A Radial Basis Function Approach to Credit Barrier Model

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Abstract

Albanese and et. al. (2003) and Avellaneda and Zhu (2001) develop the framework of Credit Barrier Model. They provide special solutions to the model in case of simple stochastic structure. The technical aspect of the model remains wide open for general stochastic structure that is crucial when the model is required to calibrate with aggregate amount of empirical data. This paper provides a technical solution to this problem with the use of radial basis function (RBF). This paper discusses the numerical implementation of the credit barrier model using the RBF method. It also demonstrates that the RBF method is numerically tractable in this problem and allows in the model richer stochastic structure capable of capturing relevant market information.

(Keywords : Credit Migration, Kolmogorov Equation, Radial Basis Function)

Introduction

Credit rating measures the perceived creditworthiness of corporate borrower. The likelihood of credit migration and default are quantified by the historical rates that are well published by rating agencies such as Moodys and Standard & Poors. Pricing of credit instruments requires the construction of effective models that are consistent with empirical migration and default probabilities and as well capture the price components of credit risk implied from derivative market. This paper discusses the credit barrier model introduced by Albanese and et. al. (2003), extending the work of Avellaneda and Zhu (2001). Changes in credit rating in this model can be described through barrier crossings by an underlying credit quality process. Under this framework, the credit barrier model attempts to achieve consistency with empirical data for both the real-world and risk-neutral measures. In the literatures, the technical aspect on this issue remains wide open for underlying process with general stochastic structure especially the model is incorporated with jump process. This paper discusses the numerical implementation of this model using an efficient technique known as Radial Basis Function method (see Hardy [1971]). The method is numerically tractable in the credit migration problem and allows great flexibility in the stochastic structure of the credit quality process. The latter is crucial when it comes to the issue of capturing aggregate amount of relevant empirical information in the model.

Credit Barrier Model

In the credit barrier model developed by Albanese and et. al. (2003) and Avellaneda and Zhu (2001), credit quality of a defaultable bond can be described by an underlying time diffusion process $x(t)$ satisfying a stochastic differential equation of the form :

$$dx = \mu(x)\, dt + \sigma(x)\, dz \tag{1}$$

The value of $x(t)$ is restricted inside the interval $[0,\infty)$ with absorbing lower boundary at zero and unattainable upper boundary at infinity. Incorporating into the Moody's credit rating scheme with seven rating classes $R_{1,2,3,4,5,6,7} = \{ Caa\text{-}C, B, Ba, Baa, A, Aa, Aaa \}$, the entire credit quality interval $[0,\infty)$ is divided into seven corresponding partitions,

$$(\theta_0 = 0, \theta_1]\ ,\ (\theta_1, \theta_2]\ ,\ (\theta_2, \theta_3]\ ,\ (\theta_3, \theta_4]\ ,\ (\theta_4, \theta_5]\ ,\ (\theta_5, \theta_6]\ ,\ (\theta_6, \theta_7 = \infty)$$

with credit barriers at ($\theta_0 = 0, \theta_1, \theta_2, \theta_3, \theta_4, \theta_5, \theta_6, \theta_7 = \infty$). Thus, the defaultable bond will have a credit rating of R_m at time t, if the underlying credit quality $x(t)$ is inside the partition $(\theta_{m-1}, \theta_m]$. The lower barrier $\theta_0 = 0$ corresponds to the level of credit default for $x(t)$ at which the credit quality process will be absorbed and the bond will immediately default.

In this model, credit migration and default can be described through barrier crossings by the underlying credit quality process. For credit migration from initial rating R_l to a new rating R_m at time t, the process starts off from $x(0) \in (\theta_{l-1}, \theta_l]$ and evolves through barriers to $x(t) \in (\theta_{m-1}, \theta_m]$. The migration probability for $R_l \to R_m$ at time t is defined to be an average value over the initial credit quality as

$$M_{lm}(t) = \langle \int_{\theta_{m-1}}^{\theta_m} dx\, U(t, x \mid \rho) \rangle_l = \int_{\theta_{m-1}}^{\theta_m} dx\, U(t, x \mid \rho_l) \tag{2}$$

where $U(t, x \mid \rho)$ is the transition density for nonzero $x(0) = \rho$ going to $x(t) = x$, and $\langle\, .\, \rangle_l$ denotes an average over the initial credit quality in $(\theta_{l-1}, \theta_l]$. For simplification purpose, we replace the average expression in (2) by taking a central value for $x(0) = \rho_l$ inside $(\theta_{l-1}, \theta_l]$. Thus, there is another set of parameters $(\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7)$ in the model that correspond to the initial credit qualities for ratings $R_{1,2,3,4,5,6,7}$ in defining the migration probabilities. In the same way and under the same parameterization, we can define the default probability before time t for initial rating R_l as

$$P_l(t) = 1 - \int_0^\infty dx\, U(t, x \mid \rho_l) \tag{3}$$

Equation (3) measures the probability of credit default for which the credit quality begins at $x(0) = \rho_l$ and hits the default barrier at $\theta_0 = 0$ before time t.

