Leaky Modes of Slab Waveguides — Asymptotic Solutions

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***Abstract*— Approximate analytic solutions of the leaky modes in two-dimensional slab waveguides are derived through an asymptotic analysis. General slab waveguides with three layers of possibly different refractive indices are studied for both the transverse electric and transverse magnetic cases. Our results are useful in the eigenmode expansion method where the leaky modes appear if a perfect matched layer (PML) is used to terminate the transverse directions of optical waveguides.**

***Index Terms*— Slab waveguides, leaky modes, asymptotic solutions, perfeclty matched layers.**

I. Introduction

FOR open waveguides, the leaky modes are useful because they can be used to partially represent the wave field related to the continuous spectrum of the radiation and evanescent modes [1], [2], [3]. Leaky modes also appear [4] when an open waveguide is terminated by a perfectly matched layer (PML) [5], [6]. With a PML, the waveguide supports a discrete sequence of modes [4], [7], [8] which can be identified as perturbations [9] of the original propagating and leaky modes and the new Berenger modes [10]. To implement the eigenmode expansion method [11], [12] with a PML [13], [14], it is necessary to calculate the PML modes. For two-dimensional step-index waveguides, the propagation constants satisfy a nonlinear equation. Some numerical methods have been proposed [15], [16], [17] to solve this equation. Since the propagation constants of the PML modes are in general complex, it is not easy to find all PML modes in a given region of the complex plane. However, if an initial guess for a propagation constant is available, we can easily obtain an accurate solution by using a nonlinear equation solver, such as Newton's method. Therefore, it is desirable to find approximate analytic solutions for the PML modes.

Rogier and De Zutter [4] found some approximate solutions of the PML modes for two-layer waveguides that are open in one side of the transverse direction. A PML is used to terminate the open transverse side of the waveguide. In this paper, we consider three-layer slab waveguides where the transverse variable is unbounded for both positive and negative directions. We find approximate solutions of the leaky modes for the original open waveguide. These are asymptotic solutions and they are very accurate for higher modes. It turns out that the PML leaky modes and the original leaky modes have identical asymptotic behaviors. Therefore, our approximate solutions can be used as initial guesses for computing the PML leaky modes by an iterative method, such as Newton's method. Asymptotic results for the transverse electric (TE) and transverse magnetic (TM) leaky modes of a three-layer slab waveguide are presented in sections 3 and 4, respectively. In section 5, we consider the two-layer semi-open waveguides studied in [4], [8] and derive some improved asymptotic results.

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II. Definition and basic properties

In this section, we consider a general two-dimensional waveguide given by a real refractive index profile $n(x)$, where x is a variable transverse to the waveguide axis in the z direction. To simplify the discussion further, we assume that the medium is homogeneous for $x < 0$ and $x > d$, where d is a constant. More specifically, we let $n(x) = n_1$ for $x < 0$ and $n(x) = n_2$ for $x > d$. For $0 < x < d$, the refractive index can be an arbitrary positive function of x, except that $\max_{0<x<d} n(x)$ should be larger than both n_1 and n_2. For slab waveguides, the leaky modes are usually defined as complex solutions of the dispersion relationship. For a general refractive index profile, the dispersion relationship cannot be written down analytically. In the following, we give a definition of the leaky modes based on a reduced nonlinear eigenvalue problem on the interval $(0, d)$.

For the planar waveguide specified by $n(x)$, a TE propagating mode has the y-component of the electric field given by $E_y = \phi(x)e^{i(\beta z-\omega t)}$, where ϕ is the mode profile, β is the real propagation constant and ω is the angular frequency. For ϕ and β^2, we have the following eigenvalue problem:

$$\frac{d^2\phi}{dx^2} + k_0^2 n^2(x)\,\phi = \beta^2\phi \quad \text{for} \ -\infty < x < \infty, \qquad (1)$$

$$\lim_{|x|\to\infty} \phi(x) = 0, \qquad (2)$$

where k_0 is the free space wavenumber. Since the medium is homogeneous for $x < 0$ and $x > d$, the above linear eigenvalue problem (linear in the eigenvalue β^2) defined on the whole x axis can be reduced to a nonlinear eigenvalue problem on the finite interval $0 < x < d$. We have

