Asymptotic Solutions of the Leaky Modes and PML Modes in a Pekeris Waveguide

Jianxin Zhu
Department of Mathematics
Zhejiang University, Hangzhou, China

Ya Yan Lu*
Department of Mathematics
City University of Hong Kong
Kowloon, Hong Kong

Abstract

Leaky modes are useful to partially represent the continuous spectrum in open waveguides. The Perfectly Matched Layer (PML) is a widely used technique for truncating unbounded domains in numerical simulations of wave propagation problems. When a Pekeris waveguide is terminated by a finite PML, it gives rise to three classes of modes corresponding to the trapped modes, the leaky modes and the modes that approximate the branch line integral along the Pekeris cut. In this paper, we derive high order asymptotic solutions for these modes in a Pekeris waveguide and a Pekeris waveguide terminated by a finite PML.

1 Introduction

As a simple model for underwater acoustics [1, 2], the Pekeris waveguide has been studied extensively. The normal mode theory [3, 4, 5] reveals that a general wave field in the Pekeris waveguide can be written down explicitly as a finite sum of the trapped modes plus an integral of the continuous spectrum along the Ewing-Jardetzky-Press (EJP) branch cut. The integral along the EJP branch cut is highly oscillatory and difficult to evaluate, but it can be re-written as the infinite sum of the leaky modes plus an integral along the

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Pekeris cut. The leaky modes are useful since they give an approximate representation of the continuous spectrum. The trapped modes and the leaky modes do not comprise a complete set due to the presence of the branch line integral along the Pekeris cut. Unfortunately, the integral along the Pekeris branch cut is still difficult to evaluate. In practical numerical implementations, a discrete approximation of the continuous spectrum is necessary. One possible approach is to assume that the square of the wavenumber is a linear function of the depth in the lower half space [6]. More often, we simply truncate the depth as in the Parabolic Equation (PE) method [7] and the step-wise coupled mode method [8]. If the depth is truncated without adding artificial absorption, we have a discrete set of modes approximating the branch line integral along the EJP cut. The convergence is extremely slow corresponding to the existence of spurious reflections from the lower bottom boundary. To reduce the spurious reflections, an artificial absorbing layer [7, 8] is often introduced in the lower part of the sea-bottom. For some problems, a large truncation depth is needed when this technique is used. For PE models, non-local transparent boundary conditions [9, 10, 11, 12, 13] can also be used. However, they require all the previous acoustic field along the bottom boundary in each marching step. Besides, the transparent boundary conditions are not suitable for the coupled mode method. Yevick and Thomson [14] applied the perfectly matched layer (PML) technique [15, 16] to PE models and demonstrated that PML is efficient at truncating the unbounded sea-bottom with minimal spurious reflections. The PML technique has been successfully used with the coupled mode method for optical waveguide simulations [17, 18].

Although the Pekeris waveguide is very simple, exact solutions of the leaky modes are not available. A perturbation result can be found in [2], but it is only useful for the lowest one or two modes. In [19], Rogier and De Zutter derived a leading order asymptotic result for leaky modes of a microstrip substrate which is similar but not identical to the Pekeris waveguide. In a recent work [20], we derived some high order asymptotic solutions for the leaky modes of a slab waveguide which contains three different layers. In section 2, we derive high order asymptotic solutions for leaky modes of the Pekeris waveguide. Our formulas are increasingly more accurate for higher modes and they give us accurate information about how the leaky modes are distributed in the complex plane. In section 3, we derive high order asymptotic solutions for the PML modes (the eigenmodes in a waveguide terminated by a finite PML) in a Pekeris waveguide. For microstrip substrates, leading order asymptotic solutions of the PML modes have been derived in [19]. The PML is an effective approach to approximate the branch line integrals. The asymptotic solutions give a clear and explicit picture on how this is achieved. As in [19], three classes of PML modes have been identified. The first and the second classes correspond to the original trapped modes and the leaky modes, respectively. The third class of modes appears to give a discrete approximation for the integral along the Pekeris cut.

To implement the step-wise coupled mode method with a PML, the PML modes have to be calculated. The PML modes are also useful to construct the Green's function of the waveguide [21]. Although an analytic equation can be written down for the eigenvalues of the PML modes, it is not an easy task to find all its solutions, since the eigenvalues are complex. Our asymptotic solutions of the PML modes are useful, because they can be used as initial guesses for solving the exact eigenvalues numerically.

