Wavelets on the Interval: A Short Survey

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Abstract. The construction of wavelets on intervals has garnered significant attention, and there are currently two primary approaches employed in this area of research. One approach involves obtaining the wavelet on the interval by reconstructing the boundary function using multi-resolution analysis, starting from wavelets defined on the real line $\mathbb{R}$. This approach was initially proposed by Meyer and subsequently refined by Cohen. More recently, Han extended this approach to encompass biorthogonal multi-wavelets. The second approach involves constructing a spline function as a scaling function from a knot sequence, which allows for the definition of the function itself on the interval. Additionally, wavelets on intervals or their extensions, such as non-uniform meshes and manifolds, have been considered in more generalized settings. Our aim is to provide a comprehensive summary of these results, offering a better understanding of the developmental trajectory of wavelets on intervals. This summary will not only facilitate further investigation in this topic but also aid in the practical application of wavelets on intervals.

Keywords: wavelets on intervals, boundary function, multi-wavelets

1 Introduction

The theory of wavelet analysis has received a great deal of attention since it was introduced by many pioneers, see[10,18,23,45,43], and many references therein. In view of their desirable properties, wavelets were soon used in signal analysis, differential equations, finite element, and many other areas, see [7], [21], [27], [28], [36], [38], [46], [47], [54], and many references therein. Typically, wavelets are bases for $L^2(\mathbb{R}^d)$ and are constructed based on the multiresolution analysis. However, in practice, such as the signal/image processing, data are defined on a bounded domain Ω, e.g., $\Omega = [0, 1]$, the unit interval. The wavelets constructed on the real line must be adapted to the bounded domain. One way is to utilize boundary/periodic extension on the data, which could bring undesirable artifacts into the data near the boundary after processing. Another way is to consider building the wavelets on the bounded domain directly, which could avoid the unnatural extension of the data. In this survey, we provide a comprehensive summary of wavelets on the interval.

1.1 An overview

A direct idea is to construct the wavelet bases of $L^2[0,1]$ based on the existing bases of $L^2(\mathbb{R})$. The first one to put this idea into practice was Meyer [44], who constructed wavelets at the boundary by the Gram-Schmidt method. These boundary wavelets, combined with those wavelets originally within the domain, formed the basis for $L^2[0,1]$. Jouini and Lemarié-Rieusset [34] further extended this construction to the case of biorthogonal wavelets. However, Meyer's construction encounters certain challenges. First of all, due to the inequality of the number of scaling functions and wavelet functions, filters Firstly, an inequality between the number of scaling functions and wavelet functions renders its filters unsuitable for wavelet packet constructions. Additionally, a more significant issue arises in practical scenarios where a wavelet function's support slightly increases, leading to an uncontrollable condition number for the orthogonalized matrix. In light of these concerns, Cohen et al. [14] and Andersson et al. [3] provided an alternative construction for the boundary wavelets to address these issues. Williams and Amaratunga [55] and Madych [42] later improved the construction of the boundary wavelets by leveraging the two-scale relation. Grivet-Talocia and Tabacco [22] proposed the construction of a biorthogonal wavelet system based on Cohen et al.'s work [14]. Altuřk and Keinert [1], on the other hand, proposed an alternative method for constructing the boundary wavelet function using a different boundary recurrence relation. In subsequent studies, many researchers dedicated their efforts to exploring multi-wavelets such as [2], [20], [24], [29], and [35]. After Chui et al. [11] introduced non-stationary wavelet frames on bounded intervals, Zhu et al. [58] proposed the construction of a multi-wavelet frame on the interval, and Han and Michelle [25] drew inspiration from the folding operation in [14] to construct a biorthogonal wavelets/framelets on the interval. Han and Michelle in [26] provided a general framework on the construction of compactly supported (bi-)orthogonal (multi-)wavelets on intervals.

Wavelets on the interval through the spline functions have undergone extensive development over a significant period of time [9], [37], [41], especially Lyche and Mørken's construction of spline wavelets [41] has provided subsequent researchers with valuable inspiration for wavelet construction on intervals. Chui and Quak [12] proposed a method for constructing a semi-orthogonal wavelet basis on $L^2[0,1]$ based on spline functions. Quak and Weyrich [50] further enhanced the results in [12] by improving the decomposition algorithm, while Jia [31] revised the result theoretically. Moreover, unlike the two approaches mentioned above, Plonka et al. [48] constructed the basis of $L^2_w[-1,1]$ using the Chebyshev transform. subsequent research on the construction of spline wavelets on intervals has been influenced by Dahmen et al. [16], who first proposed the construction of a biorthogonal spline wavelet that satisfies a stability condition, and the result is generalized to the case of multi-wavelets [15]. Such a result seems too complicated for practical applications, and there are many works on improving it, such as [8], [33], [49], and [51]. At the same time, wavelet constructions that satisfy stability conditions on the interval can be used to obtain numerical solutions to the partial differential equations, see [4], [30], [32].

Besides the development of wavelets on intervals, many researchers also focus on more general conditions and domains such as non-uniform meshes, manifolds, graphs, etc., for example, see [56]. Dahmen and Schneider [17] proposed that wavelets on the manifold can be reduced to the construction of wavelets on the local path with mild boundary conditions, and constructed n-dimensional cube that satisfies this condition. Later Bownik et al. [6] gave an alternative way to construct wavelets on the manifold. Stevenson [52] applied biorthogonal wavelets on non-uniform mesh, while Bitter and Brachtendorf [5] gave the algorithm of how to achieve spline wavelets on non-uniform grids. In addition to this, Dijkema and Stevenson [19] and Stevenson [53] proposed the construction of wavelets on the hypercube, Li et al. [39] constructed wavelets on the sphere, and Zheng and Zhuang [57] constructed wavelets/framelets on graphs to address graph-related problems.

1.2 Structure of the paper

The structure of this survey is as follows.

In Section 2, we provide a concise overview of fundamental definitions and properties of multi-resolution analysis on the real line. Additionally, we introduce the concept of biorthogonal wavelets, which offer increased degrees of freedom compared to orthogonal wavelets. Consequently, one can construct wavelet systems with higher vanishing moments while relaxing the constraint of orthogonality, allowing for the inclusion of additional properties tailored to specific applications. The section concludes with a brief delineation of the properties of multi-resolution analysis on $L^2[0,1]$.

In Section 3, we focus on the approach of restricting wavelets from the real line to the interval. We start in Section 3.1 with an overview of Meyer's construction in [44], which reduces the problem to the construction of wavelet bases on a half-space, namely $L^2[0,\infty)$. We describe how to construct biorthogonal wavelets on intervals along this line by Cohen et al., in Section 3.2, as well as those similar results from Grivet-Talocia and Tabacco [22] in Section 3.3. In Section 3.4 we explore Han and Michelle's construction [24] for biorthogonal wavelets on the interval using the folding operator as well as their direct approach [26] on how to construct compactly supported (bi-)orthogonal (multi-)wavelets on the intervals in Section 3.5.

In Section 4, we focus on another approach of constructing wavelets on the interval through spline functions. We begin in Section 4.1 by presenting Chui and Quak's construction of semi-orthogonal wavelet bases [12], which references Lyche's construction of a spline wavelet basis [41]. Additionally, we provide a brief overview of Quak and Weyrich's improvement of the decomposition algorithm [50]. In Section 4.2, we present Dahmen et al.'s construction of a biorthogonal spline wavelet basis [16], and it is worth noting that this wavelet exhibits a certain level of stability by sacrificing orthogonality while achieving higher vanishing moments. However, the construction in [16] appears to be too complicated in practice, and we present in Section 4.3 an alternative construction by Primbs [49]. In Section 4.4, we introduce Dahmen et al.'s construction of biorthogonal multi-wavelets on the interval [15], which comes from the Hermite cubics splines. After this, we describe in Section 4.5 Jia's work [33] further on this construction so that the resulting wavelets can be better used for solving partial differential equations.

In Section 5, we explore wavelets on more general domains or with more general conditions, commencing with the construction of tensor product spline wavelets. We discuss Lyche et al.'s results [40] in Section 5.1, which also provide the wavelet construction based on triangulation. In Section 5.2, we present Dahmen and Schneider's wavelet construction [17] on an n-dimensional cube. Furthermore, in Section 5.3, we discuss the results of Stevenson [52], who constructs biorthogonal wavelets on non-uniform meshes using the aforementioned triangulation technique.

We provide the conclusions and further remarks in Section 6.

2 Preliminariles

In this section, we briefly describe some basic concepts and definitions that will be used in subsequent sections and illustrate the notation.

2.1 Multi-resolution analysis on $L^2(\mathbb{R})$

We first introduce the scaling function, the multiresolution analysis (MRA), and the wavelet function.

Definition 1. *A function $\phi \in L^2(\mathbb{R})$ is called a scaling function, if the subspaces V_j of $L^2(\mathbb{R})$, defined by*

$$V_j := clos_{L^2(\mathbb{R})}\langle \phi_{j,k} : k \in \mathbb{Z}\rangle := \overline{\langle \phi_{j,k} : k \in \mathbb{Z}\rangle}, \quad j \in \mathbb{Z},$$

where $\langle\,\rangle$ denotes the linear span and $\phi_{j,k}(x) := 2^{j/2}\phi(2^j x - k)$, $j,k \in \mathbb{Z}$, satisfy the following properties:

(1) $\cdots \subset V_{-1} \subset V_0 \subset V_1 \subset \cdots$;
(2) $clos_{L^2(\mathbb{R})}(\cup_{j\in\mathbb{Z}} V_j) = L^2(\mathbb{R})$;
(3) $\cap_{j\in\mathbb{Z}} V_j = \{0\}$;
(4) $f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \Leftrightarrow f(x + 2^{-j}) \in V_j, \quad j \in \mathbb{Z}$,

and if $\{\phi(\cdot - k) : k \in \mathbb{Z}\}$ is a Riesz basis of V_0, i.e., there exist constants A and B with $0 < A \le B < \infty$ such that

$$A\|\{c_k\}\|_{l^2}^2 \le \left\| \sum_{k\in\mathbb{Z}} c_k \phi(\cdot - k) \right\|_2^2 \le B\|\{c_k\}\|_{l^2}^2, \tag{2.1}$$

for any $\{c_k\} \in l^2$. We also say that the scaling function ϕ generates a multiresolution analysis $\{V_j\}$ of $L^2(\mathbb{R})$.

Thus, one can obtain a nested sequence $\{V_j\}$ of subspaces from a scaling function ϕ, and if one consider the orthogonal complement subspace W_j of V_j related to V_{j+1}, one can have an orthogonal decomposition of $L^2(\mathbb{R})$, i.e., $L^2(\mathbb{R}) = \oplus_{j\in\mathbb{Z}} W_j$, where $\oplus$ denotes the orthogonal sum.

Definition 2. *A function $\psi \in L^2(\mathbb{R})$ is a wavelet dual to the scaling function if the subspaces W_j of $L^2(\mathbb{R})$ defined by*

$$W_j := clos_{L^2(\mathbb{R})}\langle \psi_{j,k} : k \in \mathbb{Z}\rangle, \quad j \in \mathbb{Z},$$

satisfy $V_{j+1} = V_j \oplus W_j$, $j \in \mathbb{Z}$, where $\psi_{j,k}(x) := 2^{j/2}\psi(2^j x - k)$, $j,k \in \mathbb{Z}$.

