Asymptotic Solutions of Eigenmodes in Slab Waveguides Terminated by Perfectly Matched Layers

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For numerical modeling of optical wave-guiding structures, perfectly matched layers (PMLs) are widely used to terminate the transverse variables of the waveguide. The PML modes are the eigenmodes of a waveguide terminated by PMLs and they have found important applications in the mode matching method, the coupled mode theory, etc. In this paper, we consider PML modes for two-dimensional slab waveguides. It is shown that the PML modes consist of perturbed propagating modes, perturbed leaky modes and two infinite sequences of Berenger modes. High order asymptotic solutions for the Berenger modes are derived using a systematic approach.

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1. Introduction

For numerical simulation of waves, the perfectly matched layer (PML) [1] is a very powerful and extremely popular technique for truncating unbounded domains. While PML was originally introduced in the time domain, it is particularly easy to use in the frequency domain as a complex coordinate stretching [2]. Since typical optical waveguides are open structures with unbounded transverse directions, the PML is ideal for truncating the transverse variables. The PML technique was first applied to optical waveguides in a beam propagation method [3] and a leaky mode solver [4]. It also found an important application in the mode matching (or eigenmode expansion) method [5–8]. More recently, it has been applied in a new version of the coupled mode theory [9] and a scattering matrix formalism for modeling photonic integrated circuits [10].

The mode matching method [11–14] is widely used for numerical simulation of lightwaves propagating in optical wave-guiding structures. The standard mode matching method assumes that the structure can be divided into z-invariant segments, where z is a variable along the main propagation direction. In each segment, the wave field is decomposed as a sum of the forward and backward components and these components are expanded in the eigenfunctions of the local transverse operator. For open optical waveguides, the exact eigenfunction expansion involves an integral related to the continuous spectrum of radiation and evanescent modes, and it is difficult to handle numerically. It turns out that PML is an effective method for discretizing the continuous spectrum [15]. When the transverse variables are terminated by PMLs, the waveguide supports a discrete sequence of eigenmodes (which will be called PML modes). The field in the waveguide can then be expanded in the PML modes [5–7]. The completeness of the PML modes has been studied in [16]. Meanwhile, the PML modes gives an efficient series expansion for the Green's function [17]. The new coupled mode theory [9] and scattering matrix formalism [10] also rely on the PML modes.

The PML modes are defined as the eigenfunctions of the transverse operator modified by the PMLs. Although it is often necessary to solve the PML modes numerically, it is helpful to understand their analytic properties. In particular, it is useful to know how the eigenvalues of the PML modes are distributed in the complex plane. Rogier *et al.* derived leading order asymptotic solutions for the PML modes in optical fibers [18, 19] and in two-layer waveguides with one side bounded by a perfect electric (or magnetic) conductor [19,20]. They also classified the PML modes as finite number of perturbed propagating modes, an infinite sequence of perturbed leaky modes, and a remaining sequence of modes (the Berenger modes). However, these studies do not cover the important case of three-layer slab waveguides consisting of a core, a cladding and a substrate. In an early work [21], we derived perturbation results for the PML propagating modes of a slab waveguide. When a PML of finite thickness is used, the propagating modes are slightly modified. In particular, the propagation constants are complex in general, leading to unphysical growth or attenuation along the waveguide axis. Therefore, it is necessary to use PMLs carefully, if a long propagation distance is involved. In another work [22], we derived asymptotic solutions for the leaky modes of original slab waveguides (without PMLs).

In this paper, we develop an asymptotic theory for the PML modes of a three-layer slab waveguide, where PMLs are needed for both sides of the transverse direction. Similar to the cases studied in [18, 20], the PML modes consist of finite number of perturbed propagating modes, an infinite sequence of perturbed leaky modes, and infinite number of Berenger modes. However, the Berenger modes consist of two infinite sequences with different asymptotic phase angles (in general). It turns out that the PML leaky modes have the same asymptotic solutions as the original leaky modes [22]. Each Berenge mode sequence is asymptotically identical to the Berenger modes of a two-layer waveguide. The first two-layer waveguide has the core and the cladding of the original slab waveguide, and the second two-layer waveguide has the core and the substrate. In the following sections, we justify these claims and systematically derive high order asymptotic solutions for the Berenger modes.

2. Transverse electric modes

Consider a two-dimensional slab waveguide with its axis in the z direction and a refractive index profile given by

$$n(x) = \begin{cases} n_1, & x < b_1; \\ n_0, & b_1 < x < b_2; \\ n_2, & x > b_2, \end{cases} \tag{1}$$

where n_0, n_1 and n_2 are the refractive indices of the waveguide core, the substrate and the cladding, respectively. A schematic of the waveguide is shown in Fig. 1. We assume that