$$\frac{d^2\phi}{dx^2} + k_0^2 n^2(x)\,\phi = \beta^2\phi \quad \text{for} \quad 0 < x < d, \qquad (3)$$

$$\frac{d\phi}{dx} = -i\gamma_1\,\phi \quad \text{at} \quad x = 0, \qquad (4)$$

$$\frac{d\phi}{dx} = i\gamma_2\,\phi \quad \text{at} \quad x = d, \qquad (5)$$

where γ_1 and γ_2 depend on the eigenvalue β^2 as

$$\gamma_1 = \sqrt{k_0^2 n_1^2 - \beta^2}, \quad \gamma_2 = \sqrt{k_0^2 n_2^2 - \beta^2}. \qquad (6)$$

Here, the square root of a complex number follows the usual definition. That is $\sqrt{a} = \sqrt{|a|}e^{i\theta/2}$, if $a = |a|e^{i\theta}$ for $-\pi < \theta \le \pi$. In particular, $\sqrt{-a} = i\sqrt{a}$ if $a > 0$ and $\text{Re}\sqrt{a} \ge 0$ for any complex a. While the propagating modes corresponding to real values of β satisfy both (1, 2) and (3, 4, 5), the latter nonlinear eigenvalue problem could have solutions that violate the condition (2). These solutions with complex values of β^2 are the leaky modes. On the other hand, the radiation modes and the evanescent modes (which compromise the continuous spectrum $\beta^2 < k_0^2 \max\{n_1^2, n_2^2\}$) are obtained when condition (2) is replaced by the condition that ϕ is bounded for all x, and they do not satisfy (4) and (5).

If we multiply (3) by the complex conjugate of ϕ and integrate over $(0, d)$, we obtain

$$\begin{aligned} &i\gamma_1|\phi(0)|^2 + i\gamma_2|\phi(d)|^2 \\ &= \int_0^d \left[(\beta^2 - k_0^2 n^2)|\phi|^2 + \left|\frac{d\phi}{dx}\right|^2 \right] dx. \end{aligned} \tag{7}$$

This follows from an integration by part and applying the boundary conditions (4, 5). From (7), we conclude that if β^2 is a real eigenvalue of (3, 4, 5), then $\beta^2 \ge k_0^2 \max\{n_1^2, n_2^2\}$. Otherwise, the right hand side of (7) is real and the left hand side would have a positive imaginary part. Of course, condition (2) actually implies that $\beta^2 > k_0^2 \max\{n_1^2, n_2^2\}$. In that case, the left hand side of (7) is negative. This gives rise to $\beta^2 < k_0^2 \max_{0<x<d} n^2(x)$. On the other hand, if β^2 is a complex eigenvalue of (3, 4, 5), the imaginary part of the left hand side is positive. Compared with the right hand side, we conclude that $\text{Im}\,\beta^2 > 0$. Therefore, both the real and imaginary parts of β are positive. If $\phi(x)e^{i(\beta z - \omega t)}$ is a leaky mode, it decays exponentially in the positive z direction.

For the TM mode, the y-component of the magnetic field is given by $H_y = \phi(x)e^{i(\beta z - \omega t)}$, ϕ and β^2 satisfy the following nonlinear eigenvalue problem:

$$n^2 \frac{d}{dx}\left(\frac{1}{n^2}\frac{d\phi}{dx}\right) + k_0^2 n^2 \phi = \beta^2 \phi, \quad 0 < x < d, \tag{8}$$

$$\frac{d\phi}{dx} = -i\gamma_1 \frac{n^2(0+)}{n_1^2}\phi, \quad x = 0+ \tag{9}$$

$$\frac{d\phi}{dx} = i\gamma_2 \frac{n^2(d-)}{n_2^2}\phi, \quad x = d-, \tag{10}$$

where γ_1 and γ_2 are given in (6). A leaky mode is defined as a solution of the above nonlinear eigenvalue problem with a complex eigenvalue β^2. Instead of (7), we have

$$\begin{aligned} &\frac{i\gamma_1}{n_1^2}|\phi(0)|^2 + \frac{i\gamma_2}{n_2^2}|\phi(d)|^2 \\ &= \int_0^d \left[\frac{1}{n^2}\left|\frac{d\phi}{dx}\right|^2 + \left(\frac{\beta^2}{n^2} - k_0^2\right)|\phi|^2 \right] dx. \end{aligned} \tag{11}$$

