City University of Hong Kong

Course code & title:	MA6606 Computational Linear Algebra
Session:	Semester B, 2001-2002
Time allowed:	Three hours

This paper has THREE pages. (Including this page)

Instructions to candidates:

- The paper has **seven** questions.
- Attempt only **SIX** questions.
- All questions carry equal marks.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

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1. Give a definition of the matrix 2-norm. Calculate $||A||_2$ for

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

2. For the following symmetric matrix

$$A = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{bmatrix},$$

   (a) find a Givens rotations G_1 and an upper triangular matrix R, such that $G_1 A = R$,

   (b) find a Givens rotation G_2 and a symmetric tridiagonal T, such that $G_2 A G_2^T = T$.

3. The standard LU decomposition is $A = LU$, where L is unit lower triangular, U is upper triangular. An alternative is $A = \hat{U}\hat{L}$, where $\hat{U}$ is *unit* upper triangular and $\hat{L}$ is lower triangular (not necessarily unit). Assuming that this factorization exists, write down an algorithm for $A = \hat{U}\hat{L}$. The following matrix factorization may be useful:

$$A = \begin{bmatrix} B & c \\ d^T & e \end{bmatrix} = \begin{bmatrix} I & c/e \\ 0 & 1 \end{bmatrix} \begin{bmatrix} B - cd^T/e & 0 \\ d^T & e \end{bmatrix},$$

where A is an $m \times m$ square matrix, e is a scalar, B is $(m-1) \times (m-1)$, c and d are column vectors of length $m-1$.

4. Let q_1, q_2 and q_3 be three real unit vectors (of length m) that are orthogonal to each other. Let A be a real $m \times m$ matrix and assume

$$A[q_1, q_2] = [q_1, q_2, q_3] \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix}, \quad \text{and let} \quad b = q_1 - q_2.$$

Show that

$$\min_{x \in <q_1, q_2>} ||Ax - b|| = \min_{z \in \mathbb{R}^2} \left\| \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 0 & 2 \end{bmatrix} z - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \right\|.$$

where $|| \cdot ||$ is the 2-norm, $< q_1, q_2 >$ is the vector space spanned by q_1 and q_2.

5. Consider a symmetric matrix A. Let λ_1, λ_2 be two distinct eigenvalues of A with their corresponding unit eigenvectors q_1 and q_2. Let $x_0 = q_1 + \epsilon q_2$ be an approximate eigenvector, where ϵ is a small constant. The Rayleigh quotient iteration can be used to find the next approximation x_1. Show that x_1 is proportional to $q_1 - \epsilon^3 q_2$.

6. (a) For the real vector $x = (x_1, x_2)^T$, we have two Householder reflections H_1 and H_2 such that

$$H_1 x = \begin{bmatrix} ||x|| \\ 0 \end{bmatrix}, \quad H_2 x = \begin{bmatrix} -||x|| \\ 0 \end{bmatrix},$$

where $||x||$ is the 2-norm of x. Show that the general formula for Householder reflection gives

$$H_1 = \frac{1}{||x||} \begin{bmatrix} x_1 & x_2 \\ x_2 & -x_1 \end{bmatrix}, \quad H_2 = -H_1$$

(b) Based on the 2×2 Householder reflections in (a), carry out one iteration of the QR method without shift (i.e., shift is zero) for

$$A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix},$$

where a is a constant. This leads to a new matrix $\hat{A}$ which has the same eigenvalues as A. For what values of a, the two matrices $\hat{A}$ and A are identical? Do you think the QR method without shift can fail to converge?

7. Apply the Lanczos method to

$$A = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}.$$

Find a 2×2 matrix T_2 and orthonormal vectors q_1, q_2 (choose your own q_1), such that $A[q_1, q_2] \approx [q_1, q_2]T_2$.

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