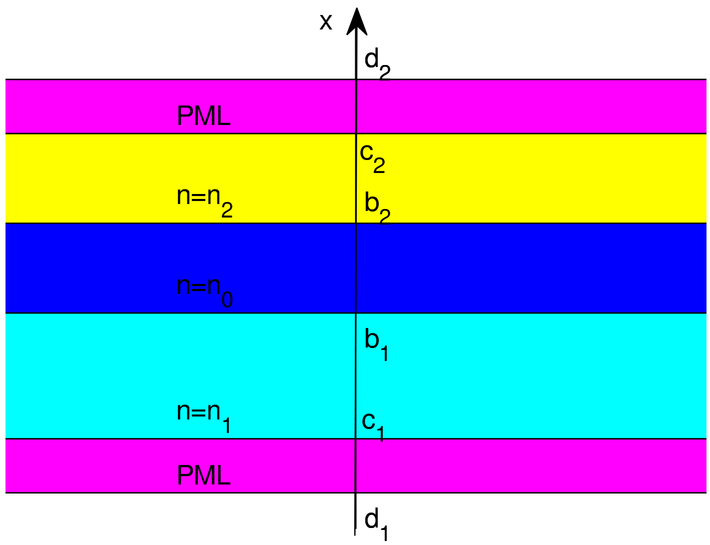

Fig. 1. A slab waveguide terminated by PMLs.

$b_1 \leq 0 \leq b_2$ and $n_0 > \max\{n_1, n_2\}$. The width of the waveguide core is $b_2 - b_1$. The PMLs are introduced by a complex coordinate stretching

$$\hat{x} = \int_0^x s(\tau)\, d\tau, \quad s(x) = 1 + i\sigma(x) \tag{2}$$

where σ (a dimensionless function) satisfies $\sigma(x) = 0$ for $c_1 \leq x \leq c_2$ and $\sigma(x) > 0$ otherwise, c_1 and c_2 satisfy the condition $c_1 \leq b_1 < b_2 \leq c_2$. The transverse variable x is terminated at $x = d_1$ and $x = d_2$, where $d_1 < c_1$ and $d_2 > c_2$. The actual PML layers correspond to the intervals (d_1, c_1) and (c_2, d_2) where $s(x) \neq 1$.

For a PML mode in the transverse electric (TE) polarization, the y component of the electric field is $E_y = \phi(x)e^{i\beta z}$, where ϕ (the mode profile) and β (the propagation constant) satisfy the following eigenvalue problem:

$$\frac{1}{s}\frac{d}{dx}\left(\frac{1}{s}\frac{d\phi}{dx}\right) + k_0^2 n^2 \phi = \beta^2 \phi, \;\; d_1 < x < d_2, \tag{3}$$

$$\phi(d_1) = \phi(d_2) = 0. \tag{4}$$

In the above, the time dependence is assumed to be $e^{-i\omega t}$ for an angular frequency ω, and k_0 is the free space wavenumber. A simple zero boundary condition is used at $x = d_1$ and $x = d_2$.

Let γ_0, γ_1 and γ_2 be given by

$$\gamma_j = \sqrt{k_0^2 n_j^2 - \beta^2}, \quad j = 0, 1, 2, \tag{5}$$

where the complex square root follows the standard definition, namely, if $a = |a|e^{i\theta}$ for $-\pi < \theta \leq \pi$, then $\sqrt{a} = \sqrt{|a|}e^{i\theta/2}$. For this choice, the negative real axis is the branch cut and the real part of $\sqrt{a}$ is always non-negative. Since the refractive index is piecewise constant, Eq. (3) is reduced to

$$\frac{d^2\phi}{d\hat{x}^2} + k_0^2 n_j^2 \phi = \beta^2 \phi \tag{6}$$

for $b_1 < x < b_2$ ($j = 0$), $d_1 < x < b_1$ ($j = 1$), and $b_2 < x < d_2$ ($j = 2$), respectively. Using the analytic solutions of these equations and matching ϕ and $d\phi/dx$ at the interfaces, we arrive at the following nonlinear equation for β^2:

$$\frac{\gamma_0 - i\gamma_1 \cot(\rho_1\gamma_1)}{\gamma_0 + i\gamma_1 \cot(\rho_1\gamma_1)} \cdot \frac{\gamma_0 - i\gamma_2 \cot(\rho_2\gamma_2)}{\gamma_0 + i\gamma_2 \cot(\rho_2\gamma_2)} = e^{2i(b_1 - b_2)\gamma_0}, \tag{7}$$

where

$$\hat{d}_j = \hat{x}(d_j) = d_j + i\int_{c_j}^{d_j} \sigma(\tau)\, d\tau, \tag{8}$$

$$\rho_1 = b_1 - \hat{d}_1, \quad \rho_2 = \hat{d}_2 - b_2. \tag{9}$$

Notice that both the real and the imaginary parts of ρ_1 and ρ_2 are positive. Therefore,

$$\rho_1 = |\rho_1|e^{i\varphi_1}, \quad \rho_2 = |\rho_2|e^{i\varphi_2}, \tag{10}$$

where φ_1, $\varphi_2 \in (0, \pi/2)$.

