City University of Hong Kong

Course code & title:	MA6606 Computational Linear Algebra
Session:	Semester B, 2002-2003
Time allowed:	Three hours

This paper has THREE pages. (Including this page)

Instructions to candidates:

- The paper has **seven** questions.
- Attempt only **SIX** questions.
- All questions carry equal marks.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

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1. Let A be the following 2×2 matrix

$$A = \begin{bmatrix} 3 & 1 \\ 2 & 1 \end{bmatrix}.$$

   (a) Find the LU decomposition with partial pivoting: $PA = LU$.

   (b) Find the QR factorization: $A = QR$.

   (c) Find $||A||_2$.

2. Let A be an $m \times m$ matrix (where $m > 2$) and b be an $m \times 1$ vector. Let c_1, c_2 be two linearly independent $m \times 1$ vectors and $K =< c_1, c_2 >$ be the two-dimensional space spanned by c_1, c_2. Describe a method to solve

$$\min_{x \in K} ||Ax - b||_2.$$

3. Describe an algorithm for solving the following bi-diagonal linear system

$$\begin{bmatrix} a_1 & c_1 & & \\ & a_2 & \ddots & \\ & & \ddots & c_{m-1} \\ & & & a_m \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

   Show that your algorithm is backward stable.

4. In the process of orthogonal reduction to upper Hessenberg form (for eigenvalue problems), we have

$$H_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0.6 & 0.8 & 0 \\ 0 & 0.8 & -0.6 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad H_1 A = \begin{bmatrix} 2 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 1 \end{bmatrix}.$$

   Finish the computation and find the upper Hessenberg matrix.

5. Consider the following 4×4 matrix:

$$A = \begin{bmatrix} 1 & & & \\ & 2 & & \\ & & 3 & \\ & & & 4 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} [1, 1, 1, 1] = \begin{bmatrix} 2 & 1 & 1 & 1 \\ 1 & 3 & 1 & 1 \\ 1 & 1 & 4 & 1 \\ 1 & 1 & 1 & 5 \end{bmatrix}.$$

   Let $\lambda_1 < \lambda_2 < \lambda_3 < \lambda_4$ be the four eigenvalues of A. Show that

$$1 < \lambda_1 < 2, \quad 2 < \lambda_2 < 3, \quad 3 < \lambda_3 < 4, \quad 4 < \lambda_4.$$

6. Let $A = QTQ^*$, where A is 4×4 real symmetric, Q is orthogonal and T is symmetric tridiagonal. Let q_1 be the first column of Q and define

$$K = [q_1, Aq_1, A^2q_1, A^3q_1].$$

Show that Q^*K is upper triangular.

7. The MINRES method is an iterative method for solving $Ax = b$, where A is symmetric, but not necessarily positive definite. The solution x_n of MINRES is obtained from solving

$$\min_{x\in\mathcal{K}_n} ||b - Ax||_2,$$

where $\mathcal{K}_n =< b, Ab, ..., A^{n-1}b >$ is the Krylov subspace. Let $p(\lambda) = 1 + c_1\lambda + c_2\lambda^2 + c_3\lambda^3$ satisfy

$$||p(A)\, b||_2 \leq ||q(A)\, b||_2,$$

where q is an arbitrary polynomial of degree ≤ 3 satisfying $q(0) = 1$. For

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 2 & 0 \\ 0 & 2 & 2 & 3 \\ 0 & 0 & 3 & 4 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

write down x_3 in terms of c_1, c_2 and c_3.

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