As discussed in Albanese and et. al. (2003), it is not possible for this model to have a good match with empirical migration and default probabilities with the diffusion process in (1) driven by a Brownian term. Specifically, we could not match both the migration probabilities of changing more than one credit rating and the default probabilities for upper initial ratings. A possible solution for this problem is to introduce jump in the credit quality process through stochastic time changes as described by the variance-gamma model (see Madan, Carr and Chang [1998]). The corresponding jump process is obtained by evaluating the original process with random time increments given by an increasing gamma process $\gamma(t,1,\nu)$ with mean rate 1 and variance rate ν in the unit of time. That is,

$$x^J(t) = x(\, \gamma(t,1,\nu)\,) \tag{4}$$

Here, calendar time increment $t \to t + \Delta t$ is mapped to a random advance in gamma time $\gamma \to \gamma + \Delta\gamma$ that generates an upward jump in the process. The size of the positive gamma increment $\Delta\gamma$ follows a gamma probability density function given by

$$\mathrm{G}(\Delta\gamma, \Delta t) = \frac{(\Delta\gamma)^{\Delta t/\nu - 1}\, e^{-\Delta\gamma/\nu}}{\Gamma(\Delta t/\nu)\, \nu^{(\Delta t/\nu)}} \tag{5}$$

with mean Δt and variance $\nu\Delta t$. If the variance rate ν approaches zero so that there is no jump, the gamma density behaves like Dirac delta function $\delta(\Delta\gamma - \Delta t)$. The transition density for the jump process can be obtained by integrating out the gamma time in the density for the original process using the gamma pdf in (5) as

$$U^J(t, x \mid \rho) = \int_0^\infty ds\, \mathrm{G}(s, t)\, U(s, x \mid \rho) \tag{6}$$

It follows that the migration and default probabilities with the introduction of jump become

$$M^J_{lm}(t) = \int_{\theta_{m-1}}^{\theta_m} dx\, U^J(t, x \mid \rho_l) \tag{7}$$

$$P^J_l(t) = 1 - \int_0^{\infty} dx\, U^J(t, x \mid \rho_l) \tag{8}$$

Accordingly, there are a total of 14 free parameters $\theta_{1,2,3,4,5,6}$, $\rho_{1,2,3,4,5,6,7}$, and ν in the credit barrier model that need to be calibrated with empirical migration and default probabilities through the use of equations (7) and (8).

For the credit quality process driven by the SDE in (1), the transition density satisfies the forward-time Kolmogorov equation,

$$\frac{\partial}{\partial t} U(t, x \mid \rho) = \frac{1}{2} \frac{\partial^2}{\partial x^2} [\, \sigma^2(x) U(t, x \mid \rho) \,] - \frac{\partial}{\partial x} [\, \mu(x) U(t, x \mid \rho) \,] \tag{9}$$

with boundary conditions $U(0, x \mid \rho) = \delta(x - \rho)$, $U(t, 0 \mid \rho) = 0$, and $U(t, \infty \mid \rho) = 0$. To demonstrate how the migration and default probabilities in (7) and (8) can be calculated, we consider first a simple structure for the credit quality process given by a standard Brownian motion $dx = dz$. This is, in essence, the form adopted by Hull and White (2001) in modeling instead the risk-neutral process. With zero drift and unit volatility in this simple case, the KG equation in (9) has an analytic solution for the transition density that satisfies the required boundary conditions. It is given by

$$U(t, x \mid \rho) = \frac{1}{\sqrt{2\pi t}} \left[\; e^{-(x-\rho)^2/2t} - e^{-(x+\rho)^2/2t} \;\right] \tag{10}$$

It follows immediately that the probability measures in (7) and (8) can both be written in terms of integrals of cumulative normal distribution functions with gamma density as

$$M^J_{lm}(t) = \psi(\theta_m - \rho_l, t) - \psi(\theta_{m-1} - \rho_l, t) - \psi(\theta_m + \rho_l, t) + \psi(\theta_{m-1} + \rho_l, t) \tag{11}$$

$$P^J_l(t) = 2\, \psi(-\rho_l, t) \qquad , \qquad \psi(\xi, t) = \int_0^{\infty} ds\, \mathrm{G}(s, t)\, \mathrm{N}(\, \xi / \sqrt{s} \,) \tag{12}$$

The gamma integral $\psi(\xi, t)$ is numerically intensive. For completeness, a closed form solution expressed in terms of special functions can be found in appendix (A11) of Madan, Carr and Chang (1998). Using equations (11) and (12), we can then calibrate the model with empirical migration and default probabilities provided, for example, by Moody's Investor Service (see Carty [1997]).

For credit quality process with general stochastic structure, it is not always possible to have an analytic solution for the KG equation. The transition density in that case must therefore be evaluated numerically. We adopt in this paper an efficient numerical technique known as the Radial Basis Function method (see Hardy [1971]) which is a meshless interpolation scheme formulated using infinitely smooth basis functions. Without loss of generality, we extend our discussion here to driftless CIR process $dx = \sqrt{x}\, dz$ so as to demonstrate the use of the RBF approach on the current problem. In fact, we have shown below that both standard Brownian motion and driftless CIR process can give very good match with empirical data.

RBF Formulation

The idea of the RBF method is to formulate an interpolation problem based on scattered points in the domain. The interpolant can be expanded through a set of radial basis functions over the radial distances from the interpolating point to individual scattered points. Recall in the credit barrier model, the value of the credit quality process is restricted inside the interval $[0,\infty)$. Suppose $x_0, x_1, x_2, \dots, x_N$ are scattered points in this domain with $x_0 = 0$ located on the lower boundary. For credit qualities ρ and x inside the same domain, the transition density can be expanded in the RBF method as

$$U(t, x \mid \rho) = \sum_{j=0}^{N} \lambda_j(t, \rho)\, \phi(\| x - x_j \|) \tag{13}$$

where $\phi(\| x - x_j \|)$ is a radial basis function over the quantity $\| x - x_j \|$ that measures the Euclidean distance between x and the scattered point x_j. Using the RBF expansion in (13) with some choice of ϕ, we can solve the KG equation in (9) by evaluating the RBF coefficients $\lambda_0(t, \rho)$, $\lambda_1(t, \rho)$, … , and $\lambda_N(t, \rho)$ such that the transition density satisfies both the PDE and the required boundary conditions.