2 Leaky Modes

We consider the two dimensional Helmholtz equation:

$$\rho \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial u}{\partial x}\right) + \rho \frac{\partial}{\partial z}\left(\frac{1}{\rho}\frac{\partial u}{\partial z}\right) + k^2 u = 0, \tag{1}$$

where z is the depth, x is the horizontal distance (the range), ρ is the density, $k = \omega/c$ is the wavenumber, ω is the angular frequency and c is the sound speed. In general, ρ and k are functions of x and z. For ocean acoustics, the pressure-release condition $u = 0$ is typically used at $z = 0$. If the ocean bottom is modeled by an unbounded fluid layer, equation (1) is valid for the half plane $z > 0$. For a range-independent waveguide where ρ and k are functions of z only, we have trapped mode solutions given as $u = \phi(z)e^{i\beta x}$, where ϕ and $\lambda = \beta^2$ satisfy the following eigenvalue problem

$$\rho \frac{d}{dz}\left(\frac{1}{\rho}\frac{d\phi}{dz}\right) + k^2\phi = \lambda\phi \quad \text{for} \quad z > 0, \tag{2}$$

$$\phi(0) = 0, \quad \lim_{z\to\infty} \phi(z) = 0. \tag{3}$$

Here, we further assume that for a large enough depth, ρ and k are constants. Therefore, for some $G > 0$, we have

$$k = k_2, \quad \rho = \rho_2, \quad z > G, \tag{4}$$

where k_2 and ρ_2 are constants. To satisfy the condition at $z = \infty$, we must have $\lambda = \beta^2 > k_2^2$ and ϕ should decay to zero (as $z \to \infty$) like $e^{-\sqrt{\lambda - k_2^2}\, z}$. This gives rise to

$$\phi_z = i\sqrt{k_2^2 - \lambda}\,\phi \quad \text{for} \quad z > G,$$

where the square root follows the standard definition, such that the square root of a negative number is a pure imaginary number with a positive imaginary part. This allows us to reduce the original eigenvalue problem to a nonlinear eigenvalue problem on the finite interval $0 < z < H$ for an arbitrary $H > G$. We have

$$\rho \frac{d}{dz}\left(\frac{1}{\rho}\frac{d\phi}{dz}\right) + k^2\phi = \lambda\phi \quad \text{for} \quad 0 < z < H, \tag{5}$$

$$\phi = 0 \quad \text{at} \quad z = 0, \tag{6}$$

$$\phi_z - i\sqrt{k_2^2 - \lambda}\,\phi = 0 \quad \text{at} \quad z = H. \tag{7}$$

If we multiply equation (5) by $\rho^{-1}\overline{\phi}$ and integrate over $(0, H)$, we get

$$\lambda \int_0^H \frac{1}{\rho}|\phi|^2\, dz - \int_0^H \frac{1}{\rho} k^2 |\phi|^2\, dz = \frac{1}{\rho(H)} i\sqrt{k_2^2 - \lambda}\, |\phi(H)|^2 - \int_0^H \frac{1}{\rho}|\phi_z|^2\, dz. \tag{8}$$

Let λ be a real eigenvalue of the system (5-7), from (8), we conclude that $i\sqrt{k_2^2 - \lambda}$ must be real, thus $\lambda \geq k_2^2$. Furthermore, the two terms in the right hand side of (8) are negative, therefore

$$\lambda \int_0^H \frac{1}{\rho}|\phi|^2\, dz < \int_0^H \frac{1}{\rho} k^2 |\phi|^2\, dz.$$

This gives rise to $\lambda < k_1^2$, where $k_1 = \max k(z)$. The system (5-7) also has complex eigenvalues corresponding to the leaky modes of the waveguide. Since the branch-cut of the standard square root is the negative real axis, we have $\mathrm{Re}\sqrt{k_2^2 - \lambda} > 0$ if λ is complex with a non-zero imaginary part. Comparing the imaginary parts of the two sides of (8), we have $\mathrm{Im}\,\lambda > 0$. Therefore, a leaky mode (which depends on x as $e^{i\sqrt{\lambda}x}$) decays exponentially in the propagation direction x.