A famous family of orthogonal wavelets is constructed by Daubechies [18] with the scaling function ϕ and ψ have support width $2N-1$ and order N vanishing moment of ψ.

2.2 Biorthogonal decomposition on $\mathbb{R}$

A scaling function $\phi \in L^2(\mathbb{R})$ is associated with a mask $\mathbf{a} := \{a_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ by the following refinement equation (two-scale relation):

$$\phi(x) = \sum_{k\in\mathbb{Z}} a_k \phi(2x-k). \tag{2.2}$$

We say that two scaling functions $\phi, \tilde{\phi}$ form a dual pair if

$$(\phi, \tilde{\phi}(\cdot - k))_{\mathbb{R}} = \delta_{0,k}, \quad k \in \mathbb{Z}, \tag{2.3}$$

where $(\cdot,\cdot)_{\mathbb{R}}$ denote the usual L^2 inner product on the whole real line, and δ is the Kronecker symbol:

$$\delta_{i,j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$

We say that ϕ is *exact* of order d if all polynomials with at most degree $d-1$ can be represented by the linear combinations of integer translation $\phi(\cdot - k)$. More precisely, define

$$\alpha_{\tilde{\phi},r}(y) := ((\cdot)^r, \tilde{\phi}(\cdot - y))_{\mathbb{R}}. \tag{2.4}$$

Thus, from the definition of a dual pair, we have

$$x^r = \sum_{k\in\mathbb{Z}} \alpha_{\tilde{\phi},r}(k)\phi(x-k), \quad r = 0,\ldots,d-1. \tag{2.5}$$

Let ϕ and $\tilde{\phi}$, with mask $\mathbf{a}$ and $\tilde{\mathbf{a}}$, generate multi-resolution analysis $\{V_j\}$ and $\{\tilde{V}_j\}$, respectively. Then, to construct the biorthogonal decomposition of $L^2(\mathbb{R})$, we need to find the complement subspace W_j and $\tilde{W}_j$ of V_j and $\tilde{V}_j$ related to V_{j+1} and $\tilde{V}_{j+1}$, respectively, satisfying

$$W_j \perp \tilde{V}_j, \quad \tilde{W}_j \perp V_j.$$

Thus,

$$W_j \perp \tilde{W}_r, \quad j \neq r.$$

These subspaces W_j and $\tilde{W}_j$ can be generated by functions:

$$\psi(x) := \sum_{k\in\mathbb{Z}} b_k \phi(2x-k), \quad \tilde{\psi}(x) := \sum_{k\in\mathbb{Z}} \tilde{b}_k \tilde{\phi}(2x-k), \tag{2.6}$$

where

$$b_k := (-1)^k \tilde{a}_{1-k}, \quad \tilde{b}_k := (-1)^k a_{1-k}, k \in \mathbb{Z}. \tag{2.7}$$

Meanwhile, as for biorthogonal decomposition of $L^2(\mathbb{R})$, it should satisfy

$$(\phi, \tilde{\psi}(\cdot - k))_{\mathbb{R}} = (\tilde{\phi}, \psi(\cdot - k))_{\mathbb{R}} = 0, \quad (\psi, \tilde{\psi}(\cdot - k))_{\mathbb{R}} = \delta_{0,k}, \quad k \in \mathbb{Z}. \tag{2.8}$$

Note that if $\tilde{\phi}$ is exact order of $\tilde{d}$, with the properties of complement subspace, then ψ has $\tilde{d}$ *vanishing moment*, i.e.,

$$\int_{\mathbb{R}} x^r \psi(x) \mathrm{d}x = 0, \quad r = 0, \dots, \tilde{d} - 1. \tag{2.9}$$

2.3 Multi-resolution analysis on $L^2[0,1]$

For a multiresolution analysis adapted to the interval $[0,1]$, the sequence of nested subspaces cannot be bi-infinite. Thus, there must exist an initial subspace $V_0^{[0,1]}$ and the nested sequence becomes:

$$V_0^{[0,1]} \subset V_1^{[0,1]} \subset \cdots$$

with

$$\mathrm{clos}_{L^2}(\cup_{j\geq 0} V_j^{[0,1]}) = L^2[0,1],$$

and the orthogonal complement subspaces $W_j^{[0,1]}$ satisfying

$$V_{j+1}^{[0,1]} = V_j^{[0,1]} \oplus W_j^{[0,1]}, \quad j \in \mathbb{N}.$$

Hence,

$$L^2[0,1] = V_0^{[0,1]} \oplus_{j\in\mathbb{N}} W_j^{[0,1]}.$$

Since outside the given interval, our target function does not have a definition, and simple truncation cannot guarantee a basis for $W_j^{[0,1]}$, one necessarily needs to reconsider/reconstruct the wavelet functions on $\mathbb{R}$ that cross the boundary.

3 Wavelets on the interval via restriction

In this section, we focus on the approach of constructing wavelets on the interval through restriction of wavelet systems on $\mathbb{R}$ and the careful treatment of the boundary elements.

3.1 Meyer's approach

We first discuss Meyer's construction in [44].

For the interval $[0,1]$, by symmetry and considering sufficiently large scales, we only need to consider the case at the left boundary, i.e., we can obtain wavelets on the interval by constructing wavelets on the half-space $L^2[0,\infty)$. We assume that ϕ and ψ are from the Daubechies orthogonal family of N order vanishing moment, and both ϕ and ψ have been shifted such that $\mathrm{supp}(\phi) = \mathrm{supp}(\psi) = [-N+1, N]$. Define

$$\phi_{j,k}^{\mathrm{half}}(x) = \begin{cases} \phi_{j,k}(x), & x \geq 0, \\ 0, & x < 0, \end{cases} \tag{3.1}$$

$$V_j^{\mathrm{half}} = \overline{\langle \phi_{j,k}^{\mathrm{half}} : k \in \mathbb{Z} \rangle}, \quad j \in \mathbb{Z}. \tag{3.2}$$

Then for all $j, k \in \mathbb{Z}$, $\mathrm{supp}(\phi_{j,k}) = [2^{-j}(-N+k+1), 2^{-j}(N+k)]$. If $k \leq -N$ or $K \geq N-1$, then the support of $\phi_{j,k}$ does not overlap with the 0-boundary. Thus, one only needs to consider $\phi_{j,k}$ for $-N+1 \leq k \leq N-2$.

As in Section 2.3, there must be an initial subspace V_0^{half}. Since $\phi_{j,k}, -N+1 \leq k \leq N-2$ are all independent, and orthogonal to $\phi_{j,m}, m \geq N-1$. Thus, we orthonormalize the $\phi_{0,k}^{\mathrm{half}}, -N+1 \leq k \leq N-2$ by Gram-Schmidt procedure. The resulted functions are denoted by $\phi_{0,k}^{\mathrm{left}}, -N+1 \leq k \leq N-2$, and we have

$$\phi_{0,k}^{\mathrm{left}} = \sum_{l=-N+1}^{N-2} A_{k,l} \phi_{0,l}^{\mathrm{half}}, \quad k = -N+1, \ldots, N-2, \tag{3.3}$$

where $A = (A_{k,l})$ is an invertible matrix with $\dim(A) = 2N-2$. That is, each $\phi_{0,k}^{\mathrm{left}}$ is the linear combination of $\phi_{0,l}^{\mathrm{half}}$, $-N+1 \leq k \leq N-2$. Consequently, $\{\phi_{0,k}^{\mathrm{left}} : -N+1 \leq k \leq N-2\} \cup \{\phi_{0,l}^{\mathrm{half}} : l \geq N-1\}$ serves as a basis of V_0^{half}.

We next consider W_j^{half} and naturally, it needs to be satified with $W_j^{\mathrm{half}} = (V_{j+1}^{\mathrm{half}}) \cap (V_j^{\mathrm{half}})^{\perp}$. Using a similar technique, we can obtain the $\psi_{j,k}^{\mathrm{half}}$. As pointed out in [14], one has the facts that (i) $\psi_{j,m}^{\mathrm{half}} \in W_j^{\mathrm{half}}$ for $m \geq N-1$; (ii) $\mathrm{Proj}_{W_j^{\mathrm{half}}} \psi_{j,k}^{\mathrm{half}} = 0$ for $-N+1 \leq k \leq -1$; and (iii) The functions

$$\tilde{\psi}_{0,k}^{\mathrm{half}} = \mathrm{Proj}_{W_0^{\mathrm{half}}} \psi_{0,k}^{\mathrm{half}} = \psi_{0,k}^{\mathrm{half}} - \sum_{l=-N+1}^{N-2} \langle \psi_{0,k}^{\mathrm{half}}, \phi_{0,l}^{\mathrm{left}} \rangle \phi_{0,l}^{\mathrm{left}}, \tag{3.4}$$

for $0 \leq k \leq N-2$ are non-vanishing, indepdenent, and orthogonal to the interior $\psi_{j,m}^{\mathrm{half}}$ with $m \geq N-1$. Thus, using the three facts, we can orthonormalize $\tilde{\psi}_{0,k}^{\mathrm{half}}$, $k = 0, \ldots, N-2$, to obtain

$$\psi_{0,k}^{\mathrm{left}} = \sum_{l=0}^{N-2} B_{k,l} \psi_{0,l}^{\mathrm{half}} + \sum_{l=-N+1}^{N-2} C_{k,l} \phi_{0,l}^{\mathrm{left}}, \quad k = 0, \ldots, N-2. \tag{3.5}$$

Therefore, $\{\psi_{0,k}^{\mathrm{left}} : 0 \leq k \leq N-2\} \cup \{\psi_{0,l}^{\mathrm{half}} : l \geq N-1\}$ is a basis of W_0^{half}.

