# City University of Hong Kong

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| Course code & title: | MA6606 Computational Linear Algebra |
| Session: | Semester A, 1999-2000 |
| Date: | January 3, 2000 |
| Time: | 6:30 pm — 9:30 pm |
| Time allowed: | Three hours |

This paper has FOUR pages. (Including this page)

Instructions to candidates:

- The paper has **eight** questions.
- Attempt only **SIX** questions.
- All questions carry equal marks.
- Start each question on a new page.
- Show all working.

Materials, aids & instruments permitted to be used during examination:

- Non-programmable portable battery operated calculator.

1. Let $v$ be a real column vector of length $m$ and
$$H = I - \frac{2}{v^T v} v v^T.$$
Describe an efficient algorithm for calculating $HA$, where $A$ is a given $m \times m$ matrix. What is the total number of required operations (accurate to the leading order term)?

2. Given four points $\{(x_i, y_i),\ i = 1, 2, 3, 4\}$ in the $xy$ plane, we seek a polynomial
$$P(x) = \alpha x^2 + \beta x + \gamma$$
such that $P(x_i) \approx y_i$. In the first approach, the coefficients $\alpha$, $\beta$ and $\gamma$ are determined from
$$\min_{\alpha,\beta,\gamma} \sum_{i=1}^{4} [P(x_i) - y_i]^2.$$
Formulate this as a matrix least squares problem in the form of $\min_z ||Az - b||$ for a matrix $A$, a vector b and $z = [\alpha, \beta, \gamma]^T$. In the second approach, the coefficients $\alpha$, $\beta$ and $\gamma$ are determined from
$$\min_{\alpha,\beta,\gamma} \left( \sum_{i=1}^{4} [P(x_i) - y_i]^2 + 4\alpha^2 + \beta^2 + \frac{1}{4}\gamma^2 \right).$$
Formulate this as another matrix least squares problem for $z$. You do not need to solve these least squares problems.

3. In the process of calculating the LU decomposition with partial pivoting for the $4 \times 4$ matrix $A$, we have
$$E_3 P_3 E_2 P_2 E_1 P_1 A = U$$
where
$$E_1 = \begin{pmatrix} 1 & & & \\ a & 1 & & \\ b & & 1 & \\ c & & & 1 \end{pmatrix}, \quad E_2 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & d & 1 & \\ & e & & 1 \end{pmatrix}, \quad E_3 = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & f & 1 \end{pmatrix}.$$
$P_1$, $P_2$, $P_3$ are permutation matrices that exchange rows:

- $P_1$ exchanges row 1 with row 3;
- $P_2$ exchanges row 2 with row 4;
- $P_3$ exchanges row 3 with row 4.

Find the matrices $L$ and $P$ in $PA = LU$.

4. Let $A$ be a symmetric positive definite tridiagonal matrix given by

$$A = \begin{pmatrix} a_1 & b_1 & & & \\ b_1 & a_2 & b_2 & & \\ & b_2 & a_3 & \ddots & \\ & & \ddots & \ddots & b_{m-1} \\ & & & b_{m-1} & a_m \end{pmatrix}.$$

It can be written as $A = LDL^T$, where

$$L = \begin{pmatrix} 1 & & & & \\ l_1 & 1 & & & \\ & l_2 & 1 & & \\ & & \ddots & \ddots & \\ & & & l_{m-1} & 1 \end{pmatrix}, \quad D = \begin{pmatrix} d_1 & & & & \\ & d_2 & & & \\ & & d_3 & & \\ & & & \ddots & \\ & & & & d_m \end{pmatrix}.$$

(a) Write down an efficient algorithm for calculating $\{l_j\}$ and $\{d_j\}$.

(b) Show that $d_j > 0$ for $j = 1, 2, ..., m$.

(c) For the case of $m = 2$, show that your algorithm is backward stable.

5. Let $Q$ be an orthogonal matrix given by

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \alpha & \gamma \\ 0 & \beta & \delta \end{pmatrix}.$$

Assume $QH_1 = H_2Q$ for two upper Hessenberg matrices $H_1$ and $H_2$ (i.e., the (3,1) entries of $H_1$ and $H_2$ are zero). If the (2,1) and (3,2) entries of $H_1$ are non-zero, show that

$$\alpha = \pm 1, \quad \beta = 0, \quad \gamma = 0, \quad \delta = \pm 1.$$

6. Given a diagonal matrix $D$ and two vectors $y$ and $z$ as follows:

$$D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_m \end{pmatrix}, \quad y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}, \quad z = \begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_m \end{pmatrix},$$

if $\lambda$ is an eigenvalue of $D + zy^T$, show that

$$1 = \sum_{j=1}^{m} \frac{y_j z_j}{\lambda - d_j}.$$

7. Let $A$ and $b$ be given by

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Find the polynomial $p(\lambda) = \lambda^2 + \alpha\lambda + \beta$, such that

$$||p(A)b||_2 = \min_{q \in P_2} ||q(A)b||_2$$

where $P_2$ is the set of all monic polynomials of degree $\leq 2$.

8. The conjugate gradient method for $Ax = b$ is

- $x_0 = 0$, $r_0 = b$, $p_0 = r_0$
- for $k = 1, 2, 3, \ldots$

$$\begin{aligned} \alpha_k &= \frac{r_{k-1}^T r_{k-1}}{p_{k-1}^T A p_{k-1}} \\ x_k &= x_{k-1} + \alpha_k p_{k-1} \\ r_k &= r_{k-1} - \alpha_k A p_{k-1} \\ \beta_k &= \frac{r_k^T r_k}{r_{k-1}^T r_{k-1}} \\ p_k &= r_k + \beta_k p_{k-1}. \end{aligned}$$

Let $\lambda_1 = 2$, $\lambda_2 = 1$ be two distinct eigenvalues of the symmetric positive definite matrix $A$ and $q_1$, $q_2$ be the corresponding unit eigenvectors. If $b = q_1 + 2q_2$, find the first and the second iterations (that is, $x_1$ and $x_2$) by the conjugate gradient method.

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