From this, we obtain the same conclusions as the TE case. That is, a real eigenvalue of (8, 9, 10) must satisfies

$$k_0^2 \max\{n_1^2, n_2^2\} \le \beta^2 < k_0^2 \max_{0<x<d} n^2(x).$$

and a complex eigenvalue must satisfy $\text{Im}(\beta^2) > 0$.

III. The TE case

In this section, we consider a slab waveguide where $n(x) = n_0$ in the core ($0 < x < d$) and derive asymptotic solutions for TE leaky modes. The nonlinear eigenvalue problem (3, 4, 5) can be reduced to the following algebraic equation for β^2 [2]:

$$\tan(\gamma_0 d) = -i\frac{\gamma_0(\gamma_1 + \gamma_2)}{\gamma_0^2 + \gamma_1\gamma_2}, \tag{12}$$

where $\gamma_0 = \sqrt{k_0^2 n_0^2 - \beta^2}$. The equation has an infinite sequence of complex solutions corresponding to the leaky modes. Here, we develop an asymptotic expression for the leaky modes under the assumption that $|\gamma_0|$ is large.

Our asymptotic results are given in terms of the Lambert W functions [18] which are multi-valued functions of the complex variable ξ such that $We^W = \xi$. For an integer p, we denote the p-th branch of the Lambert W function as $\text{LambertW}(p, \xi)$. Let δ_1, δ_2 and r be given by

$$\delta_j = k_0^2(n_0^2 - n_j^2), \quad j = 1, 2, \tag{13}$$

$$r = \frac{d}{4}(\delta_1\delta_2)^{1/4}, \tag{14}$$

let $S \in \{-1, i, 1, -i\}$ be a fourth root of 1 and $W = \text{LambertW}(p, Sr)$, we have the following two asymptotic results for the leaky modes:

$$\gamma_0 \approx \frac{2i}{d}W, \tag{15}$$

$$\gamma_0 \approx \frac{W}{a_0} - \frac{a_0 a_2}{W^2} - \frac{a_0^2 a_3}{W^3} - \frac{a_0^3 a_4}{W^4}, \tag{16}$$

where

$$\begin{aligned} &a_0 = -\frac{id}{2}, \quad a_2 = -\frac{\delta_1 + \delta_2}{8}, \quad a_3 = \frac{i(\delta_1 + \delta_2)}{4d}, \\ &a_4 = -\frac{3}{64}(\delta_1^2 + \delta_2^2) + \frac{\delta_1 + \delta_2}{2d^2}. \end{aligned} \tag{17}$$

The above results are given in terms of γ_0, we can evaluate the propagation constant by $\beta = \sqrt{k_0^2 n_0^2 - \gamma_0^2}$ once γ_0 is obtained. Since the imaginary part of β^2 should be positive, both the real and imaginary parts of W should be negative. This implies that the branch number p can be restricted to negative integers only. We also re-write the 4-th root of 1 as $S = -i^{-q}$ for $q = 0, 1, 2, 3$, then the m-th leaky mode is associated with

$$W_m = \text{LambertW}(p, -i^{-q}r), \quad m = -4(1 + p) + q \tag{18}$$

for $p = -1, -2, \ldots$. It appears that the case $m = 0$ (i.e. $p = -1$ and $q = 0$) should be removed, since it does not correspond to an actual leaky mode.