To first include the absorption boundary condition $U(t, x_0 \mid \rho) = 0$ in the solution, we eliminate $\lambda_0(t, \rho)$ in (13) and rewrite the RBF expansion as

$$U(t, x \mid \rho) = \sum_{j=1}^{N} \lambda_j(t, \rho)\, B(x, x_j) \quad , \quad B(x, x_j) = \phi(\| x - x_j \|) - \phi(\| x - x_0 \|)\, \phi_{0j} / \phi_{00} \tag{14}$$
$$\phi_{ij} = \phi(\| x_i - x_j \|)$$

The unattainable boundary condition $U(t, \infty \mid \rho) = 0$ can readily be seen in (14) with the use of bounded radial basis function for which $\phi(\infty) = 0$. It follows that the PDE for the transition density in (9) can be converted into a system of linear order differential equations for the coefficients $\boldsymbol{\lambda}(t, \rho) = (\lambda_1(t, \rho), \dots , \lambda_N(t, \rho))^T$ given by

$$\frac{\partial}{\partial t}\boldsymbol{\lambda}(t, \rho) = (\boldsymbol{B}^{-1}\boldsymbol{A})\, \boldsymbol{\lambda}(t, \rho) \tag{15}$$

where $\quad B_{ij} = \phi_{ij} - \phi_{i0}\, \phi_{0j} / \phi_{00} \quad , \quad A_{ij} = \pi_{ij} - \pi_{i0}\, \phi_{0j} / \phi_{00}$

$$\pi_{ij} = \left(\frac{1}{2} \frac{\partial^2}{\partial x^2}[\, \sigma^2(x)\, \phi(\| x - x_j \|)\,] - \frac{\partial}{\partial x}[\, \mu(x)\, \phi(\| x - x_j \|)\,] \right)_{x = x_i}$$

As both A_{ij} and B_{ij} are time independent, equation (15) can easily be solved by diagonalizing $\boldsymbol{B}^{-1}\boldsymbol{A} = \boldsymbol{H}\boldsymbol{\Lambda}\boldsymbol{H}^{-1}$ through generalized eigenvector decomposition to get

$$\boldsymbol{\lambda}(t, \rho) = (\boldsymbol{H}\, e^{\boldsymbol{\Lambda} t}\, \boldsymbol{H}^{-1})\, \boldsymbol{\lambda}(0, \rho) \tag{16}$$

where $e^{\boldsymbol{\Lambda} t}$ is diagonal with entries $e^{\Lambda_{ii} t}$. The initial values of the RBF coefficients in (16) can be derived from the boundary condition $U(0, x \mid \rho) = \delta(x - \rho)$. Consider ρ to stay inside the interval $(x_{k-1}, x_k]$, we can rewrite the boundary condition in discrete spacing as $U(0, x_i \mid \rho)\, \Delta x_i = \delta_{ik}$, where Δx_i is the size of the interval $(x_{i-1}, x_i]$. It can be shown using (14) that the initial coefficients $\boldsymbol{\lambda}(0, \rho)$ can be read off from the column vector of $\boldsymbol{B}^{-1}$ to be

$$\lambda_i(0, \rho) = (\Delta x_k)^{-1}\, (B^{-1})_{ik} \quad , \quad \text{for } \rho \in (x_{k-1}, x_k] \tag{17}$$

The transition density for the jump process can be obtained by integrating out the expression in (14) using the gamma probability density. In the RBF formulation, the gamma integral can be performed analytically as time dependence only comes from the diagonal matrix $e^{\Lambda t}$ in $\lambda_j(t, \rho)$. It can be shown that [1]

$$U^J(t, x \mid \rho) = \sum_{j=1}^{N} \lambda_j^J(t, \rho)\, B(x, x_j) \tag{18}$$

where

$$\lambda^J(t, \rho) = (\boldsymbol{H}\,\boldsymbol{\Omega}\,\boldsymbol{H}^{-1})\,\lambda(0, \rho) \tag{19}$$

The matrix $\boldsymbol{\Omega}$ in (19) is again diagonal with entries $\Omega_{ii} = (1 - \nu\Lambda_{ii})^{-t/\nu}$. Here, the gamma integral only transforms the diagonal matrix $e^{\Lambda t}$ into $\boldsymbol{\Omega}$ with the introduction of jump in the credit quality process. Using the RBF density given by (18) and (19), the migration and default probabilities defined in (7) and (8) can now be calculated for credit quality process with general stochastic structure.

For the choice of bounded radial basis function, we adopt in our analysis the infinitely smooth Gaussian function $\phi(r) = exp(-\kappa r^2)$ with shape parameter κ that can be used to regulate the accuracy of the RBF solution. In defining the matrix B_{ij} in (15) with dense points $\{ x_0, x_1, x_2, \ldots , x_N \}$, adjacent quantities ϕ_{ij}, $\phi_{i(j+1)}$, and $\phi_{(i+1)j}$ can be numerically closed render the determination of $\boldsymbol{B}^{-1}$ to be inaccurate. This can be regulated by a global dilation of the argument in $\phi_{ij} = exp(-\kappa \| x_i - x_j \|^2)$ with shape parameter κ taken in the order of $(\Delta x)^{-2}$, where Δx is the typical interval size of the spacings. The parameter at optimal accuracy can be determined by calibrating the RBF result of default probability with the exact formula in (12) for the Brownian process $dx = dz$ without jump. We obtain that the optimal shape parameter is around $\kappa \approx 0.3(\Delta x)^{-2}$. In Figure 1, we display the accuracy limits of the RBF solutions with different choices of κ in this calibration.