For the special case of a Pekeris waveguide, we assume that the horizontal interface is at $z = G$. The medium properties are constant in the layers above and below the interface. We have $\rho = \rho_1$, $k = k_1$ for $0 < z < G$ and $\rho = \rho_2$, $k = k_2$ for $z > G$. For both the trapped modes and the leaky modes, we have the following eigenvalue problem in $0 < z < H$ (where $H > G$):

$$\frac{d^2\phi(z)}{dz^2} + k_1^2\phi(z) = \lambda\phi(z), \quad 0 < z < G, \tag{9}$$

$$\frac{d^2\phi(z)}{dz^2} + k_2^2\phi(z) = \lambda\phi(z), \quad G < z < H, \tag{10}$$

$$\phi(G-) = \phi(G+), \tag{11}$$

$$\frac{1}{\rho_1}\frac{d\phi}{dz}(G-) = \frac{1}{\rho_2}\frac{d\phi}{dz}(G+), \tag{12}$$

$$\phi(0) = 0, \tag{13}$$

$$\frac{d\phi}{dz} = i\sqrt{k_2^2 - \lambda}\,\phi, \quad \text{at} \quad z = H. \tag{14}$$

The microstrip substrate studied in [19] is similar but not identical to the Pekeris waveguide. For the transverse electric (TE) case of the microstrip substrate, the conditions (11) and (13) are satisfied, but (12) is replaced by the continuity of $\partial_z\phi$ at $z = G$. For the transverse magnetic (TM) case, the conditions (11) and (12) are satisfied, but (13) is replaced by $\partial_z\phi = 0$ at $z = 0$. For the above eigenvalue problem, it is easy to show that the eigenvalue λ satisfies the following equation:

$$\frac{1}{\rho_1}\sqrt{k_1^2 - \lambda}\,\cot(\sqrt{k_1^2 - \lambda}\cdot G) = \frac{i}{\rho_2}\sqrt{k_2^2 - \lambda}. \tag{15}$$

Although Eq. (15) is quite simple, it is not clear how its infinite number of solutions are distributed in the complex plane. With an initial guess, it is easy to find the nearby exact solution to high accuracy by a numerical method. But it is difficult to find all solutions in a given region of the complex plane by a numerical method alone. An approximate formula for the leaky modes can be found in [2]. It is derived based on the assumption that $|\sqrt{k_2^2-\lambda}/\sqrt{k_1^2-\lambda}|$ is small. The formula losses its accuracy when this assumption is not valid. In the following, we derive asymptotic formulas assuming that $|\lambda - k_1^2|$ is large.

From (15), we have

$$e^{2iG\sqrt{k_1^2-\lambda}} = -\frac{1+T}{1-T}, \quad T = \frac{\rho_1}{\rho_2}\frac{\sqrt{k_2^2-\lambda}}{\sqrt{k_1^2-\lambda}}. \tag{16}$$

Therefore,

$$2iG\sqrt{k_1^2-\lambda} = i(2n-1)\pi + \ln\left(\frac{1+T}{1-T}\right) \tag{17}$$

for an integer n. For larger $|\lambda|$, $T \approx \rho_1/\rho_2$ and the imaginary part of $\ln[(1+T)/(1-T)]$ is negligible. Since the standard definition of the square root function of a complex number is used here, we have $\mathrm{Re}\sqrt{k_1^2-\lambda} \geq 0$. This implies that we can only choose $n \geq 1$. The second term in the right hand side of (17) can be expanded as follows:

$$\ln\left(\frac{1+T}{1-T}\right) = a_0 + \frac{a_1}{\lambda - k_1^2} + \frac{a_2}{(\lambda-k_1^2)^2} + \frac{a_3}{(\lambda-k_1^2)^3} + ... \tag{18}$$

where

$$\begin{aligned}
a_0 &= \ln\left(\frac{\rho_2+\rho_1}{\rho_2-\rho_1}\right), \\
a_1 &= \frac{\rho_1\rho_2}{\rho_2^2-\rho_1^2}(k_1^2-k_2^2), \\
a_2 &= \frac{\rho_1\rho_2(3\rho_1^2-\rho_2^2)}{4(\rho_2^2-\rho_1^2)^2}(k_1^2-k_2^2)^2, \\
a_3 &= \frac{\rho_1\rho_2(3\rho_2^4-10\rho_1^2\rho_2^2+15\rho_1^4)}{24(\rho_2^2-\rho_1^2)^3}(k_1^2-k_2^2)^3.
\end{aligned}$$