For a symmetric slab waveguide ($n_1 = n_2$), if we use identical PML profiles in both positive and negative x directions, Eq. (7) can be simplified. More precisely, if the slab waveguide is symmetric, we can assume $b_1 = -b_2$, $c_1 = -c_2$, $d_1 = -d_2$ and $\sigma(-x) = \sigma(x)$, then $\gamma_1 = \gamma_2$, $\hat{d}_1 = -\hat{d}_2$ and Eq. (7) is reduced to

$$\frac{\gamma_0 - i\gamma_1 \cot(\rho_1\gamma_1)}{\gamma_0 + i\gamma_1 \cot(\rho_1\gamma_1)} = \pm e^{i(b_1 - b_2)\gamma_0}. \tag{11}$$

Our objective is to find asymptotic solutions of Eq. (7) or Eq. (11) assuming that $|\beta|$ is large.

To derive asymptotic solutions, we consider a sequence of β with a convergent phase angle. More precisely, let $\{\beta_m : m = 0, 1, 2, \cdots\}$ be a sequence of solutions of Eq. (7) and $\beta_m^2 = |\beta_m^2| e^{i\theta_m}$ where θ_m is the phase angle of β_m^2, then we require

$$\lim_{m\to\infty} |\beta_m| = \infty \quad \text{and} \lim_{m\to\infty} \theta_m = \theta_*.$$

We further assume that β_m^2 is in the upper half complex plane, i.e., $0 < \theta_m < \pi$, then $k_0^2 n_j^2 - \beta_m^2$ (for $0 \le j \le 2$) is in the lower half plane, and γ_j given in (5) is in the fourth quadrant. Therefore, if $|\beta_m|$ is large, we have

$$\gamma_j = |\beta_m| e^{i(\theta_m - \pi)/2} \sqrt{1 - k_0^2 n_j^2 / \beta_m^2}. \tag{12}$$

This leads to

$$\rho_j \gamma_j = |\rho_j \beta_m| e^{i(\theta_m/2 + \varphi_j - \pi/2)} \sqrt{1 - k_0^2 n_j^2 / \beta_m^2} \tag{13}$$

for $j = 1, 2$. By considering different values of θ_*, we can simplify Eq. (7) and find the asymptotic solutions.

The first sequence corresponds to $\theta_* = \pi$. In that case, $\mathrm{Im}(\rho_j\gamma_j) \to +\infty$, thus $\cot(\rho_j\gamma_j) \to -i$ exponentially. As a result, Eq. (7) is simplified to

$$\frac{\gamma_0 - \gamma_1}{\gamma_0 + \gamma_1} \cdot \frac{\gamma_0 - \gamma_2}{\gamma_0 + \gamma_2} \approx e^{2i(b_1 - b_2)\gamma_0}. \tag{14}$$

It turns out that Eq. (14) is exactly the same transcendental equation for the leaky modes of the slab waveguide. The asymptotic solutions of the leaky modes have been derived in our previous work [22].

Two more sequences can be found for $0 < \theta_* < \pi$. From (12), it is clear that $\mathrm{Im}(\gamma_0) \to -\infty$, $\mathrm{Re}[2i(b_1 - b_2)\gamma_0] \to -\infty$, thus $e^{2i(b_1 - b_2)\gamma_0} \to 0$ exponentially. If $\theta_* \ne \pi - 2\varphi_j$ for $j = 1$ or 2, then $\mathrm{Im}(\rho_j\gamma_j) \to \pm\infty$, $\cot(\rho_j\gamma_j) \to \mp i$ exponentially, so the left hand side of (7) can be simplified, but it does not converge to 0 exponentially. On the other hand, if $\theta_* = \pi - 2\varphi_j$ for $j = 1$ or $j = 2$, Eq. (7) is approximated by

$$\gamma_0 - i\gamma_j \cot(\rho_j \gamma_j) \approx 0. \tag{15}$$

This is the same as

$$\frac{\gamma_0 - \gamma_j}{\gamma_0 + \gamma_j} \approx e^{2i\rho_j\gamma_j}. \tag{16}$$

The above equation gives rise to one sequence of Berenger modes for each j. Notice that Eq. (16) involves only two layers: the waveguide core and the substrate ($j = 1$), or the waveguide core and the cladding ($j = 2$). In fact, if we consider a two-layer waveguide consisting the core and the cladding (for $x > b_1$), use a perfectly electric conductor boundary at $x = b_1$, i.e., $\phi(b_1) = 0$, we get the following transcendental equation for the TE modes

$$\frac{\gamma_0 - i\gamma_2 \cot(\rho_2\gamma_2)}{\gamma_0 + i\gamma_2 \cot(\rho_2\gamma_2)} = -e^{2i(b_1-b_2)\gamma_0}. \tag{17}$$

For a sequence with the a converging phase angle $\theta_* \in (0, \pi)$, Eq. (17) also gives rise to Eq. (16) for $j = 2$.