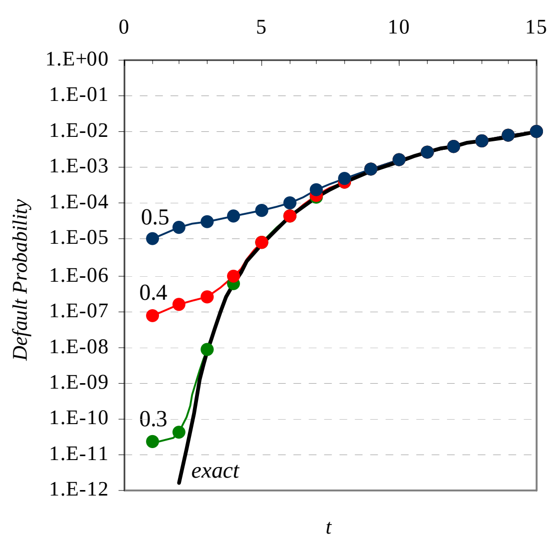

Figure 1 : Calibration of the RBF default probability with its exact formula for $dx = dz$ without jump and at $\rho = 10$. In the RBF method, we consider scattered points with equal spacings of $\Delta x = 0.1$ and $x_N = 100$. We also adopt in the RBF solutions $\kappa = 0.5(\Delta x)^{-2}$, $0.4(\Delta x)^{-2}$, and $0.3(\Delta x)^{-2}$, respectively.

[1] Note the use of the integral $\int_0^\infty ds\, \mathrm{G}(s, t)\, e^{\Lambda s} = (1 - \nu\Lambda)^{-t/\nu}$.

Model Calibration

To demonstrate the use of the RBF approach, we discuss here the cases of standard Brownian motion $dx = dz$ and driftless CIR process $dx = \sqrt{x}\, dz$ for the credit quality process in the model. As defined in (15), it is easy to show that $\pi_{ij} = ½\,\phi''_{ij}$ and $\pi_{ij} = ½\,x_i\phi''_{ij} + \phi'_{ij}$, respectively, in these two cases [2]. In our numerical analysis, we consider scattered points $x_0, x_1, x_2, \ldots , x_N$ at equal spacings of $\Delta x = 0.1$, and take $x_N = 50$ for the Brownian motion and $x_N = 200$ for the CIR process as it is more volatile. We attempt to calibrate the 14 free parameters $\theta_{1,2,3,4,5,6}$, $\rho_{1,2,3,4,5,6,7}$, and ν in the credit barrier model by comparing the RBF solution of (7) and (8) with empirical data provided by Moody's. In particular, we consider a least-square error fitting of these parameters to the data for one-year migration and default probabilities as shown in Table 1. Following an enormous numerical work, the best fit parameters for these two models are estimated to be

Standard Brownian Motion : $\theta_1 = 1.5$, $\theta_2 = 3.3$, $\theta_3 = 5.3$, $\theta_4 = 7.7$, $\theta_5 = 10.8$, $\theta_6 = 14.5$

$\rho_1 = 0.9$, $\rho_2 = 2.6$, $\rho_3 = 4.2$, $\rho_4 = 6.4$, $\rho_5 = 8.8$, $\rho_6 = 11.8$, $\rho_7 = 15.4$

$\nu = 8.2$

least-square error sum = 0.000254

Driftless CIR process : $\theta_1 = 1.3$, $\theta_2 = 5.0$, $\theta_3 = 11.4$, $\theta_4 = 21.9$, $\theta_5 = 39.7$, $\theta_6 = 66.7$

$\rho_1 = 0.8$, $\rho_2 = 3.5$, $\rho_3 = 8.2$, $\rho_4 = 16.2$, $\rho_5 = 28.5$, $\rho_6 = 47.3$, $\rho_7 = 75.5$

$\nu = 6.3$

least-square error sum = 0.000298

The least-square sums of deviation for the two models are found to be similar in the fitting. As shown also in Table 1, the above parameter sets can both generate probability values that give very good match with empirical measurements. In Table 2, we have also shown that the same parameter sets could be able to provide reasonably good match with migration and default data for a longer time horizon.

[2] Denote $\phi'_{ij} = \phi'(\| x_i - x_j \|)$ and $\phi''_{ij} = \phi''(\| x_i - x_j \|)$

Migration From \ To	*Caa-C*	*B*	*Ba*	*Baa*	*A*	*Aa*	*Aaa*	*Default*
				One-year Empirical Measurements				
Caa-C	78.3	6	1.45	0.37	0.04	0.02	0.00	13.81
B	3.54	85.2	6.52	0.69	0.14	0.04	0.00	3.87
Ba	0.46	5.57	87.08	5.11	0.44	0.09	0.02	1.25
Baa	0.07	0.68	5.25	89.16	4.22	0.27	0.04	0.31
A	0.02	0.11	0.69	5.11	91.36	2.5	0.08	0.14
Aa	0.00	0.03	0.18	0.7	6.11	91.62	1.29	0.07
Aaa	0.00	0.00	0.02	0.25	1.04	6.51	92.18	0.01
				Standard Brownian Motion				
Caa-C	78.47	5.94	1.04	0.26	0.062	0.010	0.0015	14.22
B	3.23	85.18	6.58	1.20	0.26	0.041	0.0057	3.50
Ba	0.73	5.64	86.91	4.66	0.77	0.11	0.015	1.17
Baa	0.15	0.84	4.85	89.21	4.16	0.44	0.052	0.30
A	0.032	0.17	0.69	5.17	91.24	2.41	0.22	0.072
Aa	0.0053	0.027	0.099	0.50	6.10	91.69	1.56	0.013
Aaa	0.00068	0.0033	0.012	0.053	0.40	7.00	92.53	0.0018
				Driftless CIR Process				
Caa-C	78.73	6.68	0.53	0.078	0.012	0.0013	0.00012	13.97
B	3.80	85.31	6.47	0.67	0.090	0.0092	0.00082	3.66
Ba	0.78	5.40	87.09	5.14	0.50	0.046	0.0039	1.04
Baa	0.16	0.86	5.14	89.09	4.20	0.28	0.021	0.25
A	0.030	0.15	0.66	5.12	91.35	2.49	0.14	0.052
Aa	0.0046	0.022	0.086	0.48	6.17	91.67	1.56	0.0088
Aaa	0.00052	0.0024	0.0091	0.045	0.38	6.93	92.62	0.0050

Table 1: One-year migration and default probabilities (in percent). Empirical data is taken from Exhibit 8 of Carty (1997) that corresponds to no rating withdrawal.

