Computing High-Frequency Scattered Fields by Beam Propagation Methods: A Prospective Study

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Received 01/04/2009; Accepted 02/06/2009

ABSTRACT
This paper presents some theoretical and numerical investigations concerning the fast computation of an exterior wave field to a scatterer by the Beam Propagation Method (BPM). Different models are presented and compared. It appears that the approach is able to correctly model the propagation of the propagative modes of the wave field while inaccuracies still remain for the evanescent and transition modes.

Keywords: acoustic scattering; beam propagation methods; one-way equations; fast algorithms; high-frequency

1. INTRODUCTION

The aim of this paper is to prospect the possible application of the Beam Propagation Method (BPM) for solving high-frequency scattering problems. To focus our study on special features of the BPM, we rather restrict our developments to the sound-soft scattering problems but extensions could also be considered to more general problems like sound-hard or impedance problems. Hence, the wave field u satisfies the Helmholtz equation in the unbounded domain exterior to the cylinder and can be decomposed as

$$u = u^{(i)} + u^{(s)},$$

where $u^{(i)}$ is the given incident wave, $u^{(s)}$ is the unknown scattered wave solution to the boundary-value problem

$$\begin{cases} \Delta u^{(s)} + k^2 u^{(s)} = 0, & \text{in} \quad \Omega^+, \\ u^{(s)} = -u^{(i)}, & \text{on} \quad \partial\Omega, \\ \lim_{r\to\infty} \sqrt{r}\left(\dfrac{\partial u^{(s)}}{\partial r} - iku^{(s)}\right) = 0, \end{cases}$$

where Ω is the cross-section of the cylinder, Ω^+ is the domain exterior to Ω, $\partial\Omega$ is the boundary of Ω, r is the radial variable in the polar coordinate system and $k = 2\pi/\lambda$ is the wavenumber. The radiation condition at infinity ensures the uniqueness of the solution. A brief illustration of the problem is given on Figure 1.

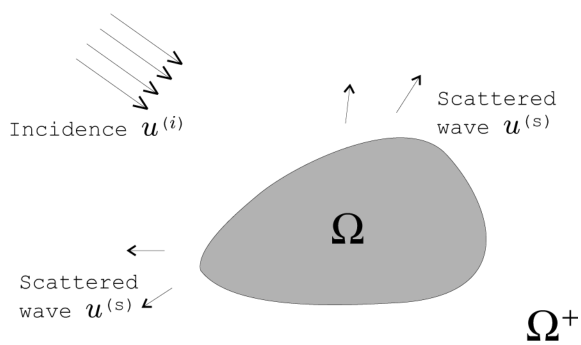


Fig.1 scattering by a two-dimensional obstacle

Various numerical methods have been developed over the past decades to solve the time-harmonic acoustic problem. One well-known possible approach is to truncate the unbounded domain Ω^+ with an Artificial Boundary Condition (ABC) and to use the Finite Element Method (FEM) in a bounded domain [1]. This method can handle scatterers with complicated boundaries. Another important method is the Boundary Integral Equation (BIE) method [2,3,4]. The problem is then reformulated as an integral equation defined on $\partial\Omega$ and solved through techniques combining an iterative linear algebra solver [5] and compression algorithms like e.g. the Fast Multilevel Multipole (FMM) technique [6]. In many applications, the angular frequency ω is extremely high, the wavenumber $k = \omega/c$ is large and the wavelength $\lambda = 2\pi/k$ is much smaller than the characteristic length of the scatterer. Then, both the ABC-FEM and the BIE have difficulties in solving such large scale problems since, in particular, the associated linear systems have a very large size. Moreover, these two methods require that an integral must be evaluated at any exterior point where the solution is desired. Both these two methods require more than $\mathcal{O}(N)$ operations for each point where N is the number of grid points used to discretize the boundary. When N is large, this is quite expensive.

The Beam Propagation Method (BPM) is an efficient approximation method widely used in optical waveguide modelling. Under the assumption that the wave field is dominated by its forward component, the BPM gives rise to one-way equations that approximate the Helmholtz equation. It is useful for waveguides that change slowly in the propagation direction. The one-way equations have only a first order derivative in the propagation direction and can be efficiently solved as an initial value problem. Operator rational approximations are involved in solving the one-way models. There are various types of rational approximations to the one-way equations. For example, the BPMs can be solved by rational approximations to the square root operator [7,8,9] or rational approximations to the exponential of the square root operator, i.e. the propagator [10,11].

In this paper, an attempt is made to apply the BPM method to the scattering problem. We develop an approximate method which can efficiently solve the scattered wavefield. Our method requires only $\mathcal{O}(1)$ operations for each point where the solution is desired.

The plan of the paper is the following. In Section 2, we use a curvilinear coordinate system to rewite the scattering problem following the parallel surfaces to the scatterer. We then explain in details in the third Section the approach by Beam Propagation Methods, developing different models and approximations as well as rational approximations for more efficiency. Section 4 proposes some numerical simulations and investigations to see how the method applies in terms of efficiency and accuracy. Finally, the last Section gives some conclusions.

2. CURVILINEAR COORDINATES SYSTEM

The scattering problem is different from the propagation problem in waveguides and in particular, the geometry itself must be treated with special attention. In the following, we first develop a coordinate transformation to apply the BPM method in the new coordinate system.

We assume that the boundary of the scatterer $\partial\Omega$ is given by

$$x = f(s), \quad y = g(s), \quad 0 \le s \le l,$$

where s is the arclength and l is the total length of $\partial\Omega$. Using the counterclockwise direction as the direction of increasing s, we have an outward unit normal vector given by

$$\vec{n}(s) = (g'(s), -f'(s)). \tag{1}$$

As illustrated in Figure 2, any point (x,y) in the exterior domain Ω^+ (assuming that Ω is convex) can be written as

$$\begin{cases} x = f(s) + \xi g'(s) \\ y = g(s) - \xi g'(s), \end{cases} \tag{2}$$