Leading order asymptotic solutions for two-layer waveguides have been derived before [20]. Higher order asymptotic solutions of Eq. (16) can be derived using a systematic approach developed in our previous work [22]. They are most conveniently given in terms of γ_j. The propagation constant β can be evaluated from $\beta^2 = k_0^2 n_j^2 - \gamma_j^2$. Furthermore, these solutions are related to the Lambert W functions [23]. For a complex number ξ, the Lambert W function $W(\xi)$ is a multi-valued function satisfying $W(\xi)e^{W(\xi)} = \xi$. For an integer p, the pth branch of the Lambert W function is denoted as LambertW(p, ξ). Let

$$\delta_j = k_0^2(n_0^2 - n_j^2), \quad W_j = \text{LambertW}(p, \pm\frac{i\rho_j}{2}\sqrt{\delta_j}),$$

then an asymptotic solution of Eq. (16) is

$$\gamma_j \approx \frac{W_j}{A_0} - \frac{A_0 A_2}{W_j^2} - \frac{A_0^2 A_3}{W_j^3} - \frac{A_0^3 A_4}{W_j^4}, \tag{18}$$

where

$$A_0 = i\rho_j, \quad A_2 = \frac{\delta_j}{4}, \quad A_3 = -\frac{\delta_j}{4A_0},$$

$$A_4 = \frac{\delta_j}{4A_0^2} - \frac{\delta_j^2}{16}.$$

The definition of W_j above involves an integer p (branch index) and a plus or minus sign. To be consistent with our assumption $0 < \theta_* < \pi$, we require that $\text{Im}(W_j) > 0$. From the properties of the Lambert W functions, this requirement leads to the following choices: we take the negative sign for $p = 0$ and both signs for $p > 0$. For a symmetric slab waveguide, if the PML is also placed symmetrically, these two sequences are identical, thus the Berenger modes are asymptotically degenerate.

Finally, we note that no asymptotic solutions exist for $-\pi < \theta_* \leq 0$. The case $\theta_* = 0$ may be considered under the assumption $0 < \theta_m < \pi$, but the left and right sides of (7) cannot balance. On the other hand, it can be shown that there is no solution sequence with $-\pi < \theta_m \leq 0$.

3. Transverse magnetic case

For a transverse magnetic (TM) mode, the y-component of the magnetic field H_y can be written as $\phi(x)e^{i\beta z}$, where the mode profile ϕ and the propagation constant β satisfy

$$\frac{n^2}{s}\frac{d}{dx}\left(\frac{1}{sn^2}\frac{d\phi}{dx}\right) + k_0^2 n^2 \phi = \beta^2 \phi, \quad d_1 < x < d_2 \tag{19}$$

$$\phi(d_1) = \phi(d_2) = 0. \tag{20}$$

As before, we assume that the PMLs are terminated by a simple zero boundary condition at $x = d_1$ and d_2. Similar to the TE case, it is easily shown that the propagation constant β satisfies the following nonlinear equation:

$$\frac{\mu_0 - i\mu_1 \cot(\rho_1\gamma_1)}{\mu_0 + i\mu_1 \cot((\rho_1\gamma_1)} \cdot \frac{\mu_0 - i\mu_2 \cot(\rho_2\gamma_2)}{\mu_0 + i\mu_2 \cot(\rho_2\gamma_2)} = e^{2i(b_1-b_2)\gamma_0}, \tag{21}$$

where $\mu_j = \gamma_j/n_j^2$ for $j = 0, 1, 2$. As in the previous section, we consider a sequence of solutions, $\beta_m^2 = |\beta_m|^2 e^{i\theta_m}$ for $m = 0, 1, 2, ...,$ such that $|\beta_m| \to \infty$ and $\theta_m \to \theta_*$ as $m \to \infty$. For different values of θ_*, we simplify Eq. (21) and find the asymptotic solutions.

The first case is $\theta_* = \pi$. After removing some exponentially small terms, Eq. (21) is reduced to

$$\frac{\mu_0 - \mu_1}{\mu_0 + \mu_1} \cdot \frac{\mu_0 - \mu_2}{\mu_0 + \mu_2} \approx e^{2i(b_1-b_2)\gamma_0}. \tag{22}$$

This is the transcendental equation of the TM leaky modes and its asymptotic solutions are already obtained [22]. Therefore, the first sequence of PML modes is asymptotically identical to the leaky modes.

For $0 < \theta_* < \pi$, we can find two sequences, for which Eq. (21) is reduced to

$$\mu_0 - i\mu_j \cot(\rho_j\gamma_j) \approx 0, \quad j = 1 \text{ or } 2. \tag{23}$$

This leads to

$$\frac{\mu_0 - \mu_j}{\mu_0 + \mu_j} \approx e^{2i\rho_j\gamma_j}, \quad j = 1 \text{ or } 2. \tag{24}$$

For the above equation, we take the logarithm for both sides, expand the right hand side in inverse powers of γ_j, and get

$$2i\rho_j\gamma_j = B_0 - \frac{B_1}{\gamma_j^2} + \frac{B_2}{\gamma_j^4} + ... \tag{25}$$

where

$$\begin{aligned} B_0 &= \ln\left(\frac{n_0^2 - n_j^2}{n_0^2 + n_j^2}\right) + (2m+1)\pi \mathrm{i}, \quad m \ge 0, \\ B_1 &= \frac{k_0^2 n_0^2 n_j^2}{n_0^2 + n_j^2}, \\ B_2 &= \frac{k_0^4 n_0^2 n_j^2 (n_j^4 - 3n_0^4)}{4(n_0^2 + n_j^2)^2}. \end{aligned}$$