Inserting (18) into (17) and taking its square, we obtain

$$\lambda - k_1^2 = A_0 + \frac{A_1}{\lambda - k_1^2} + \frac{A_2}{(\lambda-k_1^2)^2} + \frac{A_3}{(\lambda-k_1^2)^3} + ... \tag{19}$$

where

$$\begin{aligned}
A_0 &= \frac{\tilde{a}_0^2}{4G^2}, \quad \text{for} \quad \tilde{a}_0 = a_0 + i(2n-1)\pi, \\
A_1 &= \frac{\tilde{a}_0 a_1}{2G^2}, \\
A_2 &= \frac{2\tilde{a}_0 a_2 + a_1^2}{4G^2}, \\
A_3 &= \frac{\tilde{a}_0 a_3 + a_1 a_2}{2G^2}.
\end{aligned}$$

Clearly, we have $A_0 = O(n^2)$ and $A_j = O(n)$ for $j \geq 1$. Asymptotic formulas for λ can be derived iteratively from (19). To make explicit the dependence of λ on n, we denote λ by λ_n. The asymptotic formulas are:

$$\lambda_n = k_1^2 + A_0 + O\left(\frac{1}{n}\right) \tag{20}$$

$$= k_1^2 + A_0 + \frac{A_1}{A_0} + O\left(\frac{1}{n^3}\right) \tag{21}$$

$$= k_1^2 + A_0 + \frac{A_1}{A_0 + A_1/A_0} + \frac{A_2}{A_0^2} + O\left(\frac{1}{n^5}\right). \tag{22}$$

The asymptotic results in [19] are derived for a slightly different problem, but they correspond to the leading asymptotic formula (20). High order asymptotic results for the TE and TM cases of a three layer slab waveguide have been derived earlier in [20].

For a numerical example, we consider the Pekeris waveguide given by

$$G = 50\,m,\ \omega = 480,\ \rho_1 = 1000\,kg/m^3,\ \rho_2 = 1700\,kg/m^3, \tag{23}$$

$$c_1 = 1500\,m/s,\ c_2 = 1666.67\,m/s, \tag{24}$$

where $k_j = \omega/c_j$ for $j = 1, 2$. From the value of ω, we observe that the frequency is approximately 76.394 Hz. The exact values of the first six leaky modes are listed in the first column of Table 1, up to five digits. Instead of the eigenvalue λ_n, we have

Exact $\sqrt{\lambda_n}/k_1$	R.E. (20)	R.E. (21)	R.E. (22)	Frisk
0.8728 + 0.0097i	0.0161	0.0020	0.0019	0.0018
0.7287 + 0.0289i	0.0149	0.0004	0.0004	0.0041
0.4752 + 0.0657i	0.0263	0.0003	0.0002	0.0181
0.0975 + 0.4167i	0.0269	0.0002	0.0001	0.0287
0.0625 + 0.7942i	0.0066	0.0000	0.0000	0.0088
0.0541 + 1.0815i	0.0031	0.0000	0.0000	0.0058

Table 1: The first six leaky modes of a Pekeris waveguide. The first column is the exact horizontal wavenumber (scaled by k_1). The other columns are the relative errors of the asymptotic formulas (20), (21), (22) and the formula in Frisk's book [2].

shown the horizontal wavenumber $\sqrt{\lambda_n}$ and it is further scaled by k_1 for simplicity. The second, third and fourth columns of Table 1 are the relative errors of the asymptotic formulas (20), (21) and (22). Here, if $\lambda_n^{(app)}$ approximates λ_n, the relative error is defined as $|\lambda_n^{(app)} - \lambda_n|/|\lambda_n|$. The first six leaky modes actually correspond to $n = 3, 4, ..., 8$ in these formulas, presumably $n = 1$ and $n = 2$ give approximations of the two propagating modes. Our formulas become more accurate for a larger n. Relative errors of a formula in Frisk's book [2] are shown in the last column of Table 1 and the first leaky mode also corresponds to $n = 3$. Clearly, Frisk's formula is less accurate than (21) and (22), except for the first leaky mode.