To derive the results (15, 16), we re-write equation (12) as

$$e^{2id\gamma_0} = \frac{\gamma_0 + \gamma_1}{\gamma_0 - \gamma_1} \cdot \frac{\gamma_0 + \gamma_2}{\gamma_0 - \gamma_2}. \tag{19}$$

We can expand γ_1 and γ_2 in inverse power series of γ_0. That is,

$$\gamma_j = \gamma_0\left(1 - \frac{\delta_j}{\gamma_0^2}\right)^{1/2} = \gamma_0\left(1 - \frac{\delta_j}{2\gamma_0^2} - \frac{\delta_j^2}{8\gamma_0^4} + \ldots\right)$$

for $j = 1, 2$. The right hand side of (19) can also be expanded. We have

$$e^{2id\gamma_0} = \frac{16\gamma_0^4}{\delta_1\delta_2}\left(1 - \frac{\delta_1+\delta_2}{2\gamma_0^2} - \frac{\delta_1^2 - 4\delta_1\delta_2 + \delta_2^2}{16\gamma_0^4} + ...\right).$$

The fourth root of the above equation gives

$$S_0\, e^{id\gamma_0/2} = \frac{2\gamma_0}{(\delta_1\delta_2)^{1/4}}\left(1 + \frac{b_2}{\gamma_0^2} + \frac{b_4}{\gamma_0^4} + ...\right),$$

where S_0 is a fourth root of 1 and

$$b_2 = -\frac{\delta_1+\delta_2}{8}, \quad b_4 = -\frac{5(\delta_1^2+\delta_2^2) - 2\delta_1\delta_2}{128}.$$

This gives rise to

$$\frac{-iS_0 d}{4}(\delta_1\delta_2)^{1/4} = a_0\gamma_0\; e^{a_0\gamma_0}\left(1 + \frac{b_2}{\gamma_0^2} + \frac{b_4}{\gamma_0^4} + ...\right), \quad (20)$$

for $a_0 = -id/2$. We can re-write the right hand side of (20) as We^W for W given by

$$W = a_0\gamma_0 + \frac{a_2}{\gamma_0^2} + \frac{a_3}{\gamma_0^3} + \frac{a_4}{\gamma_0^4} + ... \quad (21)$$

Removing the common factor $e^{a_0\gamma_0}$, the right hand side of (20) gives

$$a_0\gamma_0\left(1 + \frac{b_2}{\gamma_0^2} + \frac{b_4}{\gamma_0^4} + ...\right)$$
$$= \left(a_0\gamma_0 + \frac{a_2}{\gamma_0^2} + \frac{a_3}{\gamma_0^3} + \frac{a_4}{\gamma_0^4} + ...\right) e^{\frac{a_2}{\gamma_0^2} + \frac{a_3}{\gamma_0^3} + \frac{a_4}{\gamma_0^4} + ...}.$$

To find the coefficients a_2, a_3 and a_4, we expand the right hand side above in an inverse power series of γ_0 and compare it with the left hand side. The results are given in (17). Since $S = -iS_0$ is still a fourth root of 1, Eq. (20) leads to $We^W = Sr$ for r given in (14). Therefore, $W = \text{LambertW}(p, Sr)$ for an integer p. To the leading order, (21) gives rise to (15). The more accurate formula (16) is obtained by inserting the leading order approximation into the higher order terms in (21).

For verify the asymptotic results, we consider two examples. The first example involves an unsymmetric slab waveguide. The parameters are $n_0 = 3.3$, $n_1 = 3.17$, $n_2 = 1.0$ and $d = 0.8\,\mu m$. The second example is a symmetric slab waveguide. We choose $n_2 = 3.17$, $d = 0.2\,\mu m$ and the same n_0 and n_1. The free space wavelength is assumed to be $\lambda = 1.55\,\mu m$. For both examples, we calculate the exact leaky modes from (12) and the approximate leaky modes from (15) and (16), based on (18) for $m = 1, 2, ..., 15$. The results are shown in Fig. 1. For more details, we list the first a few leaky modes in Table 1 for both examples. In the table, we show the exact propagation constants of the leaky modes and the relative errors of the asymptotic solutions (15) and (16). If $\tilde{\beta}$ is an approximation of the exact propagation constant β, its relative error is given by R.E. $= |\beta - \tilde{\beta}|/|\beta|$. We observe that the two asymptotic formulas are very accurate for larger m and formula (16) is indeed more accurate than (15) for smaller values of m.

