u_{j-1}^k).$$

is unconditionally unstable.

2. For the heat equation $u_t = au_{xx}$ where a is a positive constant, show that the Dufort-Frankel method

$$\frac{1}{2\Delta t}(u_j^{k+1} - u_j^{k-1}) = \frac{a}{(\Delta x)^2}(u_{j+1}^k - u_j^{k+1} - u_j^{k-1} + u_{j-1}^k).$$

is unconditionally stable.

3. Consider the following initial and boundary value problem

$$\begin{aligned} &u_t = u_{xx} - x(1-x)u \quad \text{for} \quad 0 < x < 1, \quad t > 0 \\ &u(0,t) = 0, \quad u_x(1,t) = 0, \quad t > 0, \\ &u(x,0) = f(x), \quad 0 \le x \le 1. \end{aligned}$$

If the Crank-Nicolson method is applied to the above problem with $x_j = j/3.5$ and $t_k = k/7$, find the matrix A, such that $\mathbf{v}_{k+1} = A\mathbf{v}_k$, where $\mathbf{v}_k = \left[u_1^k, u_2^k, u_3^k\right]^T$ and $u_j^k \approx u(x_j, t_k)$.

4. For the 2D heat equation on the unit square $\Omega = \{(x,y) \mid 0 < x < 1,\ 0 < y < 1\ \}$,

$$u_t = u_{xx} + u_{yy}, \quad (x,y) \in \Omega, \quad t > 0,$$

with the boundary condition $u(x,y,t) = 0$ for $(x,y) \in \partial\Omega$ and $t \ge 0$ and the initial condition

$$u(x,y,0) = \begin{cases} 1, & \text{if } (x-0.3)^2 + (y-0.4)^2 < 1/25, \\ 0, & \text{otherwise.} \end{cases}$$

Write a MATLAB program for the ADI method based on the matrix notations. Choose $\Delta x = \Delta y = 1/101$ and $\Delta t = 0.0005$, find the solutions at $t = 0.001$ and $t = 0.01$. Submit the MATLAB program and a plot of the solutions (by MATLAB command `imagesc`).