If we solve Eq. (25) iteratively, we obtain the following approximate formulas:

$$\gamma_j \approx \gamma_j^{(0)} = \frac{B_0}{2i\rho_j}, \tag{26}$$

$$\gamma_j \approx \gamma_j^{(1)} = \frac{B_0}{2i\rho_j} - \frac{2i\rho_j B_1}{B_0^2}, \tag{27}$$

$$\gamma_j \approx \gamma_j^{(2)} = \frac{1}{2i\rho_j}\left[B_0 - \frac{B_1}{(\gamma_j^{(1)})^2} + \frac{B_2}{(\gamma_j^{(1)})^4}\right]. \tag{28}$$

4. Examples

In this section, we check the accuracy of our asymptotic solutions for two examples. The first example is an unsymmetric slab waveguide. The refractive indices of the core, the substrate and the cladding are $n_0 = 3.3$, $n_1 = 3.17$ and $n_2 = 1$, respectively. The width of the core is $0.8\,\mu$m. The PMLs in the substrate and the cladding are placed at $0.8\,\mu$m and $0.4\,\mu$m from the boundaries of the core respectively, and their widths are both $0.1\,\mu$m. The geometric parameters are $d_1 = -0.9\,\mu$m, $c_1 = -0.8\,\mu$m, $b_1 = 0\,\mu$m, $b_2 = 0.8\,\mu$m, $c_2 = 1.2\,\mu$m, and $d_2 = 1.3\,\mu$m. The PML profile is given by

$$\sigma(x) = \frac{C_j\eta^3}{1+\eta^2}, \quad \eta = \frac{x - c_j}{d_j - c_j}, \quad j = 1, 2, \tag{29}$$

for x between c_j and d_j. The coefficients of $\sigma(x)$ are given by $C_1 = C_2 = 16$. The second example is a symmetric slab waveguide. The refractive indices of the core and the cladding are $n_0 = 3.4$ and $n_1 = n_2 = 1$, respectively. The widths of the core and the PMLs are $0.6\,\mu$m and $0.1\,\mu$m, respectively. The distance between the PMLs and the core is $0.4\,\mu$m. The geometric parameters are $b_2 = -b_1 = 0.3\,\mu$m, $c_2 = -c_1 = 0.7\,\mu$m, and $d_2 = -d_1 = 0.8\,\mu$m. The coefficients of $\sigma(x)$ are $C_1 = C_2 = 16$. For both examples, we assume the free space wavelength is $\lambda = 1.55\,\mu$m.

In Fig. 2, we compare the exact and approximate propagation constants of the TE Berenger modes for the two example. The approximate solutions are calculated by formula (18) and the exact solutions are obtained by solving (7) or (11). For the first example, two sequences corresponding to $j = 1$ and $j = 2$ can be identified in Fig. 2 (left). For the second example, the asymptotic solutions are doubly degenerate, but the exact solutions are not. Each asymptotic solution corresponds to two slightly different Berenger modes. To show the accuracy of the asymptotic solutions more clearly, we list some exact solutions and the relative errors of the asymptotic solutions in Table 1 and Table 2. For the second example, the two exact propagation constants corresponding to the same asymptotic solution are grouped together. For the TM case, we compare the exact and approximate propagation constants in Fig. 3 and list the exact propagation constants and relative errors of the asymptotic solutions of some Berenger modes in Tables 3 and 4. Similar to the TE case, we list the two exact values corre-