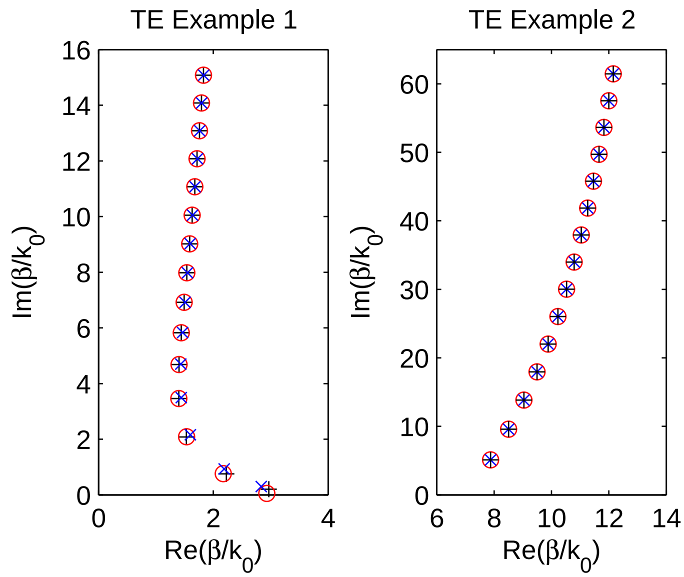

Fig. 1. Comparison of the exact (marked by "+") and approximate propagation constants of the TE leaky modes. Approximate solutions obtained from (15) and (16) are marked by "×" and "o", respectively.

Example	m	Exact β	R.E. of (15)	R.E. of (16)
1	1	12.031439+ 0.818526i	0.0545	0.05262
1	2	9.039718+ 3.053879i	0.0783	0.02494
1	3	6.214625+ 8.423266i	0.0424	0.00443
1	4	5.667104+ 14.035489i	0.0146	0.00063
1	5	5.694616+ 18.995289i	0.0067	0.00015
1	6	5.852768+ 23.611412i	0.0037	0.00005
2	1	31.927284+ 20.670091i	8.03×10^{-4}	1.34×10^{-5}
2	2	34.471744+ 38.879799i	2.75×10^{-4}	1.68×10^{-6}
2	3	36.644954+ 56.098296i	1.23×10^{-4}	3.47×10^{-7}
2	4	38.484651+ 72.821842i	6.46×10^{-5}	9.89×10^{-8}

TABLE I

EXACT PROPAGATION CONSTANTS OF THE TE LEAKY MODES AND RELATIVE ERRORS OF THE APPROXIMATE FORMULAS (15) AND (16).

IV. THE TM CASE

In this section, we derive asymptotic solutions for TM leaky modes in a slab waveguide. As before, the refractive index of the slab waveguide is n_0, n_1 and n_2 for $0 < x < d$, $x < 0$ and $x > d$, respectively, where d is the width of the core. The nonlinear eigenvalue problem (8, 9, 10) can be reduced to the following algebraic equation:

$$e^{2id\gamma_0} = \frac{\mu_0+\mu_1}{\mu_0-\mu_1}\cdot\frac{\mu_0+\mu_2}{\mu_0-\mu_2}, \quad (22)$$

where

$$\mu_j = \frac{\gamma_j}{n_j^2}, \quad \gamma_j = \sqrt{k_0^2 n_0^2 - \beta^2}, \quad j = 0, 1, 2.$$

To derive the asymptotic results, we assume that $|\gamma_0|$ is large and expand the right hand side of (22) in an inverse power series of γ_0. That is

$$e^{2id\gamma_0} = b_0\left(1 + \frac{b_2}{\gamma_0^2} + \frac{b_4}{\gamma_0^4} + ...\right), \quad (23)$$

where

$$b_0 = \frac{(n_0^2+n_1^2)(n_0^2+n_2^2)}{(n_0^2-n_1^2)(n_0^2-n_2^2)}, \tag{24}$$

$$b_2 = k_0^2 n_0^2 \left(\frac{n_1^2}{n_0^2+n_1^2} + \frac{n_2^2}{n_0^2+n_2^2} \right). \tag{25}$$