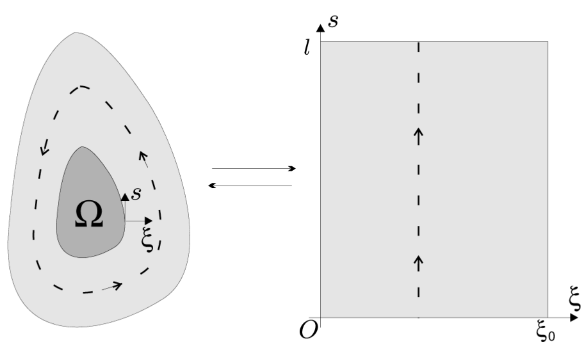


Fig.2 the curvilinear coordinate transform

for $0 \le s \le l$ and $0 \le \xi \le +\infty$. Under the new coordinate system (ξ,s), the Helmholtz equation becomes

$$u_{\xi\xi} + \frac{\gamma_\xi}{\gamma} u_\xi + \frac{1}{\gamma}\left(\frac{1}{\gamma} u_s\right)_s + k^2 u = 0,$$

where $\gamma = 1 + \xi\kappa(s)$ and κ is the curvature of the boundary $\partial\Omega$. For $u = \gamma^{-1/2}\upsilon$, we have

$$\upsilon_{\xi\xi} + \left(\frac{1}{\gamma^2}\upsilon_s\right)_s + \left(k^2 \frac{\gamma_\xi^2}{4\gamma^2} - \frac{\gamma_{ss}}{2\gamma^3} + \frac{\gamma_s^2}{4\gamma^4}\right)\upsilon = 0. \tag{3}$$

For convenience, we define an operator L by

$$L(\xi)=\frac{\partial}{\partial s}\left(\frac{1}{\gamma^2}\frac{\partial}{\partial s}\right)+k^2+\frac{\gamma_\xi^2}{4\gamma^2}-\frac{\gamma_{ss}}{2\gamma^3}+\frac{\gamma_s^2}{4\gamma^4}. \tag{4}$$

It acts on functions of s and it depends on ξ as a parameter.

3. THE BPM APPROACH

In the new coordinate system, the scattered wave propagates outwards in the increasing ξ direction. This suggests that the scattered field may be approximated by one-way models that are first order in ξ. Following the BPM for optical waveguides, we first approximate equation (3) by one-way equations and then apply operator rational approximations.

There are a few different one-way models used in the BPM. One possibility is to approximate equation (3) by $\upsilon_\xi = i\sqrt{L(\xi)}\upsilon$ using the square root operator $\sqrt{L(\xi)}$. Here, we adopt the Energy-Conserving model [12,13,14] which gives improved accuracy for slowly varying waveguide. The energy-conserving model can be derived from the continuity of the power flux and it involves a transform using the fourth root of the operator L. We have

$$\begin{cases}\phi_\xi = i\sqrt{L}\phi,\\ \phi \ = \sqrt[4]{L}\upsilon, \quad \phi(0)=\sqrt[4]{L}\upsilon(0).\end{cases} \tag{5}$$

In terms of the original function υ, the energy-conserving one-way model is

$$\upsilon_\xi=\left(i\sqrt{L}-L^{-1/4}\frac{d(L^{1/4})}{d\xi}\right)\upsilon.$$

Notice that the original boundary value problem of the Helmholtz equation for u is now approximated by an initial value problem for ϕ. As in the standard BPM, the square root (and the fourth root) of L must be approximated. For this purpose, we introduce the operator X by

$$L=k_0^2(1+X),$$

where k_0 is a reference wavenumber. Then, equation (5) becomes

$$\begin{cases} \phi_\xi = ik_0\sqrt{1+X}\phi \\ \phi(0) = -\sqrt[4]{L(0)}u^{(i)} \end{cases}.$$

Formally, for the step from ξ_j to $\xi_{j+1} = \xi_j + h$, we have

$$\phi_{j+1} = P\phi_j, \quad P = e^{ik_0 h\sqrt{1+x}}\phi_j, \tag{6}$$

where P is the one-way propagator and X is evaluated at the midpoint $\xi = \xi_j + h/2$.

The propagator P can be approximated by rational functions of X. For waveguide problem, the standard $[p/p]$ Padé approximation provides a good approximation for the propagating modes but fails to suppress the evanescent modes, while the $[(p-1)/p]$ Padé approximant can damp the evanescent modes but gives less accurate results for the propagating modes. For the current scattering problem, a proper treatment of the evanescent modes appears necessary. The propagator-θ method [15] combines the $[p/p]$ and $[(p-1)/p]$ Padé approximants by a parameter θ. With a suitable choice of p and θ, the propagator-θ method gives a better balance for approximating both the propagating and evanescent modes.

The propagator-θ approximant of degree p takes the following form

$$P \approx R_p(\theta) = \frac{1+\alpha_1 X + \ldots + \alpha_{p-1}X^{p-1} + \alpha_p X^p}{1+\beta_1 X + \ldots + \beta_{p-1}X^{p-1} + \beta_p X^p} \tag{7}$$

$$= \prod_{k=1}^{p} \frac{1+c_k X}{1+b_k X} = 1 + \sum_{k=1}^{p} \frac{a_k X}{1+b_k X} \tag{8}$$

$$= d_0 + \sum_{k=1}^{p} \frac{d_k}{1+b_k X}. \tag{9}$$

It is related to the following $[p/p]$ and $[(p-1)/p]$ Padé approximants

$$R_{p-1,p} = \frac{1+\alpha_1^{(0)}X + \quad + \alpha_{p-1}^{(0)}X^{p-1}}{1+\beta_1^{(0)}X + \quad + \beta_{p-1}^{(0)}X^{p-1} + \beta_p^{(0)}X^p} \tag{10}$$

$$R_{p,p} = \frac{1+\alpha_1^{(1)}X + \ldots + \alpha_{p-1}^{(1)}X^{p-1} + \alpha_p^{(1)}X^p}{1+\beta_1^{(1)}X + \ldots + \beta_{p-1}^{(1)}X^{p-1} + \beta_p^{(1)}X^p}. \tag{11}$$

The coefficients $\{\alpha_i^{(0)}\},\{\beta_i^{(0)}\},\{\alpha_i^{(1)}\}$ and $\{\beta_i^{(1)}\}$ can be solved from a linear system of equations assuming that the Taylor coefficients of the function $e^{ihk_0\sqrt{1+X}}$ is available. The coefficients $\{\alpha_i\}$ and $\{\beta_i\}$, $i = 1, \quad , p$ can be calculated from

$$\alpha_i = \theta\alpha_i^{(1)} + (1-\theta)\alpha_i^{(0)}, \quad \beta_i = \theta\beta_i^{(1)} + (1-\theta)\beta_i^{(0)},$$

where $\alpha_p^{(0)} = 0$ and $\theta \in [0,1]$ The expression (9) is more convenient for numerical implementation than the standard form (7). The coefficients c_k and b_k are first obtained by factorizations of the dominator and numerator of (7). Coefficients a_k and d_k can be calculated from

$$a_k = \left(c_k - b_k\right)\sum_{j\neq k}\frac{c_j - b_k}{b_j - b_k}, \tag{12}$$