So far, we have obtained the basis of V_0^{half} and W_0^{half}. We next discuss the two-scale relation for their connection to V_1^{half}. Since $\phi_{0,k}^{\text{half}} = \sum_m h_m \phi_{1,2k+m}^{\text{half}}$, $k \geq N-1$, where $h_m = (\phi, \phi_{1,m})_{\mathbb{R}}$. This equation together with (3.3) implies that

$$\phi_{0,k}^{\text{left}} = \sum_{l=-N+1}^{N-2} H_{k,l}^{\text{left}} \phi_{1,l}^{\text{left}} + \sum_{m \geq N-1} h_{k,m}^{\text{left}} \phi_{1,m}^{\text{half}}, \tag{3.6}$$

where

$$H_{k,l}^{\text{left}} = \sum_{r=-N+1}^{N-2} \sum_{m=-2k-N+1}^{-2k+N-2} A_{k,r} h_m (A^{-1})_{2k+m,l}$$

and

$$h_{k,m}^{\text{left}} = \sum_{r=-N+1}^{N-2} A_{k,r} h_{m-2r}.$$

Similarly, from $\psi_{0,k}^{\text{half}} = \sum_m g_m \phi_{1,2k+m}^{\text{half}}$, $k \geq N-1$, where $g_m = (\psi, \phi_{1,m})_{\mathbb{R}}$, together with (3.5) it follows that

$$\psi_{0,k}^{\text{left}} = \sum_{l=-N+1}^{N-2} G_{k,l}^{\text{left}} \phi_{1,l}^{\text{left}} + \sum_{m \geq N-1} g_{k,m}^{\text{left}} \phi_{1,m}^{\text{half}}, \tag{3.7}$$

where

$$G_{k,l}^{\text{left}} = \sum_{r=0}^{N-2} \sum_{-2k-N+1}^{-2k+N-2} B_{k,r} g_m (A^{-1})_{2k+m,l} + \sum_{-N+1}^{N-2} C_{k,s} H_{s,l}^{\text{left}},$$

and

$$g_{k,m}^{\text{left}} = \sum_{r=0}^{N-2} B_{k,r} g_{m-2r} + \sum_{s=-N+1}^{N-2} C_{k,s} h_{k,m}^{\text{left}}.$$

The above construction is invariant for dilation of x by 2^j. Thus we get the basis of the V_j^{half} and W_j^{half} by simply replacing 0 by j, respectively. Moreover, the two-scale relations in (3.6) and (3.7) are also valid if we replace $0, 1$ by $j, j+1$, respectively. Thus, with the help of the Gram-Schmidt method and refinement relations, we can obtain the multi-resolution analysis in half space $L^2[0, \infty)$. Due to the symmetry, we can get the multi-resolution analysis of the half-space $L^2(-\infty, 1]$ in the same way, and consequently, we obtain the wavelet basis on the interval $[0, 1]$.

However, Meyer's construction faces two primary challenges. Firstly, the number of scaling functions and wavelet functions with supports entirely contained within the interval is $2^j - 2N + 2$. However, Meyer requires $2N - 2$ scaling functions at each boundary, which does not match the number of $N - 1$ boundary wavelet functions. Consequently, in the interval $[0, 1]$, we have a total of $2^j + 2N - 2$ scaling functions but only 2^j wavelet functions. This discrepancy renders Meyer's filters unsuitable for wavelet packet construction. Furthermore, a more critical issue arises. In the Gram-Schmidt process described earlier, we need to compute the inverse of the matrix A. However, Cohen et al. in [14] pointed out that when $N \geq 5$, the condition number of the matrix A becomes excessively large. This high condition number poses a significant hindrance to the practical application of Meyer's construction. We next discuss the improvement of Meyer's construction by Cohen et al. in [14].

3.2 Cohen et al.'s improvement

Similarly to Meyer's construction [44], Cohen et al. in [14] retained those scaling functions whose support falls entirely within the interval, but in order to allow the number of scaling functions to be consistent with the number of wavelet functions, they adopted a different construction of the boundary functions. Note that the previous constructions about subsequent hierarchies are obtained based on the two-scale relation. So one just needs to reconstruct the boundary function of the initial space V_0^{half}.

Starting from the already existing interior scaling functions, Cohen et al. in [14] noticed that these functions cannot reproduce constants on $[0, \infty)$. With this in mind, they defined a boundary function ϕ^0:

$$\phi^0 := 1 - \sum_{k=N-1}^{\infty} \phi_{0,k}. \tag{3.8}$$

This ϕ^0 with those interior scaling functions would be able to reproduce all constant-valued functions in the half space. At the same time, since

$$\sum_{k=-\infty}^{\infty} \phi_{0,k} = 1, \tag{3.9}$$

Thus,

$$\phi^0 = \sum_{k=-N+1}^{N-2} \phi_{0,k}. \tag{3.10}$$

This shows that ϕ^0 is a linear combination of $\phi_{0,k}, -N+1 \leq k \leq N-2$ like the boundary function constructed by Meyer, so it also has the same properties as the original boundary function. In [14], it illustrates that such a construction does not disrupt the multi-resolution hierarchy, i.e.,

$$\langle \phi^0, \phi_{0,k} : k \geq N-1 \rangle \subset \langle \phi^0(2\cdot), \phi_{1,k} : k \geq N-1 \rangle. \tag{3.11}$$

More generally, one can construct

$$\phi^i := \sum_{k=i}^{2N-2} \binom{k}{i} \phi(\cdot + k - N + 1), i = 0, \ldots, N-2,$$

to reproduce polynomial of degree up to $N-2$, and the space

$$\overline{\langle \phi^i, \phi_{0,k} : 0 \leq i \leq N-2, k \geq N-1 \rangle}$$

then has exactly $N-1$ boundary scaling functions matching the number of $N-1$ boundary wavelet functions. However, such a construction has its limitations; that is, while Meyer's construction can reproduce polynomials of degrees up to $N-1$, this construction can, at most, represent polynomials with order not larger than $N-2$. To recover the one extra lost degree, Cohen et al. in [14] shows that one can include ϕ^{N-1} while excluding $\phi_{0,N-1}$ leading to the space V_0^{half} that can be built from

$$\overline{\langle \phi^i, \phi_{0,k} : 0 \leq i \leq N-1, k \geq N \rangle},$$

and the space W_0^{half} from

$$\overline{\langle \psi^i, \psi_{0,k} : 0 \leq i \leq N-1, k \geq N \rangle},$$

through orthonormalization, both of which have N boundary functions.

3.3 Grivet-Talocia and Tabacco's biorthogonal wavelets

By relaxing the orthogonality condition, one can have a higher degree of freedom and thus have more desired properties of the wavelet systems. We present in this section how to construct biorthogonal wavelets on the interval [0,1] based on Grivet-Talocia and. Tabacco's results in [22]. Similar to the previous process, we only consider the case at the left boundary point 0, while letting $j = 0$. Let us consider two scaling functions ϕ and $\tilde{\phi}$ that form a dual pair. Without loss of generality, we can assume $\text{supp}(\phi) = [-N+1, N] \subset \text{supp}(\tilde{\phi}) = [-\tilde{N}+1, \tilde{N}]$.

Following Meyer [44] and Cohen et al. [14], the support of all the scaling functions $\phi_{0,k}$ with $k \geq N-1$ falls entirely within the half line $[0, \infty)$. And we can reproduce all polynomials of order up to $N-2$ on a half space $L^2[0, \infty)$ by constructing $N-1$ boundary scaling functions. Since $\tilde{N} \geq N$, we can define

$$V_0^B := \langle \phi_{0,k} : 0 \leq k \leq \tilde{N}-2 \rangle, \quad V_0^I := \langle \phi_{0,k} : k \geq \tilde{N}-1 \rangle, \tag{3.12}$$

where B denotes the boundary and I denotes the interior. The scaling functions $\phi_{0,k}$ with $0 \leq k \leq \tilde{N}-2$ are then the boundary functions that we need to take care of. Note that if $\tilde{N} > N$, some of the interior functions $\phi_{0,k}$ for $N-1 \leq k \leq \tilde{N}-2$ are treated as boundary elements. Similarly, we can define:

$$\tilde{V}_0^B := \langle \tilde{\phi}_{0,k} : 0 \leq k \leq \tilde{N}-2 \rangle, \quad \tilde{V}_0^I := \langle \tilde{\phi}_{0,k} : k \geq \tilde{N}-1 \rangle, \tag{3.13}$$

and with this, we have

$$V_0^{\text{half}} = V_0^B \oplus V_0^I, \quad \tilde{V}_0^{\text{half}} = \tilde{V}_0^B \oplus \tilde{V}_0^I. \tag{3.14}$$

Since V_0^I is originally biorthogonal to $\tilde{V}_0^I$, and

$$V_0^B \perp \tilde{V}_0^I, \quad V_0^I \perp \tilde{V}_0^B, \tag{3.15}$$

Hence, the remaining problem is to reconstruct the basis of each V_0^B and $\tilde{V}_0^B$ so that the biorthogonality condition, i.e., $V_0^B \perp \tilde{V}_0^B$, is satisfied.

In fact, the new basis can be represented as a linear combination of the original basis. That is,

$$V_0^B = \langle \varphi_{0,k} : 0 \le k \le \tilde{N}-2 \rangle, \quad \tilde{V}_0^B = \langle \tilde{\varphi}_{0,k} : 0 \le k \le \tilde{N}-2 \rangle, \tag{3.16}$$

$$\varphi_{0,k} = \textstyle\sum_{k=0}^{\tilde{N}-2} A_{k,m}\phi_{0,m}, \qquad \tilde{\varphi}_{0,k} = \displaystyle\sum_{k=0}^{\tilde{N}-2} \tilde{A}_{k,m}\tilde{\phi}_{0,m}. \tag{3.17}$$

Thus, to get the entire biorthogonal system, one needs to guarantee that

$$biort(\varphi_{0,k}, \tilde{\varphi}_{0,l})_{\mathbb{R}^+} := \int_0^\infty \varphi_{0,k}(x)\tilde{\varphi}_{0,l}(x)\mathrm{d}x = \delta_{k,l}, \quad 0 \le k, l \le \tilde{N}-2. \tag{3.18}$$

Let Γ be the Gramian matrix of components [22]:

$$\Gamma_{k,l} = (\phi_{0,k}, \tilde{\phi}_{0,l})_{\mathbb{R}^+}, \quad 0 \le k, l \le \tilde{N}-2. \tag{3.19}$$

Then, the problem about the biorthogonality becomes finding two matrices A and $\tilde{A}$ such that

$$A\Gamma\tilde{A}^\top = I, \tag{3.20}$$

where $(\cdot)^\top$ is the matrix transpose and I is the identity matrix.

Utilize similar technique as in (3.4), we can obtain a wavelet basis in half space W_0^{half} and $\tilde{W}_0^{\text{half}}$. Similar to the previous illustration, we have

$$W_0^{\text{half}} = W_0^B \oplus W_0^I, \quad \tilde{W}_0^{\text{half}} = \tilde{W}_0^B \oplus \tilde{W}_0^I, \tag{3.21}$$

$$W_0^B \perp \tilde{W}_0^I, \qquad W_0^I \perp \tilde{W}_0^B. \tag{3.22}$$

The biorthogonality can be satisfied by constructing new basis for each of W_0^B and $\tilde{W}_0^B$.

We have completed the construction of biorthogonality on V_0^{half} and W_0^{half} and their dual spaces, respectively. Regarding the two-scale relations, one can utilize an approach similar to those in Section 3.1. With a similar construction for the half-space $L^2(-\infty, 1]$, we can complete the construction of biorthogonal wavelets on the interval $[0, 1]$.

3.4 Han and Michelle's folding operator

Unlike the previous approach of constructing new boundary functions to obtain (bi-)orthogonal wavelets on the interval, we can obtain wavelets on the interval by folding the original wavelets. However, such an operation requires that the original scaling function and the wavelet function satisfy either symmetry or anti-symmetry. Otherwise, biorthogonality will not be guaranteed.