Migration From \ To	*Caa-C*	*B*	*Ba*	*Baa*	*A*	*Aa*	*Aaa*	*Default*
				Two-year Empirical Measurements				
Caa-C	66.73	8.29	2.46	0.92	0.04	0.02	0.00	21.54
B	4.89	74.8	10.68	1.46	0.28	0.06	0.00	7.82
Ba	0.98	8.57	77.36	9.1	0.9	0.17	0.04	2.88
Baa	0.14	1.51	7.85	81.58	7.51	0.49	0.06	0.85
A	0.05	0.25	1.44	8.09	85.48	4.25	0.11	0.32
Aa	0.02	0.03	0.47	1.59	10.58	84.9	2.23	0.18
Aaa	0.00	0.01	0.16	0.49	2.19	10.35	86.78	0.00
				Standard Brownian Motion				
Caa-C	61.37	9.82	2.01	0.54	0.14	0.024	0.0035	26.10
B	5.57	72.80	11.44	2.40	0.55	0.093	0.014	7.12
Ba	1.41	9.87	75.81	8.53	1.59	0.25	0.034	2.51
Baa	0.31	1.69	8.74	79.72	7.80	0.95	0.12	0.66
A	0.070	0.36	1.42	9.38	83.39	4.73	0.48	0.17
Aa	0.012	0.060	0.22	1.05	11.06	84.37	3.20	0.031
Aaa	0.0016	0.0079	0.027	0.12	0.84	12.61	86.39	0.0044
				Driftless CIR Process				
Caa-C	61.79	10.52	1.04	0.17	0.027	0.0031	0.00030	26.46
B	6.75	72.95	10.97	1.37	0.20	0.022	0.00211	7.74
Ba	1.58	9.78	75.92	9.19	1.06	0.11	0.0098	2.35
Baa	0.35	1.81	9.44	79.32	7.81	0.62	0.052	0.60
A	0.071	0.34	1.42	9.49	83.34	4.88	0.33	0.13
Aa	0.011	0.052	0.20	1.05	11.34	84.11	3.21	0.023
Aaa	0.0014	0.0061	0.023	0.11	0.84	12.64	86.36	0.013
				Three-year Empirical Measurements				
Caa-C	56.8	10.12	3.88	1.04	0.03	0.00	0.00	28.15
B	5.6	65.88	13.67	2.26	0.44	0.1	0.01	12.04
Ba	1.44	10.59	69.18	12.14	1.62	0.25	0.05	4.72
Baa	0.26	2.12	9.55	75.43	10.26	0.76	0.08	1.54
A	0.07	0.45	2.19	10.26	80.42	5.8	0.18	0.63
Aa	0.02	0.09	0.76	2.53	14.54	78.68	3.09	0.29
Aaa	0.00	0.02	0.36	0.75	3.26	13.93	81.64	0.03
				Standard Brownian Motion				
Caa-C	47.99	12.16	2.87	0.84	0.22	0.041	0.0063	35.88
B	7.16	62.57	14.89	3.56	0.88	0.16	0.024	10.75
Ba	2.01	12.92	66.52	11.67	2.44	0.41	0.060	3.97
Baa	0.47	2.53	11.79	71.50	10.92	1.50	0.20	1.10
A	0.11	0.57	2.16	12.73	76.44	6.91	0.79	0.28
Aa	0.021	0.10	0.36	1.63	15.00	77.98	4.86	0.055
Aaa	0.0029	0.014	0.047	0.20	1.33	17.01	81.38	0.0081
				Driftless CIR Process				
Caa-C	48.52	12.41	1.51	0.27	0.047	0.0056	0.00058	37.24
B	8.87	62.69	13.92	2.07	0.33	0.039	0.0040	12.07
Ba	2.36	13.16	66.47	12.25	1.66	0.18	0.018	3.89
Baa	0.57	2.79	12.89	70.82	10.79	1.00	0.092	1.04
A	0.12	0.57	2.23	13.07	76.13	7.09	0.56	0.24
Aa	0.020	0.093	0.34	1.68	15.52	77.41	4.88	0.043
Aaa	0.0026	0.011	0.041	0.19	1.38	17.20	81.16	0.024

Table 2 : Two-year and three-year migration and default probabilities (in percent). Empirical data is taken from Exhibit 17 of Carty (1997) with the probability mass associated with rating withdrawal distributed across the remaining categories on a probability weighted basis.

Risk-Neutral Process

In order to deal with derivative pricings, it is necessary to bring the credit barrier model from the risk-averse real world into a risk-neutral preference. As it has been discussed in Albanese and et. al. (2003), the risk-neutral credit quality process dg can be deduced from its real-world counterpart dx in (1) through an iso-volatility transformation of credit quality that leave the volatility invariant. By Ito's Lemma, it follows immediately that such transformation from x to g must satisfy the condition

$$\sigma(x)\frac{\partial g}{\partial x} = \sigma(g) \tag{20}$$

On integrating the above expression, we find that the credit qualities under the two risk preferences can be related by the addition of a time function as

$$\Phi(g) = \Phi(x) + \chi(t) \quad , \quad \Phi(u) = \int_0^u \frac{dw}{\sigma(w)} \quad , \quad \chi(0) = 0 \tag{21}$$

The transformation is therefore specified by this time function that again needs to be calibrated with data on risk-neutral default probabilities implied by the derivative market. The initial value of $\chi(0)$ in (21) follows from the fact that the two credit qualities must coincide at current time. In this sense, the initial credit qualities $\rho_{1,2,3,4,5,6,7}$ defined previously in the real world should remain valid when dealing with the risk-neutral process.