$$d_k = -\frac{a_k}{b_k}, \quad d_0 = 1 - \sum_{k=1}^{p} d_k. \tag{13}$$

With the rational approximation (9), the propagation step (6) is approximated by

$$\begin{aligned}\phi_{j+1} = P\phi_j &\approx e^{ihk_0}\left(d_0 + \sum_{k=1}^{p}\frac{d_k}{1+b_kX}\right)\phi_j \\ &= e^{ihk_0}\left(d_0\phi_j + \sum_{k=1}^{p} d_k w_k\right),\end{aligned} \tag{14}$$

where w_k is solved from

$$(1+b_kX)w_k = \phi_i, \quad k = 1,\ 2,\ \ldots,\ p. \tag{15}$$

Since X is related to L as in (4), the discretized form of (15) is a periodic tridiagonal system. If N is the number of grid points for discretizing the s variable, the required number of operations in each step is $\mathcal{O}(N)$. Therefore, the number of operations required for each point is only of $\mathcal{O}(1)$.

The 4-th root operator $\sqrt[4]{L} = \sqrt{k_0}\sqrt[4]{1+X}$ must be evaluated when the unknown function υ is transformed to ϕ, or vice versa. If the scattered wave field is required in the entire computational domain, this operator should be evaluated after each marching step of ϕ. In order to keep $\mathcal{O}(1)$ operations, a rational approximation for $\sqrt[4]{1+X}$ is used.

For a constant ν, the function $q(x) = (1 + x)^{\nu}$ has a continued fraction expansion as

$$q(x) = 1 + \cfrac{\sigma_1 x}{1 + \cfrac{\sigma_2 x}{1 + \cfrac{\sigma_3 x}{1 + \ddots}}} \tag{16}$$

where $\sigma_1 = \nu$ and

$$\sigma_{2k} = \frac{k-\nu}{2(2k-1)}, \quad \sigma_{2k+1} = \frac{k+\nu}{2(2k+1)}, \tag{17}$$

for $k = 1, 2, \cdots$. Let $q_m(x)$ be the truncation of (16) up to the term $\sigma_m x$, we have

$$q_{2m}(x) = R_{m,m}(x), \tag{18}$$

where $R_{m,m}$ is the $[m,m]$ Padé approximant of $q(x)$. Furthermore, the sequence $\{q_{2m}(x)\}$ converges to $q(x)$ for all complex x, except for $-\infty < x < -1$ [8].

For practical computation we rearrange $q_{2m}(x)$ as

$$q_{2m}(x) = 1 + \cfrac{\sigma_1 x}{1 + \cfrac{\cdots}{1 + \cfrac{\sigma_{2m} x}{1 + i\tau}}}$$

$$= \prod_{k=1}^{m} \frac{1+\tilde{c}_k x}{1+\tilde{b}_k x} = 1 + \sum_{k=1}^{m} \frac{\tilde{a}_k x}{1+\tilde{b}_k x} \tag{19}$$

$$= \tilde{d}_0 + \sum_{k=1}^{m} \frac{\tilde{d}_k}{1+\tilde{b}_k x}. \tag{20}$$

The coefficients $\{b_k\}$ and $\{c_k\}$ can be calculated from a tridiagonal matrix related to the coefficients $\{\sigma_k\}$, and we can use the same formulas (12) and (13) to calculate coefficients $\{a_k\}$ and $\{d_k\}$.

For our case, we set $\nu = 1/4$. The initial value of ϕ can be approximately evaluated by

$$\phi_0 = \sqrt{k_0}(1+X)^{1/4} \upsilon_0 \approx \sqrt{k_0}\left(d_0 + \sum_{k=1}^{m} \frac{d_k}{1+b_k X}\right)\upsilon_0$$

$$= \sqrt{k_0}\left(d_0 \upsilon_0 + \sum_{k=1}^{m} d_k \eta_k\right), \tag{21}$$

where the operator X is evaluated at $\xi = 0$ and w_k is solved from

$$(1+\tilde{b}_k X)\eta_k = \upsilon_0, \quad k = 1, 2, \ldots, m. \tag{22}$$

The transformation from ϕ to υ, i.e.

$$\upsilon_{j+1} = k_0^{-1/2}(1+X)^{-1/4}\phi_{j+1} \tag{23}$$

can be implemented similarly. As in (14) and (15), the total number of operations needed in these approximations is also of $\mathcal{O}(N)$. As far as the scattered wave field is concerned, the number of operations spent on each point is only of $\mathcal{O}(1)$.

4. NUMERICAL RESULTS

We consider the scattering problem of a circular cylinder centered at the origin with radius a as shown in Figure 3. The boundary of the circular cross-section is given by $x = a \cos \theta$ and $y = a \sin \theta$. The outward unit normal vector is $\vec{n}(s) =$ $(\cos \theta, \sin\theta)$. In terms of ξ and s, the coordinate of a point (x, y) in Ω^+ are

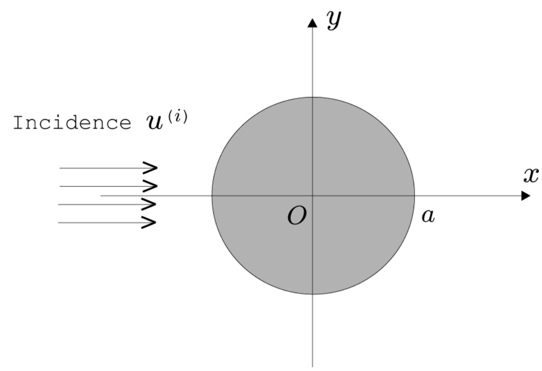


Fig.3 the scattering problem of a circular cylinder

$$x = (a+\xi)\cos\frac{s}{a}, \quad y = (a+\xi)\sin\frac{s}{a}.$$

Note that $r = a + \xi$ for this case. The operator $L(\xi)$ becomes

$$L(\xi) = \frac{a^2}{(a+\xi)^2}\frac{\partial^2}{\partial s^2} + k^2 + \frac{1}{4(a+\xi)^2}. \tag{24}$$

The incident plane wave $u^{(i)}$ is given by

$$u^{(i)} = -\exp\ (ik(x\cos\psi + y\sin\psi)), \tag{25}$$

where ψ is the incident angle. In the following, we assume that $a = 1$, $\psi = 0$ and $k = 35$. Numerical results at $\xi = 8$, i.e. $r = 9$, calculated by BPM are given in Figure 4. These results are obtained with $N = 400$, $h = 0.1$, $p = 3$, $\theta = 0.8$ and $m = 6$. The analytic solution given in the Mie series is plotted for comparison. It is clear that the BPM solution is a good approximation in the interval where the scattered wave is strong, and it is less accurate in other locations. In particular, the BPM solution has some undesirable oscillations.