Such an operation was first proposed by Cohen et al. in [14], and another way of folding was later given by Han [25]. Let us begin with Cohen et al.'s method. Define

$$f^{\text{fold}}(x) = \sum_{n\in\mathbb{Z}} [f(x-2n) + f(2n-x)]. \tag{3.23}$$

With this definition, one can know that

$$f^{\text{fold}}(-x) = f^{\text{fold}}(x), \quad f^{\text{fold}}(x+2k) = f^{\text{fold}}(x). \tag{3.24}$$

Let $\psi, \tilde{\psi}$ be two biorthogonal wavelets with corresponding scaling functions ϕ and $\tilde{\phi}$. As previously stated, to ensure that the biorthogonality is maintained after the folding is performed, we can assume (see [13]) that

$$\phi(x) = \phi(1-x) \text{ and } \psi(x) = -\psi(1-x), (\text{same for } \tilde{\phi} \text{ and } \tilde{\psi}). \tag{3.25}$$

In what follows, we assume the dual part and the original part have the same properties, and the conclusions obtained in the original part hold automatically for the dual part. From (3.24), one can derive that

$$\phi^{\text{fold}}_{j,k+2^{j+1}m}(x) = \phi^{\text{fold}}_{j,k}(x), \quad \phi^{\text{fold}}_{j,2^{j+1}-k-1}(x) = \phi^{\text{fold}}_{j,k}(x), j \geq 0, \tag{3.26}$$

which means that we only need to consider those terms with $0 \leq k \leq 2^j - 1$. Similar situation holds for $\tilde{\phi}$, ψ and $\tilde{\psi}$. As for the situation $j \leq -1$, one can show that

$$\phi^{\text{fold}}_{j,k}(x) = 2^{\frac{j}{2}} \sum_{n\in\mathbb{Z}} [\phi(2^j x - 2^{j+1}n - k) + \phi(2^{j+1}n - 2^j x - k)] = 2^{-\frac{j}{2}}, \tag{3.27}$$

and similarly, $\psi^{\text{fold}}_{j,k}(x) = 0$ for $j \leq -1$. Let the V^{fold}_j and W^{fold}_j denote the space spanned by $\phi^{\text{fold}}_{j,k}$ and $\psi^{\text{fold}}_{j,k}$, respectively (same for $\tilde{V}^{\text{fold}}_j$ and $\tilde{W}^{\text{fold}}_j$). Note that the function that is acted upon by the folding operator will be completely determined by the part in the interval $[0, 1]$, i.e.,

$$(f^{\text{fold}}, g)_{\mathbb{R}} = (f, g^{\text{fold}})_{\mathbb{R}} = (f^{\text{fold}}, g^{\text{fold}})_{[0,1]}, \forall f, g \in L^2(\mathbb{R}) \tag{3.28}$$

Thus, one can deduce that for $j \geq 0$,

$$V^{\text{fold}}_j \perp \tilde{W}^{\text{fold}}_j, \quad \tilde{V}^{\text{fold}}_j \perp W^{\text{fold}}_j, \quad V^{\text{fold}}_{j+1} = V^{\text{fold}}_j \oplus W^{\text{fold}}_j. \tag{3.29}$$

This is consistent with our discussion in Section 2.3. That is

$$L^2[0,1] = V^{\text{fold}}_0 \oplus_{j\in\mathbb{N}} W^{\text{fold}}_j = \tilde{V}^{\text{fold}}_0 \oplus_{j\in\mathbb{N}} \tilde{W}^{\text{fold}}_j. \tag{3.30}$$

Han and Michelle in [25] considered a more general folding operator and applied it to the case of multi-wavelets (multi-framelets). To distinguish between the two, we denote the function vectors on which Han and Michelle's folding operator acts as $(\cdot)^{\epsilon_1,\epsilon_2}$. Define

$$F_{c,\epsilon_1,\epsilon_2}(f) = \tilde{F} + \epsilon_1 \tilde{F}(c - \cdot) + \epsilon_2 \tilde{F}(c + 2 - \cdot) + \epsilon_1 \epsilon_2 \tilde{F}(2 + \cdot), \tag{3.31}$$

where $\tilde{F}(x) := \sum_{k\in\mathbb{Z}} f(x - 4k)$ and $\epsilon_1, \epsilon_2 \in \{-1, 1\}$ indicates symmetry or anti-symmetry. Similarly to the former folding operation in [14], such a folding operator is completely determined by the part in $[\frac{c}{2}, \frac{c}{2} + 1]$, and here we take

$c = 0$ in order to better compare it with that in [14]. Han and Michelle consider multiwavelets (multiframelets) and hence the scaling function and wavelet function are vector functions:

$$\phi = (\phi_1, \cdots, \phi_r)^\top \text{ and } \psi = (\psi_1, \cdots, \psi_s)^\top \text{ (same for } \tilde{\phi} \text{ and } \tilde{\psi}). \tag{3.32}$$

We assume they agree with the assumptions as in (3.24). Although [25] considers more scenarios, the results are similar, see [25] for more details. One can define $\Phi_j = \cup_{l=1}^r \phi_j^l$ (similar for $\tilde{\Phi}_j, \Psi_j$ and $\tilde{\Psi}_j$), where $\Phi_j^l = \{(\phi_{j,k}^l)^{\epsilon_1,\epsilon_2} : 0 \le k \le 2^j - 1\}$ and the system should have the properties similar to (3.28)–(3.30).

We can see that with the help of such a folding operator, we can simply obtain wavelets on the interval without any construction of the boundary functions in particular. However, the disadvantages of this method are obvious: on the one hand, it requires symmetry or anti-symmetry of the scaling function and wavelet function, and on the other hand, it can only be constructed if the endpoints of the target interval are integers. In the next section, we describe a more general approach which is also given by Han and Michelle in [26].

3.5 Han and Michelle's direct approach

We present Han and Michelle's recent work that utilize a direct approximation to biorthogonal wavelets on intervals without explicitly involving the dual part.

Let's start with the classical approach. The basic idea is still the same as Meyer [44]'s, by constructing wavelets in the half-space $L^2[0, \infty)$ and thus on the interval. We consider here biorthogonal multi-wavelets as in (3.32) as in the previous section. For biorthogonal multi-wavelets, their scaling and wavelet function vectors $\phi, \psi, \tilde{\phi}$, and $\tilde{\psi}$ are associated with matrix filters $a, \tilde{a}, b$, and $\tilde{b}$, respectively, whose dimension should satisfy $s = r$ and

$$\begin{bmatrix} \hat{\tilde{a}}(\xi) & \hat{\tilde{a}}(\xi + 2\pi) \\ \hat{\tilde{b}}(\xi) & \hat{\tilde{b}}(\xi + 2\pi) \end{bmatrix} \begin{bmatrix} \bar{\hat{a}}(\xi)^\top & \bar{\hat{b}}(\xi)^\top \\ \bar{\hat{a}}(\xi + \pi)^\top & \bar{\hat{b}}(\xi + \pi)^\top \end{bmatrix} = I_{2r}, \quad \xi \in \mathbb{R}, \tag{3.33}$$

where $\hat{a}(\xi) := \sum_k a(k) e^{ik \cdot \xi}$ with each $a(k)$ being a matrix of size $r \times r$ in $\mathbb{R}$.

Now we need to construct the function on the left boundary, which needs to satisfy the two-scale relation. One can obtain a two-scale relation similar to (3.6):

$$\phi^{\text{left}} = 2A_L \phi^{\text{left}}(x \cdot) + \sum_{k \ge n_\phi} A_k \phi_{1,k}, \tag{3.34}$$

where A_L, A_k are a matrices with appropriate sizes, n_ϕ denote the smallest integer such that with $\text{supp}(\phi(\cdot - k)) \subset [0, \infty)$ for all $k \ge n_\phi$. We have discussed the construction by Meyer [44] and Cohen et al. [14] for the left boundary function in the previous sections, and such a construction can follow well to the case of multi-wavelets. Let us be a little more specific. Keep in mind, we are considering multiple wavelets here, but we can consider each components in the wavelet vector ψ separately and utilize the previous results.

For A_L in (3.34), we can write it as a Jordan normal form

$$A_L = C^{-1}\mathrm{diag}(J_1, \cdots, J_r)C, \tag{3.35}$$

where

$$J_l = \begin{bmatrix} \lambda_l & 1 & 0 & \cdots & 0 \\ 0 & \lambda_l & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_l & 1 \\ 0 & 0 & \cdots & 0 & \lambda_l \end{bmatrix}, \quad \lambda_l \in \mathbb{C} \tag{3.36}$$

and define $(\phi^{\mathrm{left},l})_{1\le l\le r} = C\phi^{\mathrm{left}}$. This allows us to consider the components of the multi-wavelet independently, i.e.,

$$\begin{bmatrix} \phi^{\mathrm{left},1} \\ \vdots \\ \phi^{\mathrm{left},r} \end{bmatrix} = 2 \begin{bmatrix} J_1\phi^{\mathrm{left},1}(2\cdot) \\ \vdots \\ J_r\phi^{\mathrm{left},r}(2\cdot) \end{bmatrix} + 2\sum_{k\ge n_\phi} CA_k\phi_{0,k} \tag{3.37}$$

And after processing them separately we can then merge them and keep the original properties. Since the construction for multi-wavelets is similar, we illustrate the case with $r = 2$. For the vector functions $\phi^{\mathrm{left},1}$, $\phi^{\mathrm{left},2}$ satisfy (3.34) with A_{L_1}, A_{L_2} and $A_1(k)$, $A_2(k)$, respectively, $\phi^{\mathrm{left}} = \phi^{\mathrm{left},1} \cup \phi^{\mathrm{left},2}$ satisfies (3.34) with $A_L = \mathrm{diag}(A_{L_1}, A_{L_2})$ and $A_k = [A_1(k), A_2(k)]^\top$.

Once we obtain $\Phi = \{\phi^{\mathrm{left}}\} \cup \{\phi_{0,k} : k \ge n_\phi\}$ as a basis of V_0^{half}, we can apply Gram-Schmidt to replace the origin one with an orthogonal basis, and then imitating what we did in (3.4) and (3.5) to obtain the orthogonal multi-wavelet basis on the half space. For the biorthogonal case, similar to [22], we need to construct the biorthogonal boundary wavelets satisfying condition (3.34), which is equivalent to the fact that the Gramian matrix Γ is invertible. One can take $A = I$ and $\tilde{A} = \Gamma^{-\top}$ in (3.34) to get the dual part $\tilde{\Phi}$.

More specifically, define $n_{\tilde{\phi}}$ similar to that of n_ϕ. Consider vector function $\phi^\circ := \{\phi^{\mathrm{left}}\} \cup \{\phi_{0,k} : n_\phi \le k \le n_{\tilde{\phi}}\}$, and then we can define a vector function $\breve{\phi}^{\mathrm{left}} = \tilde{\phi}_{0,k}^{\mathrm{half}} \cup \tilde{\phi}^h$, where

$$\tilde{\phi}^h = 2\sum_{k=n_{\tilde{\phi}}}^{n_h-1} \tilde{A}_k\tilde{\phi}_{1,k} \text{ and } (\tilde{\phi}^h, \phi_{0,k})_{[0,\infty)} = 0, \forall k \ge n_{\tilde{\phi}}, \tag{3.38}$$

where $n_h = 2\max(n_{\phi^{\mathrm{left}}}, h_\phi + n_{\tilde{\phi}}) - l_{\tilde{\phi}}$, $l_{(\cdot)}$ and $h_{(\cdot)}$ denote the left and right boundary of the support of $(\cdot)$, respectively. One can start with $\eta^L = \emptyset$ and merge $\breve{\phi}^{\mathrm{left}}$ into η^L if $(\eta^L \cup \breve{\phi}^{\mathrm{left}}, \phi^\circ)$ has full rank until $\#\eta^L = \#\phi^\circ$. Thus, $\tilde{\phi}^{\mathrm{left}} = (\eta^L, \phi^\circ)^{-1}\eta^L$.