Taking the logarithm of the two sides of (23), we obtain

$$2id\gamma_0 = i2\pi m + \ln(b_0) + \frac{a_2}{\gamma_0^2} + \frac{a_4}{\gamma_0^4} + ... \tag{26}$$

where m is an integer and $a_2 = b_2$. To the leading order, we have

$$\gamma_0 \approx \frac{m\pi}{d} - \frac{i}{2d}\ln(b_0) = K_m \tag{27}$$

Since $\text{Im}(\beta^2) > 0$, the real part of γ_0 should be positive. Therefore, we choose m as positive integers. A more accurate asymptotic solution is obtained, if we keep the term a_2/γ_0^2 in (26) but approximate it by a_2/K_m^2. This gives rise to

$$\gamma_0 \approx K_m - \frac{ib_2}{2dK_m^2}. \tag{28}$$

To validate the two formulas (27) and (28), we consider the two examples studied in section 4 and compare the exact and approximate propagation constants of the TM leaky modes. As shown in Table 2, the exact propagation constants obtained from (22) are accurately approximated by the analytic solutions given in (27) and (28).

Example	m	Exact β	R.E. of (27)	R.E. of (28)
1	1	12.339851+ 0.610148i	0.0518	0.1159
1	2	10.499311+ 2.003849i	0.0761	0.0590
1	3	7.366323+ 4.194777i	0.1007	0.0442
1	4	4.1873860+ 9.152173i	0.0523	0.0138
1	5	3.112847+ 14.596377i	0.0171	0.0028
1	6	2.720526+ 19.489543i	0.0077	0.0009
1	7	2.529533+ 24.063702i	0.0043	0.0004
2	1	18.928928+ 13.977974i	0.0334	0.00225
2	2	17.503529+ 29.461225i	0.0105	0.00037
2	3	16.819679+ 45.520900i	0.0037	0.00007
2	4	16.516545+ 61.536569i	0.0016	0.00002

TABLE II

EXACT PROPAGATION CONSTANTS OF THE TM LEAKY MODES AND RELATIVE ERRORS OF THE APPROXIMATE FORMULAS (27) AND (28).

V. TWO-LAYER WAVEGUIDES

Rogier and De Zutter [4] studied the PML modes of two-layer waveguides with one side bounded by a perfect electric conductor. They have derived the leading order asymptotic solutions for both the leaky modes and the Berenger modes. In this section, we briefly study the leaky modes of these two-layer waveguides. In particular, we present some higher order asymptotic solutions.

For a two-layer waveguide, the refractive index is n_0 and n_1 for $0 < x < d$ and $x > d$, respectively. For a TE mode $E_y = \phi(x)e^{i(\beta z - \omega t)}$, the boundary condition is $\phi = 0$ at $x = 0$. The propagation constant β satisfies the following algebraic equation:

$$e^{2id\gamma_0} = -\frac{\gamma_0+\gamma_1}{\gamma_0-\gamma_1},$$

where $\gamma_j = \sqrt{k_0^2 n_j^2 - \beta^2}$ for $j = 0, 1$. Similar to section 3, we obtain

$$\pm r = We^W \tag{29}$$

where $\delta_1 = k_0^2(n_0^2 - n_1^2)$, $r = d\sqrt{\delta_1}/2$ and

$$W = a_0\gamma_0 + \frac{a_2}{\gamma_0^2} + \frac{a_3}{\gamma_0^3} + \frac{a_4}{\gamma_0^4} + ...$$

for

$$a_0 = -id, \quad a_2 = -\frac{\delta_1}{4}, \quad a_3 = \frac{i\delta_1}{4d},$$
$$a_4 = -\frac{3\delta_1^2}{32} + \frac{\delta_1}{4d^2}. \tag{30}$$

The leading order approximation derived in [4] is:

$$\gamma_0 \approx \frac{W}{a_0} = \frac{iW}{d}, \tag{31}$$

We have the following more accurate result:

$$\gamma_0 \approx \frac{W}{a_0} - \frac{a_0 a_2}{W^2} - \frac{a_0^2 a_3}{W^3} - \frac{a_0^3 a_4}{W^4}. \tag{32}$$

From (29), we see that W can be given by the Lambert W function:

$$W = W_m = \text{LambertW}\left(p, (-1)^{q+1}r\right), \quad m = -2(p+1)+q,$$

where p is the branch number and $q = 0$ or 1. Due to the requirement that $\text{Im}(\beta^2) > 0$, we only need to choose p as negative integers. It appears that the case $m = 0$ (i.e., $q = 0$ and $p = -1$) does not correspond to an actual leaky mode and it should be removed.