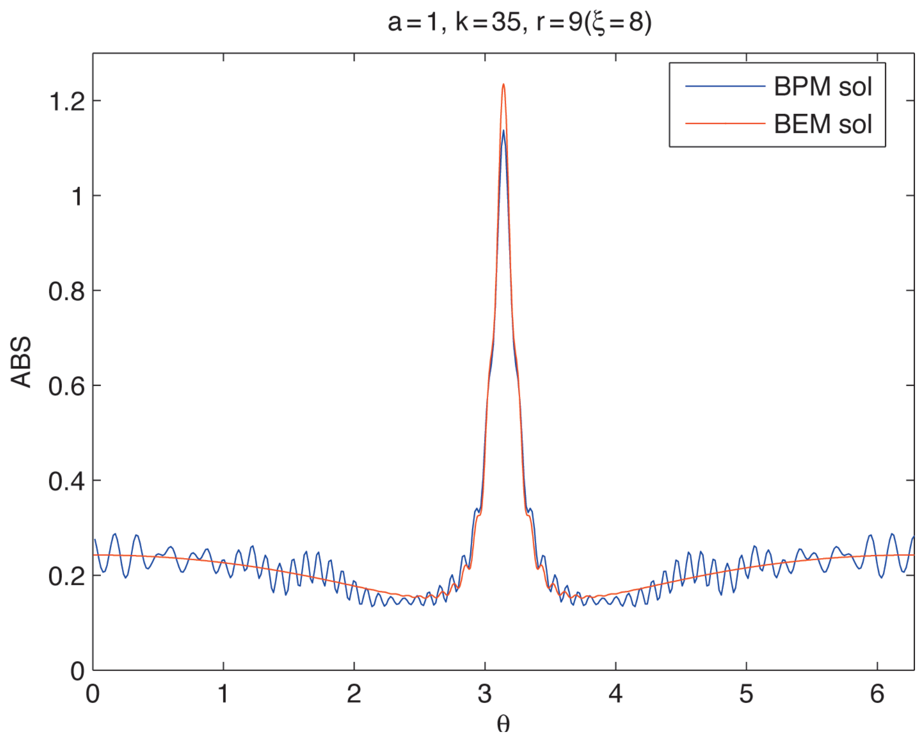


Fig.4 magnitude of the scattered wave of a circular cylinder at $r = 9$

For another example, we consider the scattering of an elliptical cylinder. The incident plane wave has an incident angle $\psi = 35°$ (as shown in Figure 5). The cross-section of the elliptical cylinder has semi-axis $a = 1$ and $b = 0.25$. The boundary is discretized by $N = 1600$ points. A reference solution is obtained by

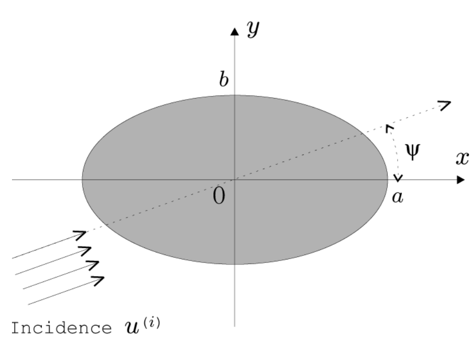


Fig.5 the scattering problem of an elliptic cylinder for plane incident wave with an incident angle ψ

the BIE method. In figure 6, we compare the reference solution with the BPM solutions at r = 9. The BPM solutions are obtained with two different stepsizes: $h = 0.1$ and $h = 0.01$. Both BPM solutions are good approximations when the scattered wave is strong, but they produce incorrect oscillations when the scattered wave is weak.

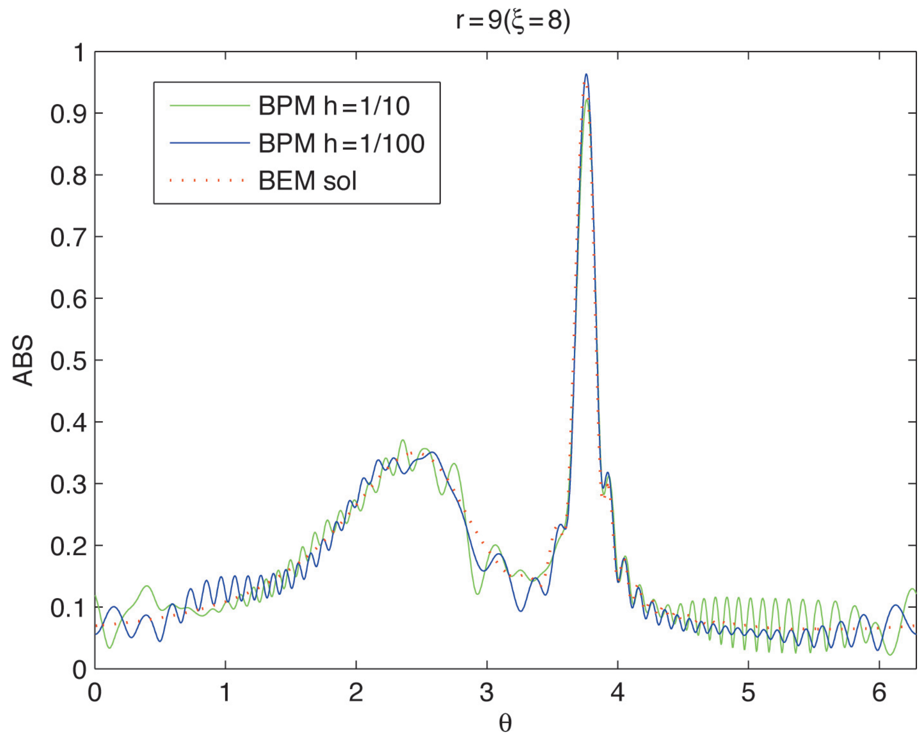


Fig.6 magnitude of the scattered wave of an elliptic cylinder for a plane incident wave with $\psi = 35°$

