City University of Hong Kong

Course code & title:	MA6606 Computational Linear Algebra
Session:	Semester A, 2003-2004
Date:	9 December 2003
Time:	18:30 — 21:30
Time allowed:	Three hours

This paper has THREE pages. (Including this page)

Instructions to candidates:

- Attempt all **SIX** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (15 marks) Find the singular value decomposition and the matrix 2-norm of

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}.$$

2. (10 marks) Find the Cholesky decomposition of

$$A = \begin{bmatrix} 4 & 2 & 2 & 0 \\ 2 & 2 & 1 & 0 \\ 2 & 1 & 5 & 2 \\ 0 & 0 & 2 & 10 \end{bmatrix}.$$

3. Consider the matrix

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & 1 \\ -2 & 1 & 0 \end{bmatrix}.$$

   (a) (10 marks) Using Householder reflections, find the matrix R in a QR factorization of A.

   (b) (10 marks) Using Householder reflections, find an upper Hessenberg matrix T, such that A and T have the same eigenvalues.

4. (20 marks) Let A be a 2×2 real symmetric positive definite matrix. It has a LDL^T decomposition:

$$A = \begin{bmatrix} a & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ l & 1 \end{bmatrix} \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} \begin{bmatrix} 1 & l \\ 0 & 1 \end{bmatrix} = LDL^T,$$

where

$$l = \frac{b}{a}, \quad d = c - bl.$$

Show that the computer results $\tilde{l}$ and $\tilde{d}$ (for l and d) correspond to the exact LDL^T decomposition of $\tilde{A}$, where the entries of $\tilde{A}$ are close to the entries of A.

5. If A has a LU decomposition, i.e. $A = LU$, where L is unit lower triangular and U is upper triangular, we define the matrix $\hat{A}$ by $\hat{A} = UL$. Show that:

   (a) (7 marks) A and $\hat{A}$ have the same eigenvalues;

   (b) (8 marks) If A is upper Hessenberg, then $\hat{A}$ is also upper Hessenberg.

6. (20 marks) Let A be a real symmetric matrix. The eigenvalues and eigenvectors of A are λ_j and $\vec{x}_j$ (for $j = 1, 2, ...$), respectively. We assume that $\lambda_1 > \lambda_2$ and the eigenvectors are unit orthogonal vectors. Let

$$\vec{q}_1 = c\,\vec{x}_1 + s\,\vec{x}_2$$

for some positive constants c and s such that $c^2 + s^2 = 1$. Show that when $\vec{q}_1$ is used as the starting vector for the Lanczos method:

$$A[\vec{q}_1, \vec{q}_2, ...] = [\vec{q}_1, \vec{q}_2, ...] \begin{bmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \beta_2 & \\ & \beta_2 & \alpha_3 & \ddots \\ & & \ddots & \ddots \end{bmatrix},$$

we have $\beta_2 = 0$. Find α_1, β_1, α_2 and $\vec{q}_2$.

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