# MA3514 — Assignment No. 6

1. Consider the equation
$$u_{xx} + u_{yy} + xu = 0$$
on the triangular region $\Omega = \{(x, y) \mid x > 0, y > 0, x + y < 1\}$, subject to the boundary condition $u|_{\partial\Omega} = g$, where $g = 1$ on the $x$-axis, $g = 2$ on the $y$-axis and $g = 3$ on the line $x + y = 1$. Let $h = 1/4$ and $u_{ij} \approx u(ih, jh)$ be the numerical solution obtained from a second order finite difference method. Define a vector $\vec{u}$ for the unknowns and find a matrix $A$ and a vector $\vec{b}$ such that $A\vec{u} = \vec{b}$.

2. Let $p(y)$, $q(y)$ and $F(x, y)$ be given functions, consider the equation
$$\partial_x^2 u + \partial_y[p(y)\partial_y u] + q(y)u = F(x, y)$$
on the unit square
$$\Omega = \{\, (x, y) \,|\, 0 < x < 1,\ 0 < y < 1 \,\}$$
with the boundary condition $u|_{\partial\Omega} = 0$, where $\partial\Omega$ is the boundary of $\Omega$. If the equation is approximated by a second order finite difference method, describe a fast solver based on Discrete Sine Transform.

3. For the ODE boundary value problem
$$u''(x) = f(x), \quad 0 < x < 1, \quad u(0) = \alpha, \quad u(1) = \beta,$$
where $f$ is a given function, $\alpha$ and $\beta$ are given constants, the second order finite difference method gives
$$A\vec{U} = \vec{b}, \quad \text{where} \quad A = \begin{bmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -2 \end{bmatrix}.$$
Show that the Gauss-Seidel iterative method converges for the system above.

4. Based on the following definitions and properties of the conjugate gradient method
$$\mathcal{K}_n =< b, Ab, A^2b, ..., A^{n-1}b >, \quad \phi(x_n) = \min_{x \in \mathcal{K}_n} \phi(x)$$
$$r_n \perp \mathcal{K}_n =< r_0, r_1, ..., r_{n-1} >=< x_1, x_2, ..., x_n >=< p_0, p_1, ..., p_{n-1} >,$$
prove the following formulae for $\alpha_n$ and $\beta_n$:
$$\alpha_n = \frac{p_{n-1}^T b}{p_{n-1}^T A p_{n-1}} = \frac{r_{n-1}^T p_{n-1}}{p_{n-1}^T A p_{n-1}} = \frac{r_{n-1}^T r_{n-1}}{p_{n-1}^T A p_{n-1}}$$
$$\beta_n = -\frac{r_n^T A p_{n-1}}{p_{n-1}^T A p_{n-1}} = \frac{r_n^T r_n}{r_{n-1}^T r_{n-1}}.$$