In going from the real-world into a risk-neutral preference, the unattainable upper boundary remains unaltered at infinity. The absorbing lower boundary at zero has been transformed into a time function of default barrier given by

$$\beta(t) = \Phi^{-1}(\ \chi(t)\ ) \quad , \quad \beta(0) = 0 \tag{22}$$

The initial value of the default barrier follows immediately from the condition that $\chi(0) = 0$. Thus, credit default occurs whenever the risk-neutral process hits this time barrier and is being absorbed. Note that $\Phi(u)$ in (21) is a one-to-one increasing function as $\sigma(w)$ is positively defined. The default barrier $\beta(t)$ is accordingly the lower boundary of the risk-neutral process. It then follows that the risk-neutral default probability for initial rating R_l can be defined as

$$Q_l(t) = 1 - \int_{\beta(t)}^{\infty} dg\ V(t, g \mid \rho_l) \tag{23}$$

where $V(t, g \mid \rho)$ is the risk-neutral transition density for nonzero $g(0) = \rho$ going to $g(t) = g$. By Ito's Lemma and with the use of the relation in (21), the risk-neutral process derived through an iso-volatility transformation can be written as

$$dg = \gamma(g, t)\ dt + \sigma(g)\ dz \tag{24}$$

where

$$\gamma(g, t) = \sigma(g)\ \chi'(t) + \mu(x)\frac{\sigma(g)}{\sigma(x)} + \frac{1}{2}\sigma(g)\ [\ \sigma'(g) - \sigma'(x)\ ]$$

The risk-neutral transition density satisfies the Kolmogorov equation,

$$\frac{\partial}{\partial t}V(t, g \mid \rho) = \frac{1}{2}\frac{\partial^2}{\partial g^2}[\ \sigma^2(g)V(t, g \mid \rho)\ ] - \frac{\partial}{\partial g}[\ \gamma(g, t)V(t, g \mid \rho)\ ] \tag{25}$$

with boundary conditions $V(0, g \mid \rho) = \delta(g - \rho)$, $V(t, \beta(t) \mid \rho) = 0$, and $V(t, \infty \mid \rho) = 0$. It must be solved numerically utilizing the RBF method as the equation involves time-dependent boundary condition.

As before, suppose $g_0, g_1, g_2, \dots, g_N$ are scattered points in the domain with $g_0 = 0$ located on the lower boundary. The risk-neutral transition density can be expanded in the RBF method as

$$V(t, g \mid \rho) = \sum_{j=0}^{N} \lambda_j(t, \rho)\ \phi(\| g - g_j \|) \tag{26}$$

To include the time-dependent absorption boundary $V(t, \beta(t) \mid \rho) = 0$ in the solution, consider a discrete approximation of the barrier function using the dense set of points $\{ g_0, g_1, g_2, \dots, g_N \}$ for which we can approximate the barrier as $\beta(t) = g_{b(t)}$ with $b(0) = 0$. It follows in this scheme that the absorption boundary can be satisfied by imposing the conditions that

$$V(t, g_0 \mid \rho) = V(t, g_1 \mid \rho) = \dots = V(t, g_{b(t)} \mid \rho) = 0 \tag{27}$$

Using these vanishing conditions, we can eliminate $\lambda_0(t, \rho), \lambda_1(t, \rho), \dots, \lambda_{b(t)}(t, \rho)$ in (26) and rewrite the RBF expansion as

$$V(t, g \mid \rho) = \sum_{j=b(t)+1}^{N} \lambda_j(t, \rho)\ D(g, g_j, t) \quad , \quad D(g, g_j, t) = \phi(\| g - g_j \|) - \sum_{h=0}^{b(t)} \sum_{k=0}^{b(t)} \phi(\| g - g_h \|)\ \Gamma_{hk}\ \phi_{kj} \tag{28}$$

where

$$\boldsymbol{\Gamma} = \begin{bmatrix} \phi_{00} & \phi_{01} & \dots & \phi_{0b} \\ \phi_{10} & \phi_{11} & \dots & \phi_{1b} \\ \vdots & \vdots & & \vdots \\ \phi_{b0} & \phi_{b1} & \dots & \phi_{bb} \end{bmatrix}^{-1}$$

Similarly, we can convert the KG equation in (25) into a system of linear order differential equations for the RBF coefficients $\boldsymbol{\lambda}(t, \rho) = (\ \lambda_{b(t)+1}(t, \rho), \dots, \lambda_N(t, \rho)\ )^T$ given by [3]

$$\frac{\partial}{\partial t}\boldsymbol{\lambda}(t, \rho) = [\ \boldsymbol{D}^{-1}(t)\boldsymbol{C}(t)\ ]\ \boldsymbol{\lambda}(t, \rho) \tag{29}$$

where

$$D_{ij} = \phi_{ij} - \sum_{h=0}^{b(t)} \sum_{k=0}^{b(t)} \phi_{ih}\ \Gamma_{hk}\ \phi_{kj} \quad , \quad C_{ij} = \omega_{ij} - \sum_{h=0}^{b(t)} \sum_{k=0}^{b(t)} \omega_{ih}\ \Gamma_{hk}\ \phi_{kj}$$

$$\omega_{ij} = \left[\ \frac{1}{2}\frac{\partial^2}{\partial g^2}[\ \sigma^2(g)\ \phi(\| g - g_j \|)\ ] - \frac{\partial}{\partial g}[\ \gamma(g, t)\ \phi(\| g - g_j \|)\ ]\ \right]_{g = g_i}$$

[3] In the discrete scheme, we assume $\beta'(t)$ is not too large such that the contribution to $\partial/\partial t$ coming from the change in $b(t)$ in the summation can presumably be neglected.