3 PML modes

To solve the Helmholtz equation (1) or its further approximations numerically, the depth z must be truncated. Under the assumption (4) that the bottom is homogeneous for $z > G$, we can terminate z by a perfectly matched layer (PML) [15] corresponding to a complex coordinate stretching [16]:

$$\hat{z} = z + i \int_0^z \sigma(\tau)\; d\tau \tag{25}$$

where σ is a continuous function satisfying $\sigma(z) = 0$ for $0 < z \le H$, $\sigma(z) > 0$ for $z > H$ and $H \ge G$. If we replace ∂_z in (1) by $\partial \hat{z} = [1 + i\sigma(z)]\partial z$, we obtain the following modified Helmholtz equation:

$$\rho \frac{\partial}{\partial x}\left(\frac{1}{\rho}\frac{\partial u}{\partial x}\right) + \frac{\rho}{1+i\sigma}\frac{\partial}{\partial z}\left(\frac{1}{\rho(1+i\sigma)}\frac{\partial u}{\partial z}\right) + k^2 u = 0. \tag{26}$$

Notice that (1) and (26) are different only if $z > H$. Both u and $\partial_z u$ are required to be continuous at $z = H$. The depth z is then terminated at $D > H$. The interval (H, D) is the actual PML layer. Equation (26) is solved with a simple boundary condition at $z = D$. We assume it is

$$u = 0 \quad \text{at} \quad z = D. \tag{27}$$

To understand the effects of a PML, we consider the special case of a Pekeris waveguide. In the following, we derive asymptotic solutions for eigenmodes in a Pekeris waveguide terminated below by a PML (i.e. the PML modes). With a PML, the eigenvalue problem becomes

$$\frac{d^2\phi}{dz^2} + k_1^2\phi = \lambda\phi, \quad 0 < z < G \tag{28}$$

$$\frac{d^2\phi}{dz^2} + k_2^2\phi = \lambda\phi, \quad G < z < H \tag{29}$$

$$\frac{1}{1+i\sigma}\frac{d}{dz}\left[\frac{1}{1+i\sigma}\cdot\frac{d\phi}{dz}\right] + k_2^2\phi = \lambda\phi, \quad H < z < D, \tag{30}$$

$$\phi(0) = 0, \tag{31}$$

$$\phi(G-) = \phi(G+), \tag{32}$$

$$\frac{1}{\rho_1}\frac{d\phi}{dz}(G-) = \frac{1}{\rho_2}\frac{d\phi}{dz}(G+), \tag{33}$$

$$\phi(H-) = \phi(H+), \tag{34}$$

$$\frac{d\phi}{dz}(H-) = \frac{d\phi}{dz}(H+), \tag{35}$$

$$\phi(D) = 0. \tag{36}$$

A nonlinear equation for the eigenvalue λ can be easily derived. It is

$$\frac{1}{\rho_1}\sqrt{k_1^2-\lambda}\,\cot\left[G\sqrt{k_1^2-\lambda}\right] + \frac{1}{\rho_2}\sqrt{k_2^2-\lambda}\,\cot\left[(\hat{D}-G)\sqrt{k_2^2-\lambda}\right] = 0, \tag{37}$$

where

$$\hat{D} = D + i \int_G^D \sigma(\tau)\, d\tau.$$

As for the leaky modes in section 2, if an initial guess is known, it is easy to find the nearby exact solution of (37) numerically. It is more difficult to find all its solutions in a given region of the complex plane. Asymptotic solutions are useful, because they give the initial guesses. For the related problem of microstrip substrates, leading order asymptotic results are available in [19]. In the following, we derive some high order asymptotic solutions.