Next, let us consider how to construct the wavelet part. For the non-dual part of the wavelet, we can obtain it directly by the direct sum property, attributing it to the fact that we only need to reconstruct a new set of boundary functions when building the biorthogonal wavelet. That is, define $m_\phi = \max(2n_\phi + h_{\tilde{a}}, 2n_\psi + h_{\tilde{b}})$ and $\psi^\circ = \{\phi^{\text{left}}(2\cdot)\} \cup \{\phi_{1,k} : n_\phi \le k \le m_\phi\}$. Thus,

$$\psi^{\text{left}} = \psi^\circ - (\psi^\circ, \tilde{\phi}^{\text{left}})\phi^{\text{left}} - \sum_{k=n_\phi}^{M_\phi - 1} (\psi^\circ, \tilde{\phi}_{0,k})\phi_{0,k}, \tag{3.39}$$

where M_ϕ is chosen to guarantee that supp $\psi^\circ \cap$ supp $\tilde{\phi}_{0,k}$ is at most a singleton.

For the biorthogonal multi-wavelets on the half line, Han and Michelle states that for each $\eta \in \{\psi^{\text{left}}\} \cup \{\psi_{0,k} : n_\phi \le k \le n_{\tilde{\phi}}\}$, there exists a unique sequence $\{c_\eta(h)\}$ such that

$$(d_\eta, g) = \begin{cases} 1, & \text{if } \eta = g \\ 0, & \text{if } g \in (\Phi \cup \Psi)\backslash\{\eta\} \end{cases} \quad \text{with } d_\eta = \sqrt{2} \sum_{h \in \Phi} c_\eta(h)\tilde{h}(2\cdot) \tag{3.40}$$

and

$$c_\eta(\phi_{0,k}) = 0, \forall k \ge m_{\tilde{\phi}}. \tag{3.41}$$

Then $\psi^{\text{left}} = \{d_\eta : \eta \in \{\psi^{\text{left}}\} \cup \{\psi_{0,k} : n_\phi \le k \le n_{\tilde{\phi}}\}\}$.

Such a construction appeared to be overly complicated, so Han and Michelle proposed a method for the construction of biorthogonal multi-wavelets on intervals that do not have to involve the explicit construction of the dual part. Define

$$\phi^{\text{left}} = \sum_{j=1}^{\infty} 2^{j-1} A_L^{j-1} g(2^j \cdot), \text{ where } g = 2 \sum_{k=n_\phi}^{\infty} A_k \phi_{0,k}, \tag{3.42}$$

and then ϕ^{left} has the form (3.34). Find ψ^{left} such that

$$\begin{bmatrix} \phi^{\text{left}}(2\cdot) \\ \phi_{0,k}(2\cdot) \end{bmatrix} = A_0 \phi^{\text{left}} + B_0 \psi^{\text{left}} + \sum_{n_\phi \le k < h_C} C_k \phi_{0,k} + \sum_{n_\psi \le k < h_D} D_k \psi_{0,k}. \tag{3.43}$$

With these, one can determine $\tilde{A}_L$ and $\tilde{B}_L$ form $A_0, B_0, \{C(k)\}_{k=n_\phi}^{h_C - 1}, \{D(k)\}_{k=n_\psi}^{h_D - 1}$ and filter $\tilde{a}, \tilde{b}$. In fact, we have obtained the dual part

$$\tilde{\phi}^{\text{left}} = \sum_{j=1}^{\infty} 2^{j-1} \tilde{A}_L^{j-1} \tilde{g}(2^j \cdot), \tag{3.44}$$

where $\tilde{g} = 2 \sum_{k=n_{\tilde{\phi}}}^{\infty} \tilde{A}_k \tilde{\phi}_{0,k}$ and

$$\tilde{\psi}^{\text{left}} = 2\tilde{B}_L \tilde{\phi}^{\text{left}}(2\cdot) + 2 \sum_{k=n_{\tilde{\phi}}}^{\infty} \tilde{B}(k)\tilde{\phi}_{1,k}, \tag{3.45}$$

where $\tilde{A}(k)$ and $\tilde{B}(k)$ are defined as

$$\tilde{A}(k) = \begin{bmatrix} 0 \\ \tilde{a}(k - 2n_\phi) \\ \vdots \\ \tilde{a}(k - 2(n_{\tilde{\phi}} - 1)) \end{bmatrix}, \quad \tilde{B}(k) = \begin{bmatrix} 0 \\ \tilde{b}(k - 2n_\psi) \\ \vdots \\ \tilde{b}(k - 2(n_{\tilde{\psi}} - 1)) \end{bmatrix} \tag{3.46}$$

In this way we obtain the biorthogonal multi-wavelet on the half-space by a direct approach.

So far, we have given results on how to obtain results on intervals by means of (bi-)orthogonal (multi-)wavelets or (multi-)framelets defined on $\mathbb{R}$. In the next section, we focus on another approach that starts from the spline function.

4 Wavelets on the interval via splines

In this section, based on the knot sequence and spline functions, we discuss the approach for wavelets on the interval from directly defining the scaling function on the interval without having to deal with the boundary.

4.1 Chui and Quak's approach

Define knot sequence $\{t_{j,k}\}$ on the interval $[0,1]$:

$$\{t_{j,k}\}_{k=-m+1}^{2^j+m-1} := \begin{cases} 0, & -m+1 \le k \le 0, \\ k/2^j, & 1 \le k \le 2^j - 1, \\ 1, & 2^j \le k \le 2^j + m - 1. \end{cases} \tag{4.1}$$

For this knot sequence, B-splines are defined as:

$$B_{j,k}^m(x) := (t_{j,k+m} - t_{j,k})[t_{j,k}, \dots, t_{j,k+m}]_t (t-x)_+^{m-1}, \tag{4.2}$$

where $[\cdot, \dots, \cdot]_t$ is the m-th divided difference of $(t-x)_+^{m-1}$ with respect to t. From this definition we know that $\operatorname{supp}(B_{j,k}^m) = [t_{j,k}, t_{j,k+m}]$ and $B_{j,k}^m = N_m(2^j x - k), 0 \le k \le 2^j - m$, where $N_m(x)$ denotes the cardinal B-spline function of order m.

The spline space of order m corresponding to this knot sequence is

$$S_j^m := \{s \in C^{m-2}[0,1] : s_{[t_{j,k}, t_{j,k+1}]} \in \Pi_{m-1}, 0 \le k \le 2^j - 1\}, \tag{4.3}$$

where Π_{m-1} denotes the space of all polynomials of degree at most $m-1$. A nested sequence of subspaces $V_j^{[0,1]}$ is obtained by setting $V_j^{[0,1]} := S_j^m$, $V_0^{[0,1]} = \Pi_{m-1}$, and the scaling function $\phi_{j,k} = B_{j,k}^m$.

It remains to consider the orthogonal complement subspaces $W_j^{[0,1]}$ satisfying $V_{j+1}^{[0,1]} = V_j^{[0,1]} \oplus W_j^{[0,1]}$. Note that $W_j^{[0,1]}$ is a subspace of S_j^{2m}. Define the spline space

$$\tilde{S}_{j+1}^{2m} := \langle B_{j+1,k}^{2m} : -m+1 \le k \le 2^{j+1} - m - 1 \rangle \tag{4.4}$$

and its subspace

$$\tilde{S}_{j+1}^{0,2m} := \langle s \in \tilde{S}_{j+1}^{2m} : s(t_{j,k}) = 0,\ 0 \le k \le 2^j \rangle. \tag{4.5}$$

In [12], Chui and Quak proved that the m-th order differential operator D_m maps $\tilde{S}_{j+1}^{0,2m}$ one-to-one onto the wavelet space $W_j^{[0,1]}$. Thus, for $2^j > 2m-1$, there exists $2^j - 2m + 2$ linearly independent interior wavelet functions belong to $W_j^{[0,1]}$:

$$\psi_{j,k} = \psi_m(2^j \cdot -l) = \frac{1}{2^{2m-1}} \sum_{l=0}^{2m-2} (-1)^l N_{2m}(l+1) B_{j+1,2k+l}^{2m,(m)},$$

where $B_{j,k}^{2m,(m)} := D_m B_{j,k}^{2m}$ and

$$\operatorname{supp}(\psi_{j,k}) = [\frac{2k}{2^{j+1}}, \frac{2k+4m-2}{2^{j+1}}], \tag{4.6}$$

which implies that $\operatorname{supp}(\psi_{j,k}) \subset [0,1]$ for $0 \le k \le 2^j - 2m + 1$.

We can see that similar to the construction of Cohen et al. in [14], we have $2^j - 2m + 2$ interior functions and need to construct $m-1$ boundary functions for both sides, then we will have totally 2^j functions as a basis. Here as before, we only consider the case at the 0-boundary. For the 0-boundary wavelets, Chui and Quak in [12] set

$$\begin{aligned} \psi_{j_0,k}(x) :=& \frac{1}{2^{2m-1}} \sum_{l=-m+1}^{-1} \alpha_{k,l} B_{j_0+1,k}^{2m,(m)}(x) \\ &+ \sum_{l=0}^{2m-2+2k} (-1)^l N_{2m}(l+1-2k) B_{j_0+1,k}^{2m,(m)}(x), \end{aligned} \tag{4.7}$$

where the coefficients $\alpha_{k,l}$ are the solution of

$$B\alpha_k = r_k, \quad -m+1 \le k \le -1 \tag{4.8}$$

with $B = (b_{i,l})_{i,l=1}^{m-1}$, $b_{i,l} = B_{j_0+1,-l}^{2m}(t_{j_0,i})$, $\alpha_k = [\alpha_{k,-m+1}, \cdots, \alpha_{k,-1}]^\top$, and $r_k = [r_{k,-m+1}, \cdots, r_{k,-1}]^\top$. Here

$$r_{k,i} = -\sum_{l=0}^{2m-2+2k} (-1)^l N_{2m}(l+1-2k) N_{2m}(2m+2i-l).$$

In the remainder of this section, we present a refinement of Quak and Weyrich's decomposition algorithm in [50] for such a construction. In general, for

$$f_{j+1}(x) = f_j(x) + g_j(x),$$

where $f_j \in V_j^{[0,1]}$ and $g_j \in V_j^{[0,1]}, j \in \mathbb{Z}_+$. Thus, we can get

$$\sum_{k=-m+1}^{2^{j+1}-1} c_k^{j+1} \phi_{j+1,k}(x) = \sum_{k=-m+1}^{2^j-1} c_k^j \phi_{j,k}(x) + \sum_{k=-m+1}^{2^j-m} d_k^j \psi_{j,k}(x). \tag{4.9}$$

That means we need the coefficients sequence $d^{N-1}, \cdots, d^{j_0}, c^{j_0}$ to reconstruct the function, where N is the initial level and j_0 is the smallest integer satisfying

$2^{j_0} > 2m-1$. The general idea is to get all the terms by recursion. But here we do not need $c^{N-1}, \cdots, c^{j_0-1}$. Computing these terms definitely reduces the arithmetic efficiency. Quak and Weyrich in [50] suggested a way to improve this situation.