To understand the limitation of our method, we expand the plane incident wave $u^{(i)}$ in a series of Bessel functions as follows:

$$u^{(i)} = -\exp(ikr\ \cos(\theta - \psi))$$
$$= -\sum_{m=-\infty}^{+\infty} i^m J_m(kr)\ \exp(im(\theta - \psi)) = \sum_{m=0}^{+\infty} u_m^{(i)},$$

where

$$u_m^{(i)} = i^m c_m J_m(kr)\ \cos(m(\theta - \psi))$$

with $c_0 = 1$ and $c_m = 2$ for $m = 1, 2, \ldots$. For this scattering problem, each term $u_m^{(i)}$ gives rise to its scattered wave

$$u_m^{(s)} = c_m \frac{J_m(ka)}{H_m^{(1)}(ka)} H_m^{(1)}(kr)\cos(m(\theta-\psi)),$$

where $H_m^{(1)}$ is the Hankel function of the first kind. Therefore, the exact scattered wave for the plane incident wave can be written in the following Mie series:

$$u^{(s)} = \sum_{m=0}^{+\infty} u_m^{(s)}. \tag{26}$$

The above infinite sum can be truncated as follows:

$$u^{(s)} \approx u^{(s),M} = \sum_{m=0}^{M} u_m^{(s)}.$$

In Figure 7, we plot the absolute values of $u_m^{(s)}$ for $\theta = 0$, $m = 0, \ldots, 60$ and at $r = 2$ and $r = 9$. We observe that the first 40 (a quantity slightly larger than ka)

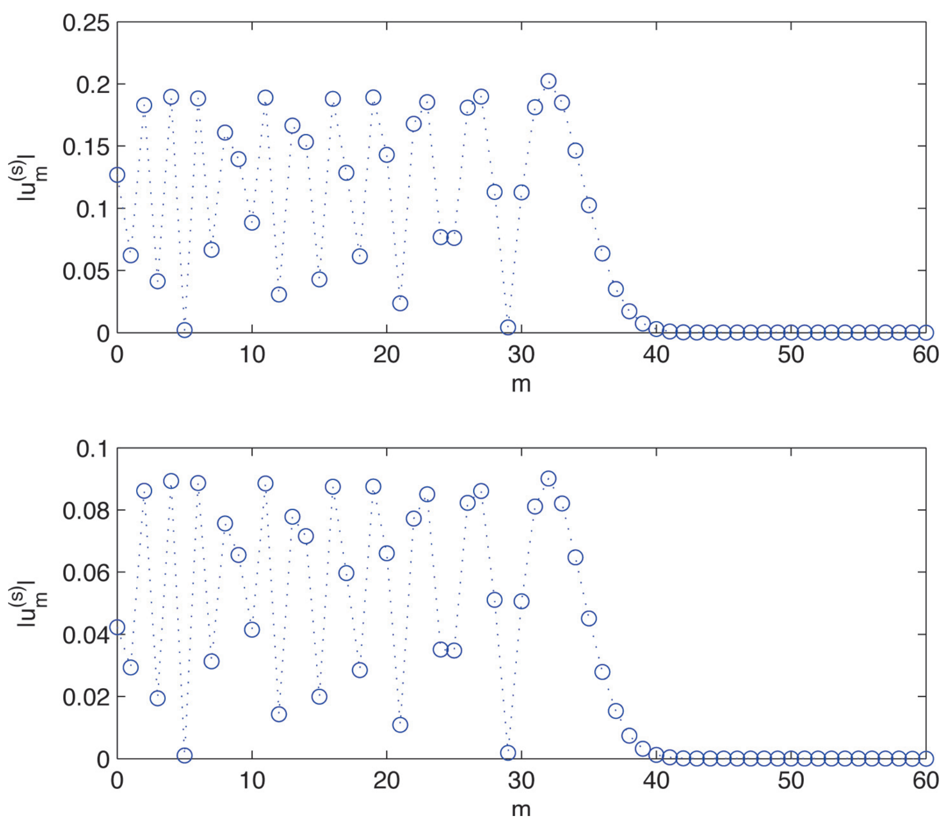


Fig.7 the magnitude of $u_m^{(s)}$ for $m = 0, \ldots, 60$ at $r = 2$ (the upper one) and $r = 9$ (the lower one) terms are significant. Therefore, M should be at least 40 if $u(^{s}),^{M}$ is to be a good approximation of $u^{(s)}$. This suggests that to obtain a BPM solution with acceptable accuracy at least the same number of terms should be properly modelled.

For this example, the E-C model can be solved analytically. In fact, the exact solution $u_m = u_m^{(s)}$ is separable and

$$L(\xi)u_m = \left[-m^2 \frac{a^2}{r^2} + k^2 + \frac{1}{4r^2} \right] u_m = \left[\frac{1-4m^2}{4r^2} + k^2 \right] u_m.$$

That is, the operator $L(\xi)$ becomes a scalar multiplicator on u_m. Thus, its square root can be evaluated directly (without using rational approximations). Following the decomposition of the exact solution (26), we can decompose ϕ (of the E-C model) as

$$\phi = \sum_{m=0}^{\infty} \phi_m.$$

Then, the propagation step from ξ_j to ξ_{j+1} is reduced to:

$$\begin{aligned}\phi_{m,j+1} &\approx \exp\left(ih\sqrt{L\left(\xi_{j+1/2}\right)}\right)\phi_{m,j} \\ &= \exp\left(ih\sqrt{\frac{1-4m^2}{4\left(a+\xi_{j+1/2}\right)^2} + k^2}\right)\phi_{m,j}.\end{aligned}$$

The transformation between ϕ and υ can be implemented in a similar manner.

Next, we compare the truncated analytic solution $u^{(s),M}$ with the analytic solution (E-C) of the E-C model and the fully numerical E-C BPM solution (RAtoEC). The solution E-C is also truncated to M terms and the numerical solution RAtoEC follows a starting field corresponding to the incident field truncated to M terms. These solutions are compared for $M = 30$ and $M = 40$. For $M = 30$, Figure 8 indicates that both E-C and RAtoEC coincide with the exact solution $u^{(s),30}$ at $r = 2$ and $r = 9$. This, however, is not the case for $M = 40$ as shown in Figure 9. It appears that the one-way E-C equation cannot accurately model the modes $u_m^{(s)}$ for large m.