Among solutions of (37), we consider a sequence that has a convergent phase angle. That is, $\lambda = \lambda_n = |\lambda_n| e^{i\theta_n}$, where $-\pi < \theta_n \le \pi$ and

$$\lim_{n\to\infty} |\lambda_n| = \infty, \quad \lim_{n\to\infty} \theta_n = \theta_*.$$

Notice that

$$\cot(y) \to \mp i \quad \text{if} \quad \text{Im}(y) \to \pm\infty,$$

and the convergence is exponentially fast. For a general θ_*, if the imaginary parts of both $G\sqrt{k_1^2 - \lambda}$ and $(\hat{D} - G)\sqrt{\lambda_2^2 - \lambda}$ tend to infinity, then (37) cannot be satisfied, since ρ_1 cannot equal $\pm\rho_2$.

For a sequence in the upper half of the complex plane, we have $0 \le \theta_n \le \pi$, then

$$G\sqrt{k_1^2 - \lambda_n} = G|\lambda_n|^{1/2} e^{i(\theta_n - \pi)/2} \left(1 - \frac{k_1^2}{\lambda_n}\right)^{1/2} \tag{38}$$

and

$$(\hat{D} - G)\sqrt{k_2^2 - \lambda_n} = |\hat{D} - G| \cdot |\lambda_n|^{1/2} e^{i[(\theta_n - \pi)/2 + \varphi]} \left(1 - \frac{k_2^2}{\lambda_n}\right)^{1/2}, \tag{39}$$

where

$$\hat{D} - G = D - G + i \int_G^D \sigma(\tau)\, d\tau = |\hat{D} - G| e^{i\varphi} \tag{40}$$

and $0 < \varphi < \pi/2$. Therefore, we need to consider only two cases: (1) $\theta_* = \pi$; (2) $\theta_* = \pi - 2\varphi$.

If $\theta_n \to \theta_* = \pi$, then

$$\text{Im}\left[(\hat{D} - G)\sqrt{k_2^2 - \lambda_n}\right] \to +\infty.$$

Thus, (37) is reduced to

$$\frac{1}{\rho_1}\sqrt{k_1^2 - \lambda}\, \cot\left[G\sqrt{k_1^2 - \lambda}\right] + \frac{1}{\rho_2}\sqrt{k_2^2 - \lambda}\,(-i + h.o.t) = 0,$$

where $h.o.t.$ denote some exponentially small terms, since the convergence (of the second cotangent term to $-i$) is exponentially fast. For larger $|\lambda|$, this equation is essentially the same as (15). Therefore, we have the same asymptotic formulas (20-22) as the original leaky modes. We conclude that the PML gives rise to a sequence of modes which are approximations of the original leaky modes.

If $\theta_n \to \theta_* = \pi - 2\varphi$, we have

$$\mathrm{Im}\left[G\sqrt{k_1^2 - \lambda_n}\right] \to -\infty.$$

Thus, (37) is reduced to

$$\frac{1}{\rho_1}\sqrt{k_1^2-\lambda}\,(i + h.o.t.) + \frac{1}{\rho_2}\sqrt{k_2^2-\lambda}\,\cot\left[(\hat{D}-G)\sqrt{k_2^2-\lambda}\right] = 0. \tag{41}$$

This gives rise to

$$e^{2i(\hat{D}-G)\sqrt{k_2^2-\lambda}} = \frac{1-T(1+h.o.t.)}{1+T(1+h.o.t.)},$$

where T is defined in (16). To derive an asymptotic formula, the exponentially small terms above can be ignored. We have

$$2i(\hat{D}-G)\sqrt{k_2^2-\lambda} = 2n\pi i - \ln\left(\frac{1+T}{1-T}\right) + h.o.t.$$

where n is an integer. From (39), we can see that $(\hat{D}-G)\sqrt{k_2^2-\lambda}$ is nearly a real number, thus we should require $n \geq 1$. Using the power series (17), the above is written as

$$2i(\hat{D}-G)\sqrt{k_2^2-\lambda} = 2n\pi i - \left[a_0 + \frac{a_1}{\lambda-k_1^2} + \frac{a_2}{(\lambda-k_1^2)^2} + \frac{a_3}{(\lambda-k_1^2)^3} + \ldots\right].$$