Consider the dual pair of the scaling function $\phi_{j,k}$ and wavelet function $\psi_{j,k}$, and denote them as $\tilde{\phi}_{j,k}$ and $\tilde{\psi}_{j,k}$, respectively. The existence of biorthogonal spline wavelets, we shall discuss them soon in the next subsection. Since

$$f_N(x) = \sum_{k=-m+1}^{2^{N-1}} c_k^N \phi_{N,k}(x) = \sum_{k=-m+1}^{2^{N-1}} a_k^N \tilde{\phi}_{N,k}(x) \tag{4.10}$$

Then we have $a^j = C^j c^j$, where $C^j = ((\phi_{j,k}, \phi_{j,l})_{[0,1]})_{k,l=-m+1}^{2^j-1}$. Similar we can get $b^j = D^j d^j$ from decomposition with wavelet functions and the dual pair. Let

$$\begin{aligned} P^j &= ((\tilde{\phi}_{j+1,\tilde{k}}, \phi_{j,k})_{[0,1]})_{\tilde{k}=-m+1;k=-m+1}^{2^{j+1}-1;2^j-1}, \\ Q^j &= ((\tilde{\phi}_{j+1,\tilde{k}}, \psi_{j,k})_{[0,1]})_{\tilde{k}=-m+1,k=-m+1}^{2^{j+1}-1;2^j-m}. \end{aligned} \tag{4.11}$$

Firstly, one can get a^N from $a^N = C^N c^N$ and then for each level $j_0 \le j \le N-1$, we can compute $a^j = (P^j)^\top a^{j+1}, b^j = (Q^j)^\top a^{j+1}$. Thus, we can get c^{j_0} from $C^{j_0} c^{j_0} = a^{j_0}$ and $d^j, j_0 \le j \le N-1$ from $D^j d^j = b^j$.

Here, we can see that in Quak and Weyrich's algorithm, we have involved the dual functions of the scaling functions and wavelet functions. Thanks to the biorthogonal wavelets, we can use different sets of functions in the decomposition and reconstruction, which gives a great deal of freedom. In the subsequent sections, we will refer to Dahmen et al.'s results in [16] to illustrate how to obtain biorthogonal wavelets from the spline functions.

4.2 Dahmen et al.'s biorthogonal spline wavelets

We already know how to construct multi-resolution analyses on intervals based on knot sequences from Chui and Quak's results in [12]. In [16], Dahmen et al. considered the case of biorthogonal spline wavelets on the interval.

In [16], Dahmen et al. used cardinal B-splines with order m as scaling functions ϕ. With this definition, ϕ is centered on $\mu(m)/2$, where $\mu(m) := m \bmod 2$. From [16], we know that for each m and $\tilde{m} \ge m, \tilde{m} \in \mathbb{N}$ such that $m + \tilde{m}$ is even, there exists $\tilde{\phi} \in L^2(\mathbb{R})$ is the dual pair of ϕ with order $\tilde{m}$. Suppose that we already have the multi-resolution space on the interval $[0,1]$:

$$\Phi_j' := \{\phi_{j,k}^{\text{left}} : k \in \Delta_j^{\text{left}}\} \cup \{\phi_{j,k} : k \in \Delta_j^0\} \cup \{\phi_{j,k}^{\text{right}} : k \in \Delta_j^{\text{right}}\}, \tag{4.12}$$

$$\tilde{\Phi}_j' := \{\tilde{\phi}_{j,k}^{\text{left}} : k \in \tilde{\Delta}_j^{\text{left}}\} \cup \{\tilde{\phi}_{j,k} : k \in \tilde{\Delta}_j^0\} \cup \{\tilde{\phi}_{j,k}^{\text{right}} : k \in \tilde{\Delta}_j^{\text{right}}\}, \tag{4.13}$$

where Δ and Δ denote the domain of k. Due to the construction of boundary functions and the nature of the dual pair, we know that only the boundary part does not satisfy the biorthogonality condition. That is, as in [22], if we define

$$V_j^L := \langle \phi_{j,k}^{\text{left}} : k \in \Delta_j^{\text{left}} \rangle, \quad V_j^I := \langle \phi_{j,k} : k \in \Delta_j^0 \rangle, \tag{4.14}$$

$$\tilde{V}_j^L := \langle \tilde{\phi}_{j,k}^{\text{left}} : k \in \tilde{\Delta}_j^{\text{left}} \rangle, \quad \tilde{V}_j^I := \langle \tilde{\phi}_{j,k} : k \in \tilde{\Delta}_j^0 \rangle, \tag{4.15}$$

then we have something similar to (3.15). Thus, things remained to do is similar to those in Section 3.3. But [22] doesn't give specific methods. The more specific method is given in [16].

Like (3.20), we need transform matrix A_j and $\tilde{A}_j$ such that

$$A_j \Gamma_j (\tilde{A}_j)^\top = I. \tag{4.16}$$

We know that if the matrix Γ_j is non-singular, then by taking

$$A_j = I, \quad \tilde{A}_j = \Gamma_j^{-\top}, \tag{4.17}$$

we can have the biorthogonality. Define

$$\Gamma_{j,L} := (\Phi_j^{\text{left}}, \tilde{\Phi}_j^{\text{left}})_{[0,1]} := ((\phi_{j,k}^{\text{left}}, \tilde{\phi}_{j,m}^{\text{left}})_{[0,1]})_{k,m \in \tilde{\Delta}_j^{\text{left}}}, \tag{4.18}$$

and the case on the right boundary can be constructed similarly. Here we use $\tilde{\Delta}_j^{\text{left}}$ rather than Δ_j^{left} since $\#\tilde{\Delta}_j^{\text{left}} \geq \#\Delta_j^{\text{left}}$. Dahmen et al. in [16] proved that $\Gamma_{j,L}$ is a matrix independent of j and can be expressed as

$$\Gamma_L = (2^{(\frac{1}{2}+k)j} (\phi_{j,l-m+r}^{\text{left}}, (\cdot)^k)_{[0,1]})_{k,r=0}^{\tilde{m}-1} := \Gamma_L(m, \tilde{m}, l, 0), \tag{4.19}$$

where l is the left boundary of Δ_j^0. It can be defined more generally,

$$\Gamma_L(m, \tilde{m}, l, \nu) = (2^{(\frac{1}{2}+k)j} (\phi_{j,l-m+r}^{\text{left}}, (\cdot)^{k+\nu})_{[0,1]})_{k,r=0}^{\tilde{m}-1}. \tag{4.20}$$

In [16], $\Gamma_L(m, \tilde{m}, l, 0)$ is non-singular if and only if $\Gamma_L(m-1, \tilde{m}, l-\mu(m-1), 1)$ is non-singular. Thus, by repeating this operation $m-1$ times, we can get

$$\det(\Gamma_L) \neq 0 \text{ if and only if } \det(\Gamma_L(1, \tilde{m}, \hat{l}, m-1)) \neq 0, \tag{4.21}$$

where $\hat{l} = l - \sum_{k=1}^{m-1} \mu(k)$. Note that for $m = 1$, $\phi(x) = \chi_{[0,1)}$, and we can reduce the matrix to a non-singular Vandermonde matrix. Thus, we get the transform matrix that we were looking for.

Let us now turn our attention to the wavelet space. First of all, according to the two-scale relations, Dahmen et al. in [16] defined the matrices $M_{j,0}$ and $\tilde{M}_{j,0}$ satisfying

$$\Phi_j^\top = \Phi_{j+1}^\top M_{j,0}, \tilde{\Phi}_j^\top = \tilde{\Phi}_{j+1}^\top \tilde{M}_{j,0}. \tag{4.22}$$

Then let $\breve{M}_{j,1}$ be some stable completion of $M_{j,0}$ and $G_j = (M_{j,0}, M_{j,1})^{-1}$ has the form $G_j = (\tilde{M}_{j,0}^\top, \breve{G}_{j,1}^\top)^\top$. $M_{j,1}$ is defined as $M_{j,1} := (I - M_{j,0}\tilde{M}_{j,0}^\top)\breve{M}_{j,1}$. Thus,

$$\Psi_j^\top = \Phi_{j+1}^\top M_{j,1}, \tilde{\Psi}_j^\top = \tilde{\Phi}_{j+1}^\top \breve{G}_{j,1}^\top. \tag{4.23}$$

So far, we have completed the construction of biorthogonal spline wavelets on the interval as in [16]. Such a construction suffers from a similar problem as Meyer's construction in [44]. The condition numbers of the matrices involved are too large for some larger m and $\tilde{m}$. In the next section, we present Primbs' refinement of this construction in [49], which is based on Chui and Quak's construction in [12].

4.3 Primbs's improvement

With the scaling functions $\phi_{j,k} = 2^{\frac{j}{2}} B^m_{j,k}$ already established by Chui and Quak [12], see Section 4.1, We have obtained basis that can represent all polynomials of order up to $m-1$ on the interval $[0,1]$. We can also obtain the dual pair $\tilde{\phi}$ of order $\tilde{m}$ as in Section 4.2. We know that the domain of the scaling function ϕ is completely within the interval $[0,1]$, but this is not the case for its dual function $\tilde{\phi}$, which we need to handle at the boundary as before.

In order to satisfy biorthogonality, we still need to ensure that the Gramian matrix $\Gamma_{j,L}$ defined as (4.18) is non-singular. Unlike the previous proof procedure, Primbs makes the matrix become an upper triangular matrix by a special construction of the boundary function of the dual part. That is,

$$\Gamma_L = \begin{bmatrix} I_{m-1} & A \\ O & \Gamma'_L \end{bmatrix}, \tag{4.24}$$

where Γ'_L is an upper triangular matrix with $\det \Gamma'_L = \frac{m-1}{\tilde{m}+m-1}$. In this way, we easily obtain that the matrix is invertible, while by repeating (4.17), we can obtain biorthogonality on the interval. The biorthogonal basis obtained by Primbs by such manipulation has a better Riesz boundary than the original one.