To gain a better understanding, we compare the solutions for a single m. That is, we compare $u_m^{(s)}$ with the corresponding analytic solution of the E-C model and the fully numerical E-C BPM solution (with a single mode incident wave). In Figure 10-12, we compare these solutions $u_m^{(s)}$ for $m = 32$, 34, 35, 36, 37 and 40 for $r \in [1,9]$. It is clear that the solutions of the E-C model are quite different

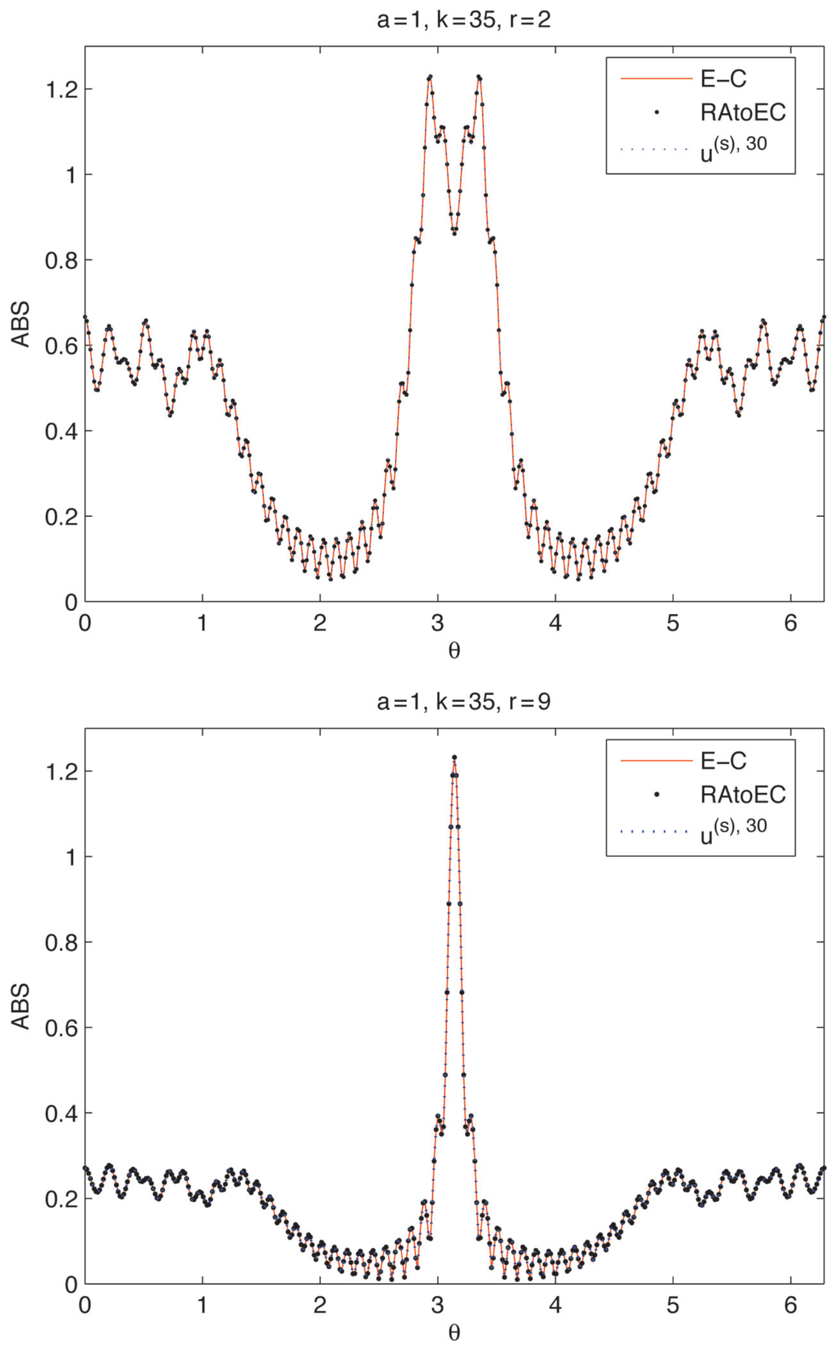
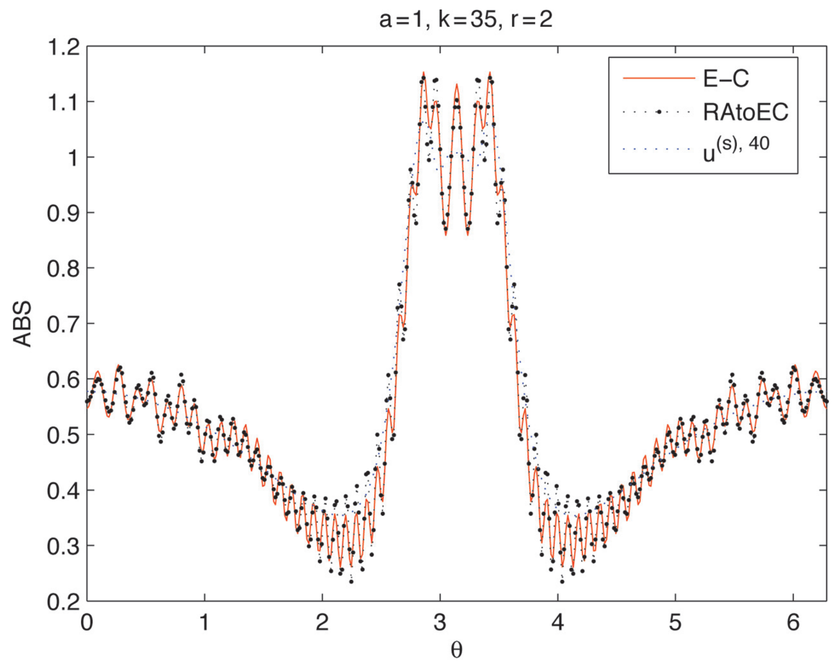


Fig.8 magnitude of the scattered wave of a circular cylinder for the incident field truncated to 30 terms



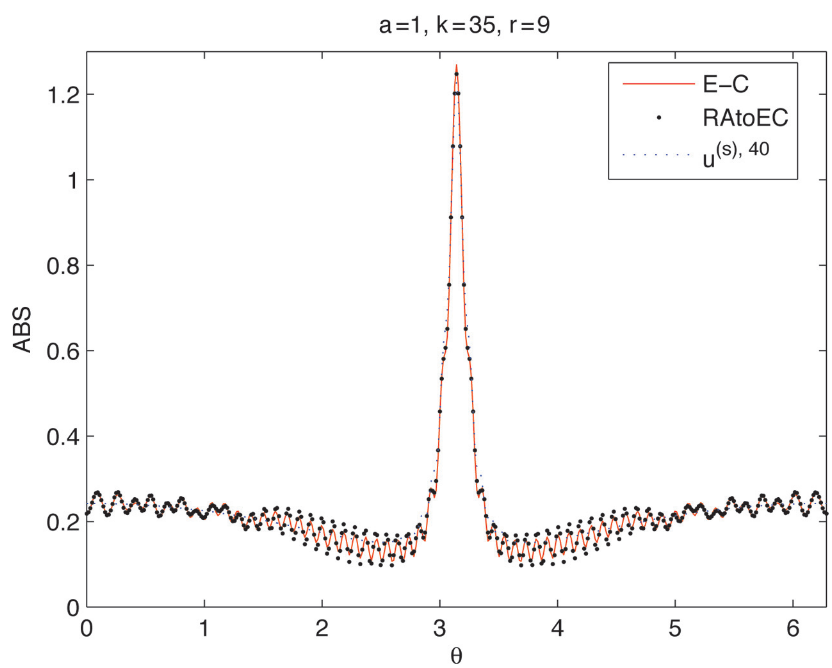


Fig.9 magnitude of the scattered wave of a circular cylinder for the incident field truncated to 40 terms