The square of the above gives rise to

$$\lambda - k_1^2 = B_0 + \frac{B_1}{\lambda-k_1^2} + \frac{B_2}{(\lambda-k_1^2)^2} + \frac{B_3}{(\lambda-k_1^2)^3} + \ldots$$

where

$$\begin{aligned}
B_0 &= k_2^2 - k_1^2 + \frac{\hat{a}_0^2}{4(\hat{D}-G)^2}, \quad \text{for} \quad \hat{a}_0 = a_0 - 2n\pi i \\
B_1 &= \frac{\hat{a}_0 a_1}{2(\hat{D}-G)^2} \\
B_2 &= \frac{2\hat{a}_0 a_2 + a_1^2}{4(\hat{D}-G)^2} \\
B_3 &= \frac{\hat{a}_0 a_3 + a_1 a_2}{2(\hat{D}-G)^2}.
\end{aligned}$$

Therefore, we have the following asymptotic formulas:

$$\lambda_n^{(2)} = k_1^2 + B_0 + O\left(\frac{1}{n}\right) \tag{42}$$

$$= k_1^2 + B_0 + \frac{B_1}{B_0} + O\left(\frac{1}{n^3}\right) \tag{43}$$

$$= k_1^2 + B_0 + \frac{B_1}{B_0 + B_1/B_0} + \frac{B_2}{B_0^2} + O\left(\frac{1}{n^5}\right). \tag{44}$$

In [19], Rogier and De Zutter derived leading order asymptotic results corresponding to (42) for microstrip substrates.

The possibility of a sequence in the lower half of the complex plane should also be considered. If $\lambda_n = |\lambda_n| e^{i\theta_n}$ and $-\pi < \theta_n \le 0$, then (38) and (39) become

$$\begin{aligned} G\sqrt{k_1^2 - \lambda_n} &= G|\lambda_n|^{1/2} e^{i(\theta_n+\pi)/2} \left(1 - \frac{k_1^2}{\lambda_n}\right)^{1/2} \\ (\hat{D} - G)\sqrt{k_2^2 - \lambda_n} &= |\hat{D} - G| \cdot |\lambda_n|^{1/2} e^{i[(\theta_n+\pi)/2+\varphi]} \left(1 - \frac{k_2^2}{\lambda_n}\right)^{1/2}, \end{aligned}$$

where φ is the phase angle of $\hat{D} - G$ as in (40). If $\theta_n \to \theta_*$ and $|\lambda_n| \to \infty$, we look for conditions on θ_* such that the imaginary parts of the above two terms do not both tend to infinity. Since

$$0 < \varphi < \frac{\theta_n + \pi}{2} + \varphi < \frac{\pi}{2} + \varphi < \pi,$$

it is clear that the imaginary part of the second term always tend to infinity. Therefore, the only possibility is $\theta_* = -\pi$. This leads to the same equation (15) as the case of $\lim \theta_n = \pi$. However, as we have discussed in section 2, equation (15) has no solution sequence in the lower half plane.

To verify the asymptotic formulas, we consider the Pekeris waveguide defined in (23) and (24), and use the following PML parameters:

$$H = 70\,m, \quad D = 80\,m, \quad \sigma(z) = \frac{10\tau^3}{1+\tau^2}, \quad \tau = \frac{z-H}{D-H} \quad \text{for} \quad z > H.$$

In Fig. 1, we show eigenvalues near k_1^2. The exact eigenvalues are calculated from (37) and they are marked by the small circles. These solutions are independently verified using a direct numerical discretization of (28-36). The two circles on the real line (near 1) are the propagating modes. The asymptotic solutions (21) and (43) are marked by the small $+$ and $\times$, respectively. In Table 2, we list the first a few exact solutions of (37) and the relative errors of the asymptotic solutions (20), (21), (42) and (43). It is clear that the asymptotic formulas are extremely accurate.