Since both C_{ij} and D_{ij} are time dependent in this case, equation (29) can only be solved iteratively through time evolution of the coefficients generated by

$$\lambda(t + \Delta t, \rho) \cong \lambda(t, \rho) + \Delta t\ [\ \boldsymbol{D}^{-1}(t)\boldsymbol{C}(t)\ ]\ \lambda(t, \rho) \tag{30}$$

The initial coefficients can again be derived from the boundary $V(0, g \mid \rho) = \delta(g - \rho)$. For ρ to stay inside the interval $(x_{k-1}, x_k]$, it can be shown similarly that $\lambda_i(0, \rho) = (\Delta g_k)^{-1}\ (D^{-1}(0))_{ik}$.

As before, the transition density with jump can be obtained by integrating out the gamma time in (28) with the same variance rate ν. Here, the gamma integral can only be performed numerically as the transition density is generated iteratively. The task is then enormous as the resulting jump default probability would follow from another integral on credit quality. To ease the numerical burden in such calculation, we apply the gamma integration to the default probability given by (23) and introduce an upper cutoff time in the integral as

$$Q^J_l(t) = \int_0^\infty ds\ \mathrm{G}(s, t)\ Q_l(s) = \int_0^c ds\ \mathrm{G}(s, t)\ Q_l(s) + E(c) \quad , \quad E(c) \approx \int_c^\infty ds\ \mathrm{G}(s, t) > 0 \tag{31}$$

The cutoff correction $E(c)$ in (31) is positively defined and can be approximated by its upper limit as we expect $Q_l(s) \approx 1$ for $s \geq c$. This will be useful in fixing the cutoff value in numerical analysis under defined accuracy.

Calibration of Risk-Neutral Process

As discussed in previous section, the transformation of credit quality from the real-world into a risk-neutral preference is completely specified by the time function $\chi(t)$ that can be calibrated with empirical data on risk-neutral default probabilities. We consider here the implied values from market term structure of credit spread rates for bonds as shown in Table 3. As the evaluation of credit spreads from quoted yield spreads for high rating bonds are overwhelmed by other factors, we calibrate the time function here using only the data from low rating bonds. For risk-neutral default probability with initial rating R_l, the market implied value can be derived using the simple expression

$$Q_l^{market}(t) = \frac{1 - e^{-S_l(t)\, t}}{1 - rc_l} \tag{32}$$

where $S_l(t)$ is the market credit spread rate of term t for bond with rating R_l, and rc_l is the average recovery rate for bond with the same rating.

	Credit Spread Rates (*basis points*)				*Recovery Rates* (*percent*)
	1 year	2 years	3 years	5 years	
Caa-C	1801	1704	1607	1460	38.02
B	898	848	799	675	37.54
Ba	588	563	537	485	39.05
Baa	92	104	112	120	49.24
	Implied Risk-Neutral Default Probabilities (*percent*)				
Caa-C	26.60	46.60	61.72	83.59	
B	13.74	24.98	34.11	45.88	
Ba	9.38	17.46	24.41	35.33	
Baa	1.80	4.07	6.49	11.45	

Table 3 : Implied risk-neutral default probabilities derived from the market credit spread rates and the average recovery rates for bonds with different ratings. Credit spread data is tax-adjusted market rates taken from Albanese and et. al. (2003). The recovery rates are historical values given by Altman and Kishore (1998).

In case of a standard Brownian motion $dx = dz$, it is easy to show that the iso-volatility transformation of credit quality is given by $g = x + \chi(t)$ with $\chi(t) = 0$, and the risk-neutral process is found to be $dg = \chi'(t)\, dt + dz$ with default barrier $\beta(t) = \chi(t)$. The risk-neutral transition density can be obtained by performing the numerical iteration generated by (30) with $\omega_{ij} = ½\, \phi''_{ij} - \chi'(t)\phi'_{ij}$. However, there is a simple solution to this problem by considering the function $U(t, g - \chi(t) \mid \rho)$ defined through the real-world density in (10). It can be shown that the function satisfies the same KG equation and boundary conditions for the risk-neutral density in (25). It then follows that the risk-neutral transition density in this model is simply given by [4]

$$V(t, g \mid \rho) = U(t, g - \chi(t) \mid \rho) \tag{33}$$

[4] This is also true for Brownian motion $dx = \mu\, dt + \sigma\, dz$ with constant drift and volatility.

Furthermore, we can use (33) to show in this model that default probability is invariant under an iso-volatility transformation. They are therefore identical in the real-world and risk-neutral preferences, and immediately for the jump measure

$$Q^J_l(t) = P^J_l(t) = 2\ \psi(-\rho_l, t) \tag{34}$$

that is independent of the time function $\chi(t)$. As shown in Table 5, the risk-neutral default probabilities in this model deviate from the implied values under this general expression with initial credit qualities $\rho_{1,2,3,4}$ and variance rate ν calibrated previously. It does not seem to fit the derivative data despite that it gives very good match in the real-world measures.