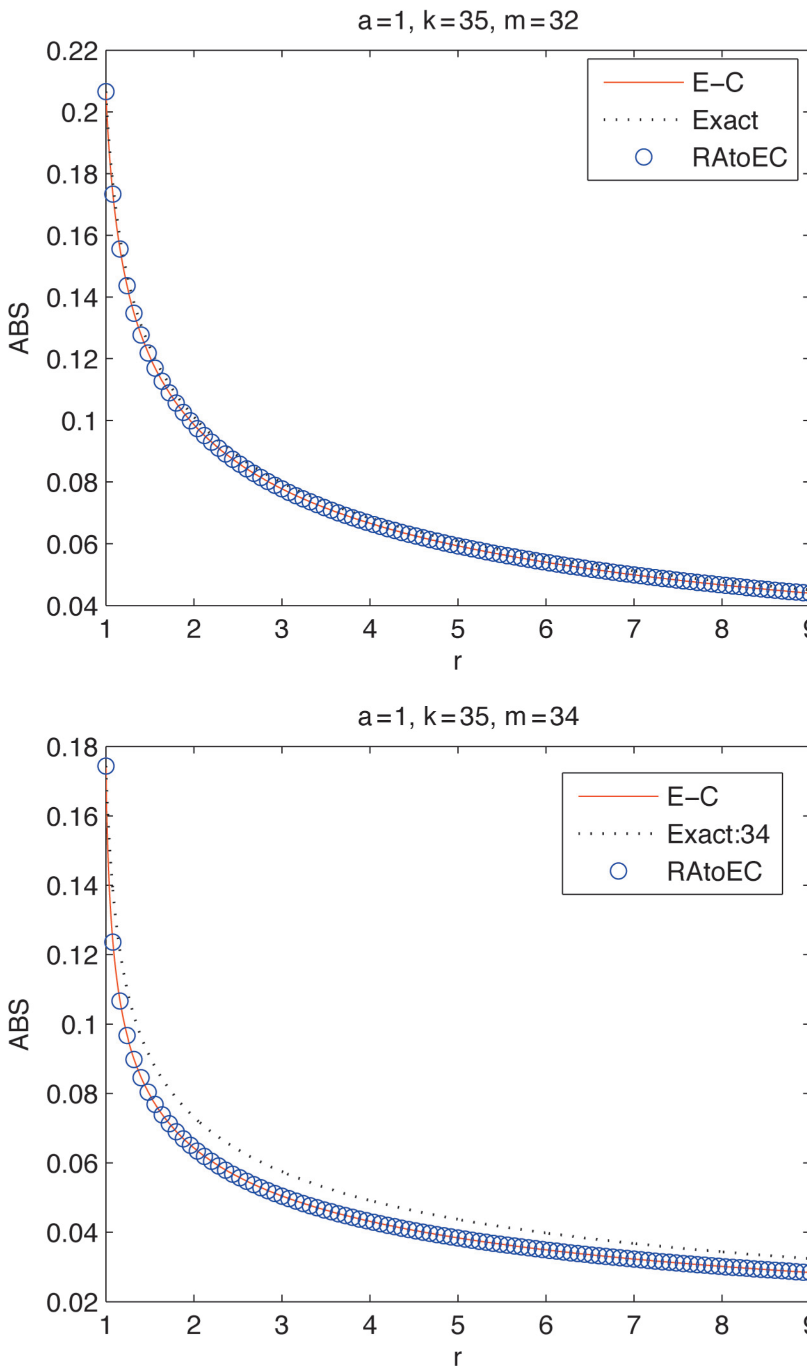


Fig.10 magnitude of the scattered waves $u_m^{(s)}$ (along a certain direction) for a circular cylinder with $m = 32$ and 34