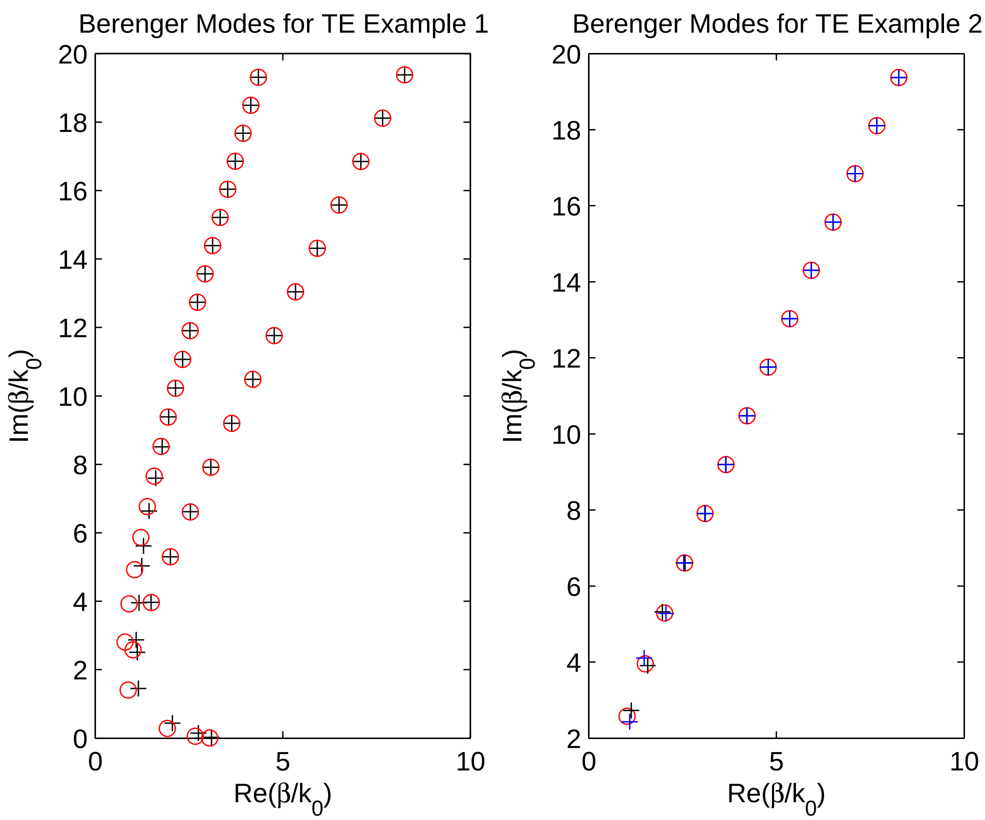

Fig. 2. Comparison of the exact (marked by "+") and approximate (marked by "o") propagation constants of the TE Berenger modes.

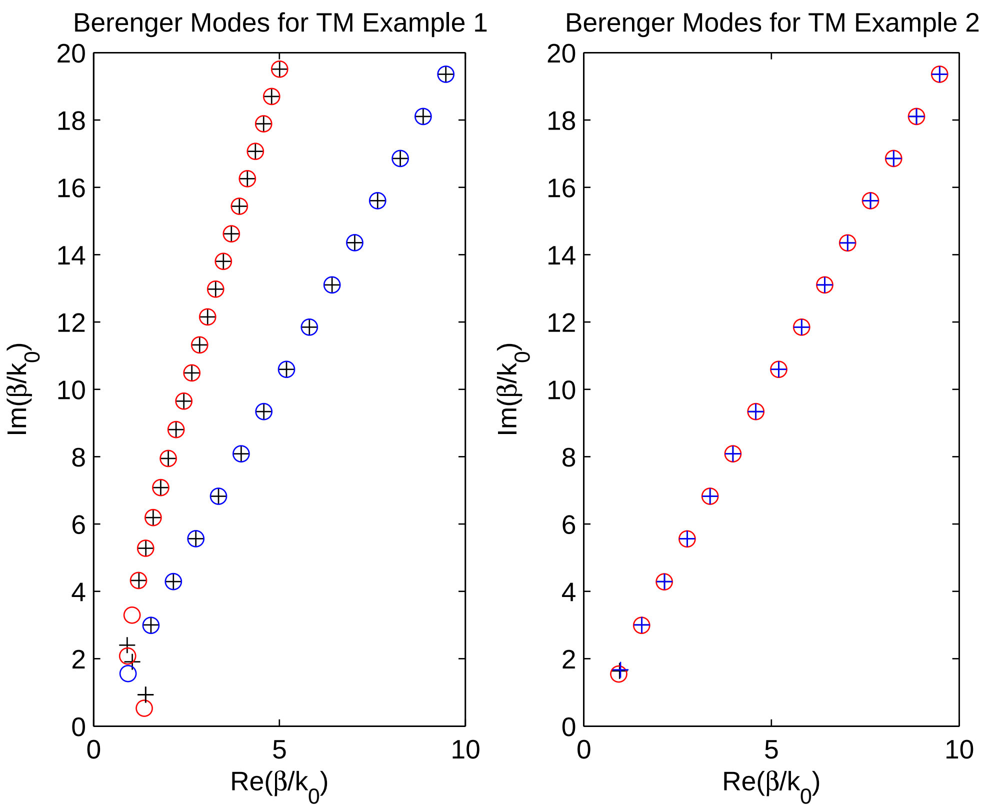

Fig. 3. Comparison of the exact (marked by "+") and approximate (marked by "o") propagation constants of the TM Berenger modes.

Sequence	Exact β_m/k_0	R.E. of (18)
1	1.43929148+ 6.63289452i	0.21×10^{-1}
1	1.61322522+ 7.59678078i	0.89×10^{-2}
1	1.78855145+ 8.51026440i	0.37×10^{-2}
1	1.95999515+ 9.38560677i	0.16×10^{-2}
1	2.13878608+10.23243809i	0.63×10^{-3}
1	2.33151054+11.06866741i	0.21×10^{-3}
1	2.52955914+11.90377903i	0.70×10^{-4}
1	2.72842216+12.73636055i	0.22×10^{-4}
1	2.92805862+13.56549835i	0.71×10^{-5}
2	1.12165964+ 2.50906348i	0.47×10^{-1}
2	1.49026048+ 3.96589470i	0.12×10^{-2}
2	2.00763631+ 5.30041599i	0.16×10^{-3}
2	2.54001164+ 6.61296082i	0.62×10^{-4}
2	3.08547425+ 7.91153908i	0.27×10^{-4}
2	3.64048886+ 9.20100880i	0.13×10^{-4}
2	4.20275879+10.48410130i	0.69×10^{-5}

Table 1. Example 1: exact propagation constants of the TE Berenger modes and relative errors of formula (18).

sponding to the same asymptotic solution together for the second example. However, each of the last four rows of Table 4 represents the two exact solutions which are not distinguishable for the digits listed.

