City University of Hong Kong

Course code & title:	MA3514 Numerical Methods for Differential Equations
Session:	Semester B, 2005-2006
Date:	9 May 2006
Time:	14:00–17:00
Time allowed:	Three hours

This paper has **three** pages. (Including this page)

Instructions to candidates:

- Attempt **all** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (15 marks) Apply the midpoint method (one of the 2nd order Runge-Kutta methods) to

$$\begin{aligned} &y'' = -y^2 + y' + t, \quad t > 0 \\ &y(0) = 1, \quad y'(0) = 2, \end{aligned}$$

with time step $h = 0.1$, find an approximation to $y(0.1)$.

2. (15 marks) Consider the differential equation:

$$y'' + q(x)y = r(x),$$

which is valid for $x \neq 0$. The function $r(x)$ is continuous for all x. The function $q(x)$ is not continuous at $x = 0$, but continuous for all $x \neq 0$. The left and right limits of $q(x)$ exist at $x = 0$, but they are not the same. That is, $q(0-) \neq q(0+)$, where $q(0-) = \lim_{x\to 0-} q(x)$ and $q(0+) = \lim_{x\to 0+} q(x)$. It is known that y satisfies the following conditions $x = 0$:

$$y(0-) = y(0+), \quad y'(0-) = \rho\, y'(0+),$$

where ρ is a given constant. For a given grid size Δx and $x_j = j\Delta x$, find a finite difference approximation at $x = 0$. That is, find a, b, c and d such that

$$ay_{-1} + by_0 + cy_1 = d,$$

where $y_j \approx y(x_j)$ for $j = -1, 0, 1$.

3. (15 marks) Consider the initial and boundary value problem

$$\begin{aligned} &u_t = (1 + x)u_{xx}, \quad 0 < x < 1, \quad t > 0, \\ &u(0, t) = u(1, t) = 0, \quad t \geq 0, \\ &u(x, 0) = 1 - 2|x - 0.5|, \quad 0 < x < 1. \end{aligned}$$

Apply the Crank-Nicolson method with $\Delta x = 1/4$ and $\Delta t = 1/4$, write down the matrix A and vector b, such that

$$A \begin{bmatrix} u_1^1 \\ u_2^1 \\ u_3^1 \end{bmatrix} = b,$$

where $u_j^k \approx u(x_j, t_k)$ for $x_j = j\Delta x$, $t_k = k\Delta t$. You do not need to solve the system.

4. (15 marks) Consider the first order hyperbolic equation

$$u_t + (1+x)u_x = 0, \quad -\infty < x < +\infty, \quad t > 0,$$

satisfying the initial condition $u(x,0) = f(x)$, where

$$f(x) = \begin{cases} x & 0 \le x < 1 \\ 0 & \text{otherwise.} \end{cases}$$

Apply the Lax-Wendroff method with $\Delta x = 0.2$ and $\Delta t = 0.1$, find u_0^1 which approximates $u(x_0, t_1)$, where $x_j = j\Delta x$, $t_k = k\Delta t$.

5. (20 marks) Consider the equation

$$u_{xx} + u_{yy} + xu = y$$

in the triangular region bounded by the x-axis, the y-axis and the line $x+y=1$. The boundary conditions are $u=1$ on the x-axis, $u=2$ on the y-axis and $u=3$ on the line $x+y=1$. For a second order finite difference method with $\Delta x = \Delta y = h = 1/4$, find the matrix A and vector b, such that

$$A \begin{bmatrix} u_{11} \\ u_{21} \\ u_{12} \end{bmatrix} = b,$$

where $u_{ij} \approx u(ih, jh)$. You do not need to solve the system.

6. Consider the Maxwell's equations in one spatial dimension:

$$\frac{\partial E}{\partial t} = -\frac{1}{\epsilon}\frac{\partial H}{\partial z}, \quad \frac{\partial H}{\partial t} = -\frac{1}{\mu}\frac{\partial E}{\partial z}, \tag{1}$$

where E and H are scalars. For simplicity, we also assume that ϵ and μ are constants.

(a) (10 marks) If we discretize E by $E_j^{k+1/2} \approx E(z_j, t_{k+1/2})$ and discretize H by $H_{j+1/2}^k \approx H(z_{j+1/2}, t_k)$, where

$$z_j = z_0 + j\Delta z, \; z_{j+1/2} = z_0 + (j+0.5)\Delta z, \; t_k = t_0 + k\Delta t, \; t_{k+1/2} = t_0 + (k+0.5)\Delta t,$$

write down a 2nd order finite difference scheme for (1) using central difference approximations.

(b) (10 marks) Analyze the stability of your method by assuming

$$E_j^{k+1/2} = E_* \rho^{k+1/2} e^{ij\beta\Delta z}, \quad H_{j+1/2}^k = H_* \rho^k e^{i(j+0.5)\beta\Delta z},$$

where E_* and H_* are some constants.

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