To demonstrate the use of the RBF approach in the current problem, we continue the discussion on the case of a driftless CIR process $dx = \sqrt{x}\, dz$. It can be shown that the iso-volatility transformation of credit quality is given by

$$\sqrt{g} = \sqrt{x} + \frac{1}{2}\chi(t) \tag{35}$$

and the risk-neutral process is calculated to be

$$dg = \gamma(g, t)\, dt + \sqrt{g}\, dz \quad , \quad \gamma(g, t) = \sqrt{g}\ \chi'(t) - \frac{1}{4}\frac{\chi(t)}{2\sqrt{g} - \chi(t)} \tag{36}$$

with default barrier $\beta(t) = ¼\, \chi^2(t)$. Note here that $\sigma(w) = +\sqrt{w}$ is only defined for $w \geq 0$ in (21), the transformation in (35) will cover the entire domain of $x \in [0,\infty)$ only if $\chi(t)$ is positively defined. In this model, the risk-neutral transition density must be solved iteratively utilizing the time evolution equation in (30) with

$$\omega_{ij} = \frac{1}{2}\, g_i\, \phi''_{ij} + [1 - \gamma(g_i, t)\,]\phi'_{ij} - \phi_{ij}\frac{\partial}{\partial g}\gamma(g_i, t)$$

The risk-neutral default probability with jump in (31) can then be calculated by introducing a cutoff time in the integration that also ceases the iteration. The cutoff time c is fixed below according to our current limit on computing speed, and the cutoff correction can be estimated using the equation in (31). To save large amount of numerical work, we parameterize the time function in a simple polynomial form $\chi(t) = at^{\alpha}$ that satisfies the initial condition $\chi(0) = 0$ and with $a \geq 0$ for $\chi(t)$ being positive. With this simple parameterization, we have compared below the risk-neutral default probabilities evaluated based on quadratic ($\alpha = 1$) and cubic ($\alpha = 3/2$) barriers in order to manifest the essence of $\chi(t)$ in matching with the implied values in contrast to the previous case of a standard Brownian process.

The iteration time increment in (30) is chosen to be $\Delta t = 0.00001$ in order to ensure numerical stability in the solution. In the parameterization of $\chi(t)$ above, the default barrier is given by the function $\beta(t) = ¼\, a^2 t^{2\alpha}$ that goes up in power order of iteration time. As we have $\beta(t) = g_{b(t)}$ or $b(t) = \beta(t)/\Delta g$ at equal spacings, the size of Γ in (28), given by $(b(t) + 1)\times(b(t) + 1)$, also grows in the same pace. This renders the iteration speed to slow down progressively as the transformation matrix [$\boldsymbol{D}^{-1}(t)\boldsymbol{C}(t)$] is defined in term of matrix multiplication involving Γ. For an iterative solution to be feasible, it is therefore essential to terminate the iteration by a proper cutoff time c at which the error involved is acceptable. With variance rate of $\nu = 6.3$ in this model, the cutoff corrections in the calculation of the jump default probabilities in (31) are given in Table 4 as

Cutoff Values	*Cutoff Corrections* (*percent*) 1 year	2 years	3 years	5 years
10	1.67	3.96	6.84	14.20
15	0.59	1.47	2.66	6.11
20	0.22	0.57	1.06	2.63
25	0.085	0.22	0.44	1.14
30	0.034	0.092	0.18	0.50
35	0.014	0.038	0.077	0.22
40	0.0056	0.016	0.033	0.098
45	0.0023	0.0066	0.014	0.043
50	0.0010	0.0028	0.0061	0.019

Table 4 : The estimated cutoff corrections in calculating the jump risk-neutral default probabilities of different time horizons.

In our numerical analysis, we consider scattered points $g_0, g_1, g_2, \dots, g_N$ at equal spacings of $\Delta g = 0.2$ and take $g_N = 100$ away from the initial credit qualities $\rho_1 = 0.8$, $\rho_2 = 3.5$, $\rho_3 = 8.2$, and $\rho_4 = 16.2$ to be considered. We choose the cutoff time $c = 20$ in consideration of our processor speed limit and the location of the barrier $\beta(c)$ relative to g_N. Table 5 shows the estimated risk-neutral default probabilities based on both the quadratic and cubic barrier functions with, for example, $a = 0.1$ for initial credit ratings of { *Caa-C*, *B*, *Ba*, *Baa* }. The results demonstrate clearly the essential role of $\chi(t)$ in defining the risk-neutral measure in the credit barrier model as a variety of term structures can presumably be generated. It is therefore worth investigating a richer functional structure for $\chi(t)$ as well as the stochastic structure for the underlying credit quality process in the model. As the RBF implementation allows great flexibility in these structures through the terms π_{ij} and ω_{ij}, there exists no great difficulty in generalizing the model that would be capable of capturing both the migration and derivative datas. This will be addressed in forthcoming work.

	Risk-Neutral Default Probabilities (*percent*) 1 year	2 years	3 years	5 years
	Standard Brownian Motion			
Caa-C	14.22	26.10	35.88	50.38
B	3.50	7.12	10.75	17.78
Ba	1.17	2.51	3.97	7.16
Baa	0.30	0.66	1.10	2.15
	Driftless CIR Process			
	$\chi(t) = 0.1\,t$ (quadratic barrier)			
Caa-C	13.45	25.47	35.78	51.73
B	3.26	6.94	10.88	19.17
Ba	0.93	2.16	3.68	7.54
Baa	0.37	0.92	1.66	3.83
	$\chi(t) = 0.1\,t^{3/2}$ (cubic barrier)			
Caa-C	14.99	28.34	39.76	57.26
B	4.54	9.60	14.94	25.85
Ba	1.54	3.49	5.79	11.31
Baa	0.46	1.13	2.04	4.68

Table 5 : Risk-neutral default probabilities based on different credit quality processes.

Conclusions

Credit barrier model has provided a framework for the pricing of instruments written on creditworthiness of corporate borrower. The success of the credit barrier model relies on accurate calibration with empirical data from both the rating agencies and derivatives market. We discuss here the numerical implementation of this model using the Radial Basis Function method and demonstrate a great capability in making a good match with empirical data. The RBF method is shown to be numerically tractable in this problem and allows great flexibility in the stochastic and functional structures of the model. This is crucial when it comes to the issue of capturing aggregate amount of relevant empirical information in the model.

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