Primbs' subsequent thoughts on obtaining the basis of the complementary space agree with Dahmen et al.'s approach [16]. Similar to (4.22), consider the relation between neighboring hierarchies. More specifically, in Primbs's construction, it has

$$\phi^{\text{left}}_{j,k} = \frac{1}{\sqrt{2}} \sum_{l=1}^{2m-2} m^{\text{left}}_{l,k} \phi^{\text{left}}_{j+1,l}, \tag{4.25}$$

with

$$m^{\text{left}}_{l,k} := \begin{cases} 0, & 1 \le l \le 2k-m-1, \\ a_{l_1+l+m-2k}, & 2k-m \le l \le 2k, \\ 0, & 2k+1 \le l \le 3\tilde{m}+2m-5, \end{cases} \tag{4.26}$$

where l_1 is the left boundary of supp ϕ and $a_k = 2^{1-m}\begin{pmatrix} m \\ k+\lfloor \frac{m}{2} \rfloor \end{pmatrix}$. By symmetry we can obtain the matrix M_R. Thus, $M_{j,0}$ has the form

$$M_{j,0} = \frac{1}{\sqrt{2}} \begin{bmatrix} M_L & & \\ & A_j & \\ & & M_R \end{bmatrix} \begin{matrix} \} \, m+2\tilde{m}-3 \\ \\ \} \, m+2\tilde{m}-3 \end{matrix} \tag{4.27}$$

where

$$(A_j)_{n,k} := \begin{cases} a_{l_1+1+n-2k}, & -1 \le n-2k \le m-1, \\ 0, & \text{otherwise.} \end{cases} \tag{4.28}$$

Then as in (4.23), we obtain biorthogonal wavelets on the interval.

4.4 Dahmen et al.'s biorthogonal multi-wavelets

We next consider the case of multi-wavelets as before, and it is worth mentioning that Dahmen et al.'s results in [15] is the first to construct multi-wavelets on the interval. Although it considers only the dyadic case, the results undoubtedly have a great impact on subsequent research on multi-wavelets on the interval. Multi-wavelets have received a lot of attention due to their desirable properties that normal wavelets can not have. We give a specific example for multi-wavelets on the interval in this section, while the systematic method we have already stated in Section 3.5.

Consider the cubic splines given by

$$\phi_1(t) := \begin{cases} (t+1)^2(-2t+1), & t \in [-1,0], \\ (1-t)^2(2t+1), & t \in [0,1], \\ 0, & \text{otherwise,} \end{cases} \tag{4.29}$$

$$\phi_2(t) := \begin{cases} (t+1)^2 t, & t \in [-1,0], \\ (1-t)^2 t, & t \in [0,1], \\ 0, & \text{otherwise.} \end{cases} \tag{4.30}$$

For the scaling function vector $\phi = (\phi_1, \phi_2)^\top$ defined on the whole real line, it satisfies the two-scale relation

$$\phi(x) = \sum_{k\in\mathbb{Z}} A_k \phi(2x-k), \quad x \in \mathbb{R}, \tag{4.31}$$

where

$$A_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ -\frac{1}{8} & -\frac{1}{8} \end{bmatrix},\ A_0 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix},\ A_2 = \begin{bmatrix} \frac{1}{2} & -\frac{3}{4} \\ \frac{1}{8} & -\frac{1}{8} \end{bmatrix}, \tag{4.32}$$

and $A_k = 0$ for other cases.

For the dual pair $\tilde{\phi}$ satisfies

$$(\phi, \tilde{\phi}(\cdot - k))_{\mathbb{R}} = \delta_{0,k} I_2, \quad k \in \mathbb{Z}, \tag{4.33}$$

we can get the dual matrices $\tilde{A}_k$ in (4.31) for $\tilde{\phi}$ with supp $\tilde{A} = \{-2, \cdots, 2\}$:

$$\tilde{A}_{-2} = \begin{bmatrix} \frac{-7}{64} & \frac{-5}{64} \\ \frac{87}{128} & \frac{31}{64} \end{bmatrix}, \tilde{A}_{-1} = \begin{bmatrix} \frac{1}{2} & \frac{3}{16} \\ \frac{-99}{32} & \frac{-37}{32} \end{bmatrix},\ \tilde{A}_0 = \begin{bmatrix} \frac{39}{32} & 0 \\ 0 & \frac{15}{8} \end{bmatrix},\ \tilde{A}_1 = \begin{bmatrix} \frac{1}{2} & \frac{-3}{16} \\ \frac{99}{32} & \frac{-37}{32} \end{bmatrix}, \tilde{A}_2 = \begin{bmatrix} \frac{-7}{64} & \frac{5}{64} \\ \frac{-87}{128} & \frac{31}{64} \end{bmatrix},$$

By (4.33), we also have

$$\sum_{k\in\mathbb{Z}} A_k \tilde{A}_{k+2m}^\top = 2\delta_{0,m} I_2. \tag{4.34}$$

Let us consider the case on the interval. As in the previous idea, we need to reconstruct the boundary functions. Define

$$\phi_{j,r}^{\mathrm{left}}(x) = \sum_{m=0}^{2} (\alpha_{\tilde{\phi},r}(k))^{\top} \phi_{j,m}^{\mathrm{half}}(x), \quad 0 \le r \le 3, \tag{4.35}$$

where $\alpha_{\tilde{\phi},r}$ is the vector form of (2.4) and then we can get the left boundary vectors

$$\Phi_{j,1}^{\mathrm{left}} = \begin{bmatrix} \phi_{j,0}^{\mathrm{left}} \\ \phi_{j,1}^{\mathrm{left}} \end{bmatrix}, \quad \Phi_{j,2}^{\mathrm{left}} = \begin{bmatrix} \phi_{j,2}^{\mathrm{left}} \\ \phi_{j,3}^{\mathrm{left}} \end{bmatrix}. \tag{4.36}$$

The right boundary vector can be obtained in the same way. Thus, together with the domain and the part that is completely in the interval, a set of bases that can represent all polynomials of order up to 3 in the interval $[0,1]$ is obtained.

As with the previous construction of biorthogonal wavelets, the new boundary functions do not satisfy biorthogonality. Thus we need the matrix defined in (4.18) to be invertible, but here the multi-wavelet is not a generalized case but a special case from the cubic spline, thus we can obtain $\det(\Gamma_{j,L}) = \frac{16}{15}$ directly by computation, see details in [15]. The method of constructing biorthogonal multi-wavelets is the same as [16]; see Sections 4.2 and 4.3.

4.5 Jia's Hermite cubic spline wavelets on the interval

Dahmen et al.'s construction in [15] appears to be complicated as far as practical applications are concerned, and Jia in [33] constructed a Hermite cubic spline wavelet basis on the interval that is more suitable for finding numerical solutions to differential equations.

Since it aims to apply to differential equations, Jia in [33] further required that

$$(\psi_1', \phi_m'(\cdot - k)) = (\psi_2', \phi_m'(\cdot - k)) = 0, \ m = 1, 2, \forall k \in \mathbb{Z} \tag{4.37}$$

so that one has

$$\int_0^1 w'(x) v'(x) \mathrm{d}x = 0, \quad \forall w \in \Psi_j, v \in \Phi_j, \tag{4.38}$$

where

$$\Phi_j := \{\phi_1(2^j \cdot -k) : 1 \le k \le 2^j - 1\} \cup \{\phi_2(2^j \cdot -k)|_{[0,1]} : 0 \le k \le 2^j\}, \tag{4.39}$$

and

$$\Psi_j := \{\psi_1(2^j \cdot -k) : 1 \le k \le 2^j - 1\} \cup \{\psi_2(2^j \cdot -k)|_{[0,1]} : 0 \le k \le 2^j\}. \tag{4.40}$$

With this, one can show that

$$\psi_1(x) = -2\phi_1(2x+1) + 4\phi_1(2x) - 2\phi_1(2x-1) - 21\phi_2(2x+1) + 21\phi_2(2x-1) \tag{4.41}$$

and

$$\psi_2(x) = \phi_1(2x+1) - \phi_1(2x-1) + 9\phi_2(x) + 12\phi_2(2x) + 9\phi_2(2x-1). \quad (4.42)$$

Let $H^1(0,1) := \{u : u \in L^2(0,1) \text{ and } u' \in L^2(0,1)\}$ and $H_0^1(0,1)$ be the closure of $\{u : u \in C[0,1] \cap C^1(0,1) \text{ and } u(0) = u(1) = 0\}$ in the space $H^1(0,1)$. With above, one can get the decomposition of the space $H_0^1(0,1)$:

$$H_0^1(0,1) = V_1 \oplus \sum_{j \in \mathbb{Z}_+} W_j. \quad (4.43)$$

In Jia's construction, V_1 is spanned by $\phi_{1,k}(x), k = 1,2,3,4$:

$$\phi_{1,1}(x) = \sqrt{\tfrac{5}{24}}\phi_1(2x-1),\ \phi_{1,2}(x) = \sqrt{\frac{15}{4}}\phi_2(2x),$$
$$\phi_{1,3}(x) = \sqrt{\tfrac{15}{8}}\phi_2(2x-1),\ \phi_{1,4}(x) = \sqrt{\frac{15}{4}}\phi_2(2x-2),$$

where the coefficients are chosen such that $\|\phi'_{1,k}\|_{L^2[0,1]} = 1, k = 1,2,3,4$. Do the same normalization for $\psi_{j,k}$ and then let $g_k = \phi_{1,k}$ for $k = 1,2,3,4$ and $g_{2^{j+1}+k} = \psi_{j,k}$ for $n \in \mathbb{Z}_+$.

For the Dirichlet boundary problem

$$\begin{cases} -u'' + u = f \text{ on } (0,1), \\ u(0) = u(1) = 0, \end{cases} \quad (4.44)$$

we can solve it by the weak form of the equation:

$$(u,v) + (u',v') = (f,v). \quad (4.45)$$

With the obtained wavelet basis, we can transform the problem into

$$\sum_{k=1}^{2^{j+1}} [(g'_j, g'_k) + (g_j, g_k)]\eta_k = (g_j, f), 1 \le j \le 2^{j+1}, \quad (4.46)$$

where $u_n = \sum_{k=1}^{2^{j+1}} \eta_k g_k$.

There are many other applications of wavelets on the interval to differential or integral equations, e.g., [4], [21], [27].

5 Extensions

We have discussed wavelets on the interval in the previous sessions on mainly two approaches. We next discuss some more generalized situations, such as considering the domains to be manifolds, spheres, etc. Although in the previous sections, we considered basically the case of the interval $[0,1]$, we can obtain wavelets on any interval $I = [a,b]$ by operations such as translation and dilation. We consider higher dimensional cases in this section, but ultimately, it still needs to base on the one-dimensional cases.