4 Conclusions

In this paper, we derived high order asymptotic solutions for the leaky modes and PML modes in a Pekeris waveguide. For a Pekeris waveguide terminated by a finite PML, three different classes of modes are identified. They correspond to the finite number of trapped modes, the infinite sequence of leaky modes and an infinite sequence of modes which approximate the branch line integral along the Pekeris cut. Asymptotic formulas for the two infinite sequences are established. For both sequences, the asymptotic formulas

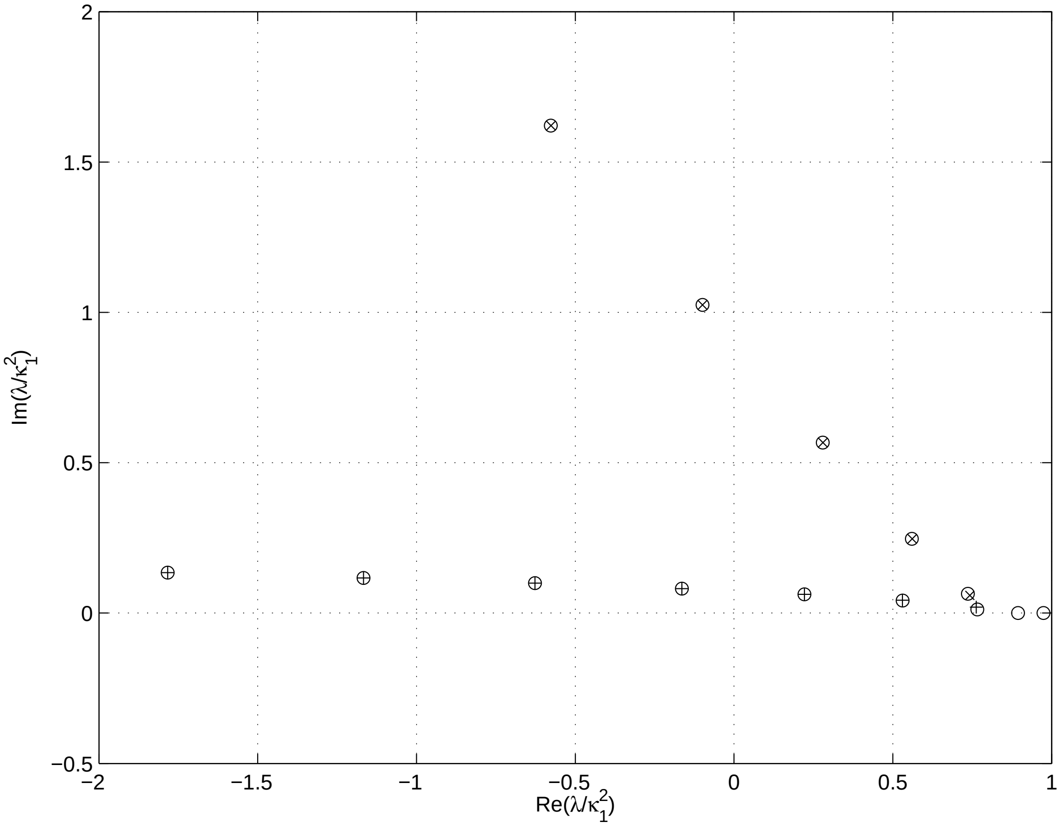

Figure 1: Comparison of exact and approximate eigenvalues for a Pekeris waveguide with a PML.

appear to be extremely accurate. To use a PML in the step-wise coupled mode method for solving a general range-dependent problems, an efficient method to calculate the PML modes is needed. Asymptotic solutions are useful, because they provide good approximations which can be improved (if necessary) by a direct numerical method.

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Exact $\sqrt{\lambda_n}/k_1$	R.E. (20)	R.E. (21)	Exact $\sqrt{\lambda_n}/k_1$	R.E. (42)	R.E. (43)
0.8752 + 0.0068i	0.0196	0.0056	0.8588 + 0.0374i	0.0137	0.0055
0.7293 + 0.0285i	0.0155	0.0011	0.7654 + 0.1612i	0.0150	0.0009
0.4751 + 0.0655i	0.0266	0.0007	0.6752 + 0.4199i	0.0121	0.0002
0.0975 + 0.4167i	0.0268	0.0002	0.6820 + 0.7513i	0.0060	0.0000
0.0625 + 0.7942i	0.0066	0.0000	0.7562 + 1.0717i	0.0030	0.0000

Table 2: Normal modes in a Pekeris waveguide terminated below by a PML and relative errors of the asymptotic formulas (20), (21), (42) and (43). The modes related to the leaky modes are listed in the left. The first a few modes of the third class are listed in the right. Both the exact horizontal wavenumber (scaled by k_1) and the relative errors of the asymptotic formulas are listed.

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