MA3514 — Assignment No. 5

1. For the scalar advection equation $u_t + a\,u_x = 0$, where a is a non-zero constant, the Lax-Friedrichs method is

$$\frac{1}{\Delta t}\left(u_j^{k+1} - \frac{u_{j-1}^k + u_{j+1}^k}{2}\right) + a\,\frac{u_{j+1}^k - u_{j-1}^k}{2\Delta x} = 0.$$

   and the Leapfrog method is

$$\frac{u_j^{k+1} - u_j^{k-1}}{2\Delta t} + a\,\frac{u_{j+1}^k - u_{j-1}^k}{2\Delta x} = 0.$$

   Find the stability condition for each method.

2. For the Burgers' equation

$$u_t + uu_x = 0, \quad -\infty < x < \infty, \quad t > 0$$

   with initial condition

$$u(x,0) = e^{-10(4x-1)^2},$$

   use MATLAB to find a numerical solution at $t = 0.3$ based on Lax-Wendroff method and $\Delta x = 0.01$. For that, you need to truncate x to a finite interval and use zero boundary conditions, and also choose proper Δt. Submit your MATLAB program and a plot of the solution.

3. For the following equation

$$u_{tt} = u_{xx} + 2u_x.$$

   If second order central difference approximations are used to approximate u_{tt}, u_{xx} and u_x, analyze the stability property of this method.

4. Derive an explicit central difference scheme for the solution of

$$u_{tt} = (1+x)u_{xx}, \quad 0 < x < 1, \quad t > 0,$$

   with the boundary and initial conditions

$$u(x,0) = x(1-x),\ u_t(x,0) = 0,\ u_x(0,t) = u(0,t),\ u(1,t) = 0.$$

   You should discretize x as $x_j = (j-0.5)\Delta x$, with $\Delta x = 1/(n+0.5)$ where n is an integer. Now, for $t_k = k\Delta t$ and $u_j^k \approx u(x_j, t_k)$,

   (a) write down the initial conditions (for the first two time levels): u_j^0 and u_j^1;

   (b) write down a formula for u_j^{k+1} (assuming that u_j^k for all j are known) and pay special attention to $j = 1$ and $j = n$.