5. Conclusion

In the previous sections, we analyzed the PML modes of two-dimensional slab waveguides. The PML modes can be classified as the PML propagating modes, the PML leaky modes, and the Berenger modes. The PML propagating modes are perturbations of the true propagating modes. The asymptotic solutions of the PML modes are derived by considering a sequence of propagation constants β_m satisfying $|\beta_m| \to \infty$ and $\theta_m \to \theta_*$ where θ_m is the phase angle of β_m^2 (or $\theta_m/2$ is the phase angle of β_m). The PML leaky modes are asymptotically identical to the true leaky modes, and they correspond to $\theta_* = \pi$. High order asymptotic solutions for the leaky modes of slab waveguides are available in our previous work [22]. For the Berenger modes, we find two sequences corresponding to $\theta_* = \pi - 2\varphi_j$ for $j = 1$ and 2,

Exact β/k_0	R.E. of (18)
1.96391106+ 5.32063246i	0.11×10^{-1}
2.05728030+ 5.28033113i	0.71×10^{-2}
2.53330898+ 6.60819817i	0.27×10^{-2}
2.56865978+ 6.60561514i	0.23×10^{-2}
3.09196258+ 7.90452147i	0.71×10^{-3}
3.10386252+ 7.90639448i	0.71×10^{-3}
3.65136361+ 9.19430542i	0.19×10^{-3}
3.65505052+ 9.19561386i	0.21×10^{-3}
4.21504927+10.47774410i	0.49×10^{-4}
4.21612167+10.47832680i	0.60×10^{-4}
4.78345537+11.75629902i	0.11×10^{-4}
4.78375196+11.75651836i	0.18×10^{-4}
5.35612345+13.03115463i	0.20×10^{-5}
5.35620165+13.03122997i	0.63×10^{-5}

Table 2. Example 2: exact propagation constants of the TE Berenger modes and relative errors of formula (18).

where φ_j is related to the location and profile of the PML in the substrate or the cladding. It is interesting to note that these two sequences are asymptotically identical to the Berenger modes in two-layer waveguides consisting of only the core and the substrate, or only the core and the cladding. For these two sequences, we derived high order asymptotic solutions based on a systematic approach.

We study the analytic properties of the PML modes because they have important applications as described in [5–10, 17]. Our results should also be useful for numerical implementation of the mode matching method. For two-dimensional waveguides with a piecewise constant refractive index, it is possible to avoid discretizing the transverse variable, write down the eigenfunctions analytically, and solve the eigenvalues from a transcendental equation. A number of numerical methods have been developed to solve the eigenvalues from this transcendental equation [24–26], but our asymptotic solutions make the task much easier, since they provide excellent initial guesses. With these initial guesses, the exact eigenvalues can be easily found by solving the nonlinear equation with Newton's method or any other nonlinear equation solver.

Sequence	Exact β/k_0	R.E. of (28)
1	1.22010469+ 4.32303238i	0.23×10^{-2}
1	1.40473378+ 5.28413486i	0.46×10^{-3}
1	1.60215116+ 6.19549084i	0.95×10^{-4}
1	1.80584180+ 7.08042812i	0.15×10^{-4}
1	2.01256752+ 7.94794035i	0.86×10^{-5}
2	1.54685795+ 3.00455475i	0.39×10^{-2}
2	2.14756656+ 4.29343843i	0.73×10^{-3}
2	2.75336146+ 5.56421614i	0.21×10^{-3}
2	3.36158633+ 6.82692719i	0.76×10^{-4}
2	3.97119999+ 8.08533478i	0.33×10^{-4}
2	4.58168840+ 9.34117889i	0.16×10^{-4}
2	5.19276524+10.59537601i	0.87×10^{-5}

Table 3. Example 1: Exact propagation constants and relative errors of formula (28) for TM Berenger modes.

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References

1. J. P. Berenger, "A perfectly matched layer for the absorption of electromagnetic-waves," J. Comput. Phys. **114**(2), 185-200 (1994).
2. W. C. Chew and W. H. Weedon, "A 3D perfectly matched medium from modified Maxwells equations with stretched coordinates," Microwave and Optical Technology Letters **7**(13), 599-604 (1994).
3. W. P. Huang, C. L. Xu, W. Lui, and K. Yokoyama, "The perfectly matched layer (PML) boundary condition for the beam propagation method," IEEE Photon. Technol. Lett. **8**(5), 649-651 (1996).
4. W. P. Huang, C. L. Xu, W. Lui, and K. Yokoyama, "The perfectly matched layer boundary condition for modal analysis of optical waveguides: leaky mode calculations," IEEE Photon. Technol. Lett. **8**(5), 652-654 (1996).

