City University of Hong Kong

Course code & title:	MA6606 Computational Linear Algebra
Session:	Semester A, 2005-2006
Date:	15 December 2005
Time:	18:30 — 21:30
Time allowed:	Three hours

This paper has THREE pages. (Including this page)

Instructions to candidates:

- Attempt all **seven** questions.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. (15 marks) Find the eigenvalues, the singular values and the 2-norm of the matrix

$$A = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

2. (15 marks) For the matrix A below, find the LU decomposition with partial pivoting, i.e. $PA = LU$.

$$A = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{bmatrix}.$$

3. (15 marks) Let matrix A and vector b be given by

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \\ 0 & 2 \\ 0 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

   (a) Find a QR factorization of A.

   (b) Solve the least squares problem

$$\min_x ||Ax - b||.$$

   You need to calculate both the minimum value of $||Ax - b||$ and the value of x at which the minimum is obtained.

4. (15 marks) Find an orthogonal matrix Q and an upper Hessenberg matrix H, such that $A = QHQ^*$, where

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 2 & 1 \\ 1 & 0 & 2 \end{bmatrix}.$$

5. (15 marks) Consider the matrix

$$T_0 = \begin{bmatrix} a & b \\ \epsilon & d \end{bmatrix},$$

where ϵ is small and a step of the QR method

$$T_0 - sI = QR, \quad T_1 = sI + RQ$$

with $s = d$. Find the matrix T_1 and show that the $(2,1)$ entry of T_1 is $O(\epsilon^2)$.

6. (15 marks) Consider the Lanczos method

$$A[q_1, q_2, ...] = [q_1, q_2, ...] \begin{bmatrix} \alpha_1 & \beta_1 & & \\ \beta_1 & \alpha_2 & \beta_2 & \\ & \beta_2 & \alpha_3 & \ddots \\ & & \ddots & \ddots \end{bmatrix}.$$

where A is a symmetric matrix, $q_1 = b/||b||$ for some non-zero vector b. If $\beta_1 \neq 0$ and $\beta_2 = 0$, show that

$$f(A)b = ||b||\ [q_1, q_2]\, f(T_2) \begin{bmatrix} 1 \\ 0 \end{bmatrix},$$

where $T_2 = \begin{bmatrix} \alpha_1 & \beta_1 \\ \beta_1 & \alpha_2 \end{bmatrix}$ and $f(z)$ is a polynomial of z.

7. (10 marks) Consider the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

satisfying $|ad| > |bc|$. The LU decomposition of A can be computed using three operations. Show that the algorithm for computing the LU decomposition of A is backward stable.

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