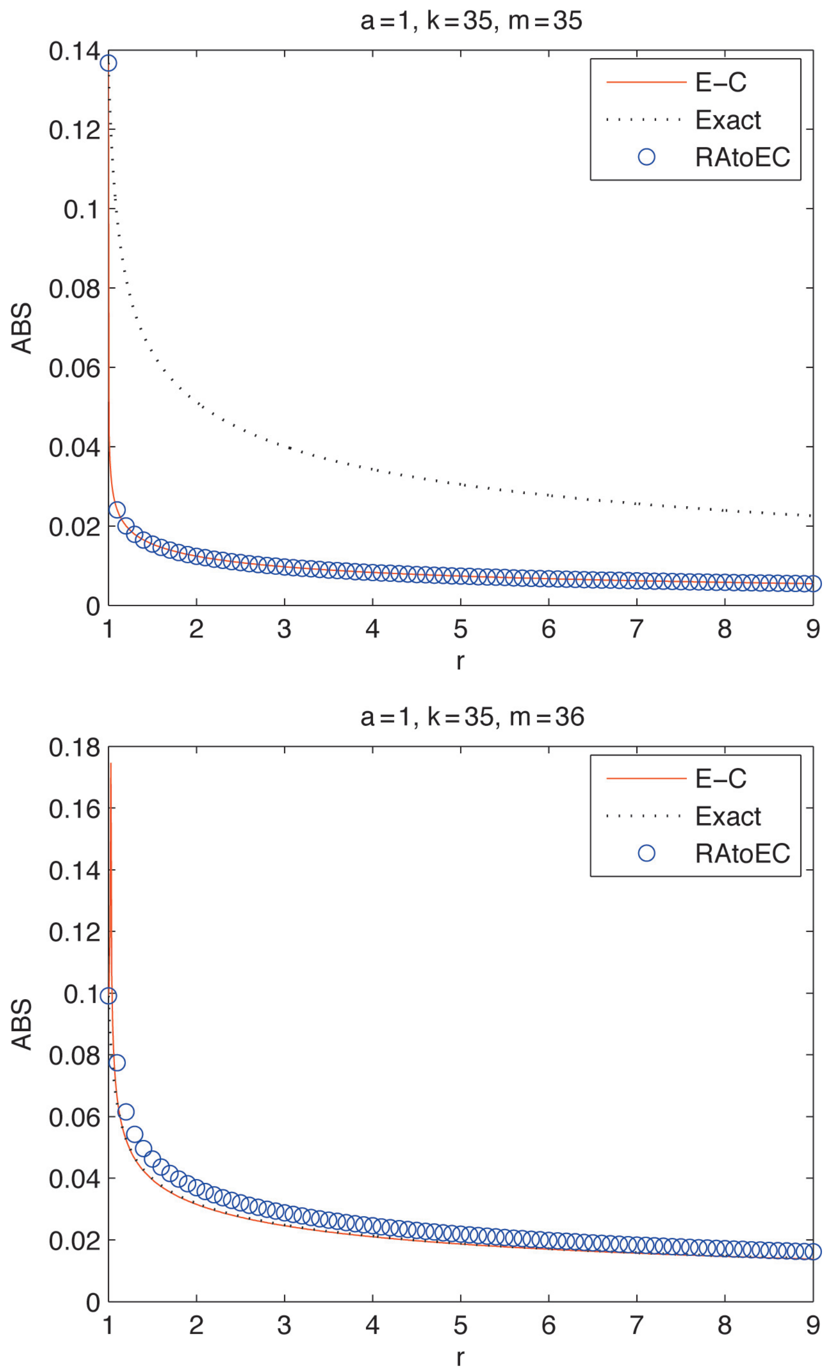


Fig.11 Magnitude of the scattered waves $u_m^{(s)}$ (along a certain direction) for a circular cylinder with $m = 35$ and 36

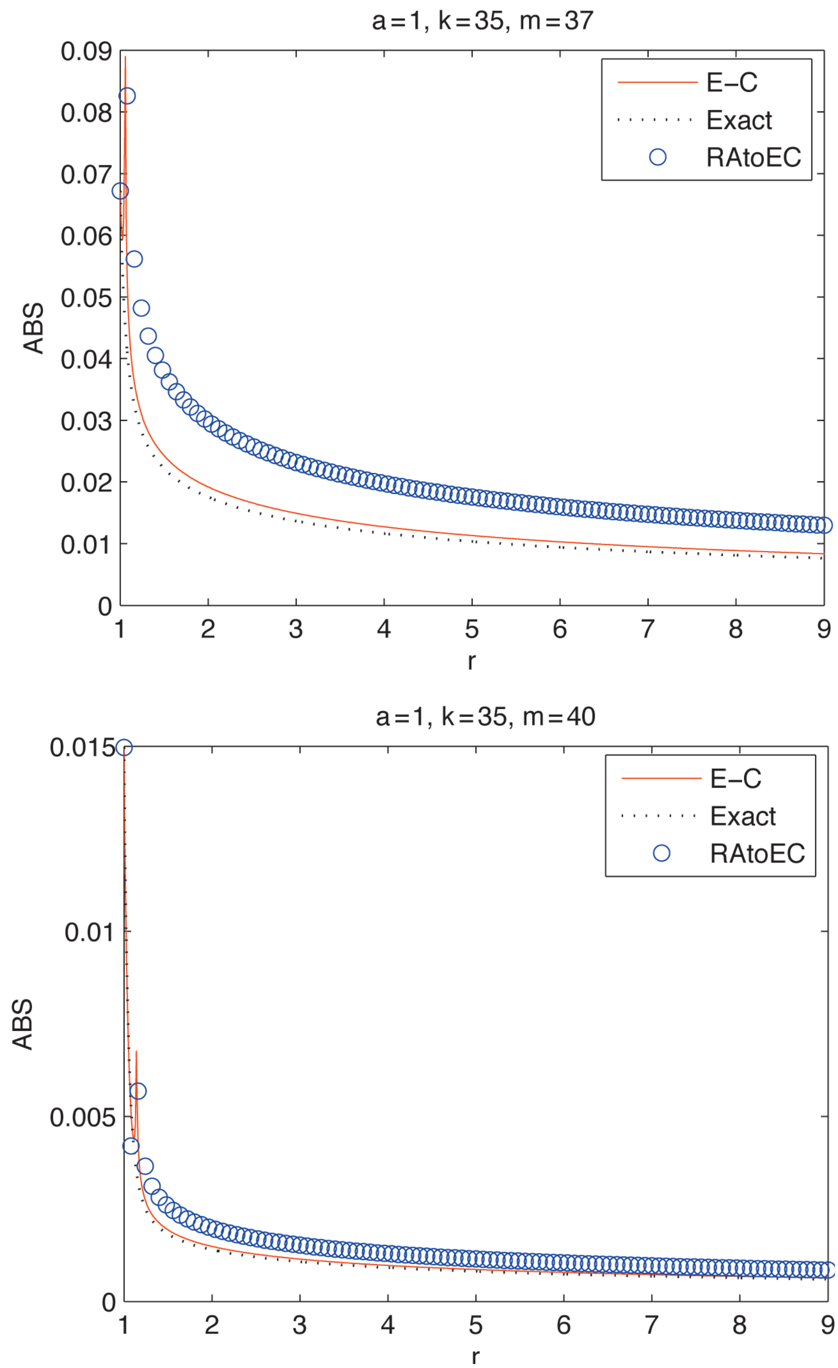


Fig.12 magnitude of the scattered waves $u_m^{(s)}$ (along a certain direction) for a circular cylinder with $m = 37$ and 40

from $u_m^{(s)}$ for $m = 34$ and $m = 35$, and the rational approximations used with the E-C model give poor results for $m \geq 36$. A possible explanation is that $u_m^{(s)}$ has a strong evanescent behavior near the scatterer and this is difficult to model by the E-C equation and rational approximations. In conclusion, the accuracy of E-C one-way model is limited, since it fails to approximate all the modes that are important in the scattered wave. Nevertheless, the method does give a rough approximation very efficiently.

5. CONCLUSION

In this paper, we develop and implement the Beam Propagation Method for 2-D scattering problems associated with acoustic sound-soft cylinders. This is an approximate method for scattering problems in the high frequency regime. The method is very efficient, since the required number of operations is $\mathcal{O}(1)$ for each point where a solution is calculated. However, the accuracy of this method is limited. The BPM approach cannot accurately compute the scattered wave from a circular cylinder. Nevertheless, the methods may be useful to large scale scattering problems for which a more accurate solution is difficult to get. Moreover, these new propagation models can be helpful in building efficiently an approximate exterior wave field used in the background of FEM for reducing pollution effects like e.g. in [16, 17]. In particular, access to high-order one-way exterior models would be valuable for accuracy improvement.

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