5.1 Tensor products and triangulation

Let $V_1 = V_0 \oplus W_0$ and $V_1' = V_0' \oplus W_0'$ be decomposition of spline space. Thus we can divide the tensor product space into four pieces, i.e.,

$$V_1 \otimes V_1' = (V_0 \otimes V_0') \oplus (V_0 \otimes W_0') \oplus (W_0 \otimes V_0') \oplus (W_0 \otimes W_0'). \tag{5.1}$$

For any given $f_1 \in V_1 \otimes V_1'$, it can also be decomposed as

$$f_1 = f_0 + g_0 + g_1 + g_2, \tag{5.2}$$

where $f_0 \in V_0 \otimes V_0'$, $g_0 \in V_0 \otimes W_0'$, $g_1 \in W_0 \otimes V_0'$, and $g_2 \in W_0 \otimes W_0$. Since both $\{\phi_{0,k}\}$ and $\{\psi_{0,k}\}$ form the basis of V_1, Lyche et al. in [40] defined

$$(\Phi_0^\top, \Psi_0^\top) = \Phi_1^\top (P, Q) = \Phi_1^\top M, \tag{5.3}$$

where Φ_0, Ψ_0 is the basis vector of V_0, W_0, respectively. One can write f_1 with the basis and coefficients matrix C_1, i.e.,

$$f_1(x, y) = \Phi_1^\top(x) C_1 \Phi_1'(y). \tag{5.4}$$

Similar to (5.4), we can define C_0, D_0, D_1, D_2 for f_0, g_0, g_1, g_2, respectively. With C_1, P, Q, P', Q', we can get the value of the coefficients matrix C_0, D_0, D_1, D_2.

For the case of triangulation, we give the definition of triangle first. A triangle is the convex hull of three non-collinear points $[x_1, x_2, x_3]$ with three edges $[x_1, x_2], [x_1, x_3]$ and $[x_2, x_3]$. Let $\mathcal{T}_0 = \{T_1, \ldots, T_M\}$ and $\Omega = \cup_{i=1}^n T_i$, and then we can define the triangulation.

Definition 3. *$\mathcal{T}_0$ is a triangulation if*

(1) the intersection $T_i \cap T_j$ is either empty or corresponds to a common vertex or a common edge, $i \neq j$;
(2) the number of boundary edges incident on a boundary vertex is two;
(3) the region Ω is simply connected.

Let V_0 denote the linear space of piecewise linear functions over $\mathcal{T}_0$, we need to construct $V_1 \supset V_0$, and thus get the wavelet space W_0. In fact, Lyche et al. in [40] made a more careful division of $\mathcal{T}_0$ by letting y_1, y_2, y_3 be the midpoints of $[x_1, x_2], [x_1, x_3], [x_2, x_3]$, respectively, so that one can get a new triangulation $\mathcal{T}_1$, and the V_1 generated by this $\mathcal{T}_1$ is what one need to construct. The construction of wavelets is then corresponding to the midpoint of each edge in $\mathcal{T}_0$.

In this section, we briefly introduce the construction of tensor-product and non-tensor-product wavelet systems in the bi-variate case, which establishes the basis for later studies. The construction of wavelets on manifolds by Dahmen and Schneider [17] in Section 5.2 can be attributed to the construction of wavelets on the n-dimensional cube which uses tensor products, while the construction of wavelets on non-uniform meshes by Stevenson [52] in Section 5.3 deals with triangulation, and the essence is to divide the domain by triangles.

5.2 Manifolds

In this section, let us define $\Omega = [0,1]^n$ and for $Z \subset \{0,1\}$,

$$[0,1]_Z = \begin{cases} [0,1], \text{ if } Z = \{\emptyset\}, \\ [-1,1], \text{ if } Z = \{0\}, \\ [0,2], \text{ if } Z = \{1\}, \\ [-1,2], \text{ if } Z = \{0,1\}. \end{cases} \tag{5.5}$$

In this way one can define different extensions of the unit interval. Further, define for $Z = Z_1 \otimes \cdots \otimes Z_n$,

$$\Omega_Z = [0,1]_{Z_1} \otimes \cdots \otimes [0,1]_{Z_n}, \tag{5.6}$$

where $\otimes$ denote the tensor product, and $\tilde{Z} = \{0,1\}^n \backslash Z$.

With these, Dahmen and Schneider in [17] stated that Sobolev spaces $H^s(\zeta)$ on manifolds ζ can be isomorphized to explicit product spaces, i.e.,

$$T : H^s(\zeta) \to H^s(\zeta_1)_{Z_1} \otimes \cdots \otimes H^s(\zeta_N)_{Z_N}, \tag{5.7}$$

and T extends isomorphism for the dual

$$H^{-s}(\zeta) \equiv H^{-s}(\zeta_1)_{\tilde{Z}_1} \otimes \cdots \otimes H^s(\zeta_N)_{\tilde{Z}_N}. \tag{5.8}$$

One can deduce that constructing a wavelet on a manifold can be weakened to constructing a wavelet on each local path that satisfies certain boundary conditions. The latter, in turn, are smooth parameterized images, so one only needs to construct the wavelet bases on the n-dimensional cube.

The paper [17] focus on how to construct wavelets that satisfy the boundary conditions, so we will not go into details, while for wavelets on the n-dimensional cube, [17] takes a simple tensor product approach to obtain them, which we have already talked about in the previous section. In the next section, we refer to Stevenson's result in [52] on the construction of biorthogonal wavelets on non-uniform meshes.

5.3 Non-uniform meshes

In this section, we follow Stevenson's idea in [52] to construct biorthogonal wavelets on non-uniform meshes.

Let us first define n-simplex $\mathcal{T}$

$$\mathcal{T} = \{\lambda \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0\}, \tag{5.9}$$

and if $n = 2$, $\mathcal{T}$ is associated with a triangle. For $I \subset \mathcal{T}$, Stevenson consider $\Phi = \{\phi_\lambda : \lambda \in I\}$ satisfies the following properties:

(1) $\phi_\lambda \in C(\mathcal{T})$;

(2) $\phi_\lambda(\mu) = \phi_{\pi(\lambda)}(\pi(\mu))$ for any permutation $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$;
(3) ϕ_λ vanishes for e does not include λ;
(4) $\{\phi_\lambda|_e : \lambda \in I \cap e\}$ is independent.

Here, e is the face of $\mathcal{T}$. The intent of such a construction is clear: one hopes to build the scaling function basis on the triangle. Below we consider the case $n = 2$ (In fact, Stevenson also just considered the case $n \leq 2$). We know that our scaling function basis needs to satisfy the two-scale relation. The most straightforward construction is to take the midpoint of each edge as we mentioned in Section 5.1. Stevenson gave a more general result: Suppose we divide $\mathcal{T}$ into $\{\mathcal{T}_i : 1 \leq i \leq 2^n\}$ and let

$$I^{(r)} = \cup_{i=1}^{n} B_i^{-1}(I), \text{ where } B_i : \mathcal{T}_i \to \mathcal{T}. \tag{5.10}$$

Then, for

$$\phi_\lambda^{(r)}(\mu) = \begin{cases} \phi_{B_i(\lambda)}(B_i(\mu)), & \text{if } \lambda, \mu \in \mathcal{T}_i, \\ 0, & \text{otherwise,} \end{cases} \tag{5.11}$$

we have $\Phi_j^{(r)} = 2^{-\frac{n}{2}}\Phi_{j+1}$, which implies that $\langle\Phi\rangle \subset \langle\Phi^{(r)}\rangle$. Stevenson also deduced that $\langle\Phi\rangle = P_{d-1,m}(\mathcal{T})$, where $P_{d-1,m}$ denote the space of piecewise polynomials on $\mathcal{T}$ of degree $d-1$ with respect to binary division of m repetitions.

For better understanding, Stevenson gave a specific example: $\Phi = \Delta^{d-1,m} = \{\delta_\lambda^{d-1,m} : \lambda \in I_{(d-1)2^m}\} \subset P_{d-1,m}$, where $I_q = \{\lambda \in \mathcal{T} : \lambda_i/q \in \mathbb{N}\}$, and

$$\delta_\lambda^{d-1,m}(\mu) = \begin{cases} 1, & \lambda = \mu, \\ 0, & \text{otherwise.} \end{cases} \tag{5.12}$$

In this special case, we have $(\Delta^{d-1,m})^{(r)} = \Delta^{d-1,m}$. Following Dahmen et al.'s process in Section 4.2, we can obtain the wavelet space $\Psi^{d-1,m}$ satisfies

$$P_{d-1,m+1}(\mathcal{T}) = P_{d-1,m}(\mathcal{T}) \oplus \Psi^{d-1,m}, \tag{5.13}$$

where

$$\Psi^{d-1,m} := \{\delta_\lambda^{d-1,m+1} : \lambda \in I_{(d-1)2^{m+1}} \backslash I_{(d-1)2^m}\}. \tag{5.14}$$

In this way, we can construct the wavelet basis on a triangulation, and for the case of non-uniform meshes, we simply decompose them into the case of a triangulation, which is a method often used in finite elements analysis.

6 Conclusion

To summarize, starting from orthogonal wavelets defined on $\mathbb{R}$, we can construct wavelets on the interval. By symmetry, we only need to consider constructing the base of the half-space $L^2[0,\infty)$ and only need to deal with one side of the boundary. In order to construct wavelets on the interval, we need to consider the multi-resolution analysis on the interval. We can see that in Section 3, we spend a lot on how to construct the base of the initial space, which is from $\phi_{0,k}$.

We can categorize the scaling functions into three types. The first is where the domain is completely within the target interval, the second is those that do not intersect with the interval, and the third is those that overlap with the boundary. We do not need to deal with the first two categories but need to focus on the third. We can make modifications to the third class of scaling functions so that the new boundary functions and the first class can form a basis for V_0. Once we have this, we can then obtain the basis of the space V_1 from the two-scale relation, while the wavelet space W_0 satisfies $V_1 = V_0 \oplus W_0$, from which we can deduce the wavelets on the interval from the projection.

The case of biorthogonality is similar, and for functions that satisfy biorthogonality on $\mathbb{R}$, we again only need to focus on the case at the boundary since functions of the first type automatically satisfy biorthogonality. We need to use the matrices C and $\tilde{C}$ to transform the boundary functions. As we mentioned in Sections 3.3 and 4.2, it is equivalent to (4.16). Moreover, if the Gramian matrices Γ are invertible, we can use the construction of (4.17) to achieve the biorthogonality of two bases on the interval.

We mentioned another method of constructing biorthogonal multi-wavelets on the interval in Section 3.4 by Han and Michelle. It requires symmetry or anti-symmetry properties for the scaling function and the wavelet function. Although it is straightforward and simple, it is not generalized enough. In Section 3.5, we present Han and Michelle's direct approach, which gives a generalized construction method for constructing (bi-)orthogonal (multi-)wavelets on the interval.

Besides by means of (bi-)orthogonal (multi-)wavelets defined on $\mathbb{R}$, with the help of spline functions, we can also construct bases in the space V_0, and the spline functions obtained by means of knot sequences defined on intervals have domains already on the interval so that there is no need to process the boundaries anymore. Chui and Quak showed that we can map the subspace of V_1 to the wavelet space with the help of differential operators. However, similar to Meyer's construction earlier, we still have to process at the boundary for wavelets, including the (bi-)orthogonal (multi-)wavelets case from the spline functions.

Wavelets on the interval can be considered in more general settings, such as manifolds, non-uniform grids, and hypercubes. The most straightforward approach to constructing wavelets on these special intervals is through tensor products, but the disadvantage of such an approach is the lack of directionality. There are many researchers working on non-tensor-product wavelets as well. By relaxing the orthogonality, the biorthogonal (multi-)wavelets constructed on the interval are able to possess more properties, such as fewer boundary conditions and more stability, which allows the wavelets on the interval to be generalized to other situations.

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