For a TM mode $H_y = \phi(x)e^{i(\beta z - \omega t)}$, the boundary condition is $d\phi/dx = 0$ at $x = 0$. The propagation constant satisfies the following equation

$$e^{2id\gamma_0} = \frac{\mu_0+\mu_1}{\mu_0-\mu_1}, \tag{33}$$

where $\mu_j = \gamma_j/n_j^2$. After expanding the right hand side in an inverse power series of γ_0 and finding the logarithm of both sides, we obtain

$$2id\gamma_0 = i(2m+1)\pi + \ln(b_0) + \frac{\hat{a}_2}{\gamma_0^2} + \frac{\hat{a}_4}{\gamma_0^4} + ...,$$

where m is an integer and

$$b_0 = \frac{n_0^2+n_1^2}{n_0^2-n_1^2}, \quad \hat{a}_2 = \frac{k_0^2 n_0^2 n_1^2}{n_0^2+n_1^2},$$
$$\hat{a}_4 = \frac{k_0^4 n_0^2 n_1^2 (3n_0^4 - n_1^4)}{4(n_0^2+n_1^2)^2}.$$

The leading order asymptotic result already obtained in [4] is

$$\gamma_0 \approx \frac{(2m+1)\pi}{2d} - \frac{i}{2d}\ln(b_0) = K_m. \tag{34}$$

Since the real part of γ_0 should be positive, we choose $m \geq 0$, although the case $m = 0$ usually does not correspond to an actual leaky mode. To obtain a more accurate result, we approximate γ_0 in the term $\hat{a}_2/\gamma_0^2$ by K_m. This leads to

$$\gamma_0 \approx K_m - \frac{i\hat{a}_2}{2dK_m^2}. \tag{35}$$

As an example, we consider a two-layer structure with $n_0 = 3.3$, $n_1 = 3.17$, $d = 0.8$ and a free space wavelength $\lambda = 1.55\ \mu m$. The propagation constants of the first a few leaky modes for both TE and TM cases are shown in Table 3. Once

(a)

m	TE case: Exact β	R.E. of (31)	R.E. of (32)
1	12.25923478+ 0.56782788i	0.0062	0.000466
2	9.79997245+ 1.97333786i	0.0048	0.000068
3	5.89890942+ 5.67502430i	0.0049	0.000025
4	4.19929208+ 11.66037670i	0.0016	0.000004

(b)

m	TM case: Exact β	R.E. of (34)	R.E. of (35)
1	11.54611350+ 1.20167970i	0.0611	0.0654
2	9.21454135+ 2.77661975i	0.0806	0.0400
3	5.31357742+ 6.09705053i	0.0798	0.0209
4	3.28098424+ 11.88797517i	0.0236	0.0038
5	2.70211408+ 17.07346121i	0.0093	0.0010
6	2.45651514+ 21.79542598i	0.0047	0.0004

TABLE III

(A): EXACT PROPAGATION CONSTANTS OF THE TE LEAKY MODES AND RELATIVE ERRORS OF THE APPROXIMATE FORMULAS (31) AND (32). (B): EXACT PROPAGATION CONSTANTS OF THE TM LEAKY MODES AND RELATIVE ERRORS OF THE APPROXIMATE FORMULAS (34) AND (35).

again, we observe that the higher order formulas (32) and (35) are indeed more accurate than the leading order formulas (31) and (34), for TE and TM cases, respectively.

VI. CONCLUSIONS

We have derived highly accurate asymptotic solutions for leaky modes in two-dimensional slab waveguides. The derivation follows a systematic approach that utilizes inverse power series of $\gamma_0 = \sqrt{k_0^2 n_0^2 - \beta^2}$, assuming that $|\gamma_0|$ is large. The accuracy of the asymptotic formulas increases when $|\gamma_0|$ is increased. Our results are useful in the eigenmode expansion method [13] when PMLs used. In such a method, it is necessary to compute the perturbed leaky modes and the Berenger modes [10] in each segment of the structure. Our approximate analytic solutions can be used to reduce the effort of computing these modes.

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