Exact β/k_0	R.E. of (28)
0.95506150+ 1.64228368i	0.50×10^{-1}
0.97763562+ 1.67133546i	0.67×10^{-1}
1.54965425+ 3.00401425i	0.44×10^{-2}
1.54854381+ 3.00354362i	0.41×10^{-2}
2.14972377+ 4.29245186i	0.79×10^{-3}
2.14974236+ 4.29241705i	0.78×10^{-3}
2.75550723+ 5.56318760i	0.22×10^{-3}
2.75550866+ 5.56318760i	0.22×10^{-3}
3.36372614+ 6.82588911i	0.82×10^{-4}
3.97333694+ 8.08429241i	0.36×10^{-4}
4.58382416+ 9.34013462i	0.17×10^{-4}
5.19490004+10.59433079i	0.93×10^{-5}

Table 4. Example 2: Exact propagation constants and relative errors of formula (28) for TM Berenger modes.

5. H. Derudder, D. De Zutter, and F. Olyslager, "Analysis of waveguide discontinuities using perfectly matched layers," Electronics Letters **34**(22), 2138-2140 (1998).
6. P. Bienstman and R. Baets, "Optical modelling of photonic crystals and VCSELs using eigenmode expansion and perfectly matched layers," Opt. Quant. Electron. **33**(4-5), 327-341 (2001).
7. P. Bienstman, H. Derudder, R. Baets, F. Olyslager, and D. De Zutter, "Analysis of cylindrical waveguide discontinuities using vectorial eigenmodes and perfectly matched layers," IEEE Trans. Microwave Theory Tech. **49**(2), 349-354 (2001).
8. J. Mu and W. P. Huang, "Simulation of three-dimensional waveguide discontinuities by a full-vector mode-matching method based on finite-difference schemes," Opt. Express **16**, 18152-18163 (2008).
9. W. P. Huang and J. Mu, "Complex coupled-mode theory for optical waveguides," Opt. Express **17**(21), 19134-19152 (2009).
10. W. P. Huang, L. Han, and J. Mu, "A rigorous circuit model for simulation of large-scale photonic integrated circuits," IEEE Photonics Journal **4**(5), 1622–1638 (2012).
11. T. E. Rozzi, "Rigorous analysis of step discontinuity in a planar dielectric waveguide," IEEE Trans. Microwave Theory Tech. **26**(10), 738-746 (1978).
12. R. Mittra, Y. L. Hou, and V. Jamnejad, "Analysis of open dielectric wave-guides using

mode-matching technique and variational-methods," IEEE Trans. Microwave Theory Tech. **28**(1), 36-43 (1980).

13. G. Sztefka and H. P. Nolting, "Bidirectional eigenmode propagation for large refractive-index steps," IEEE Photon. Technol. Lett. **5**(5), 554-557 (1993).
14. J. Willems, J. Haes, and R. Baets, "The bidirectional mode expansion method for 2-dimensional wave-guides - the TM case," Opt. Quant. Electron. **27**(10), 995-1007 (1995).
15. F. Olyslager, "Discretization of continuous spectra based on perfectly matched layers," SIAM Journal on Applied Mathematics, **64**(4), 1408-1433 (2004).
16. L. F. Knockaert and D. De Zutter, "On the completeness of eigenmodes in a parallel plate waveguide with a perfectly matched layer termination," IEEE Transactions on Antennas and Propagation **50**(11), 1650-1653 (2002).
17. H. Derudder, F. Olyslager, and D. De Zutter, "An efficient series expansion for the 2-D Green's function of a microstrip substrate using perfectly matched layers," IEEE Microwave and Guided Wave Letters **9**(12), 505-507 (1999).
18. H. Rogier and D. De Zutter, "Berenger and leaky modes in optical fibers terminated with a perfectly matched layer," J. Lightw. Technol. **20**(7), 1141-1148 (2002).
19. H. Rogier, L. Knockaert, and D. De Zutter, "Fast calculation of the propagation constants of leaky and Berenger modes of planar and circular dielectric waveguides terminated by a perfectly matched layer," Microwave and Optical Technology Letters **37**(3), 167-171 (2003).
20. H. Rogier and D. De Zutter, "Berenger and leaky modes in microstrip substrates terminated by a perfectly matched layer," IEEE Trans. Microwave Theory Tech. **49**(4), 712-715 (2001).
21. Y. Y. Lu and J. Zhu, "Propagating modes in optical waveguides terminated by perfectly matched layers," IEEE Photon. Technol. Lett. **17**, 2601-2603 (2005).
22. J. Zhu and Y. Y. Lu, "Leaky modes of slab waveguides – asymptotic solutions," J. Lightw. Technol. **24**(3), 1619-1623 (2006).
23. R. M. Corless, G. H. Gonnet, D.E.G. Hare, D. J. Jeffrey, and D. E. Knuth, "On the Lambert W function," Advances in Computational Mathematics **5**(4), 329-359 (1996).
24. R. Z. L. Ye and D. Yevick, "Noniterative calculation of complex propagation constants in planar waveguides," J. Opt. Soc. Am. A **18**(11), 2819-2822 (2001).
25. S. B. Gaal, H. J. W. M. Hoekstra, and P. V. Lambeck, "Determining PML modes in 2-D stratified media," J. Lightw. Technol. **21**(1), 293-298 (2003).
26. L. Knockaert, H. Rogier, and D. De Zutter, "An FFT-based signal identification approach for obtaining the propagation constants of the leaky modes in layered media," AEU-International Journal of Electronics and Communications **59**(4), 230-238 (2005).