Lecture Notes for Finance Mathematics MA3180

Chapter One Partial Derivatives

1.1 Functions of several variables

(1) *Basic idea.*

Functions of single variables:

$$y = \sin x \qquad y = \cos x + x^2, \qquad y = e^x + \log(x)$$

and the general form

$$y = f(x)\,.$$

Functions of several variables:

$$u = \sin x + \sin y, \qquad u = x^2 + y^2$$

and the general form:

$$u = f(x, y) \quad \text{or} \quad u = f(x, y, z)\,.$$

(2) *Elementary functions.*

Example 1 Linear function: $z = ax + by + c$. The graph of a linear function is a *plane* in three-dimensional space.

$$z = -3x - 2y + 6\,.$$

and the graph of $z = 0$ is the xy-plane.

Straightline: intersection of two planes, represented by a system of two equations. For example,

$$\begin{cases} z = -3x - 2y + 6 \\ z = 0 \end{cases}$$

Example 2 The function $z = z(x, y)$ defined by

$$x^2 + y^2 + z^2 = a^2\,.$$

Geometrically, this defines a sphere with the center $(0, 0, 0)$ and radius a. Solving the equation for z gives

$$z = \pm\sqrt{a^2 - x^2 - y^2}\,.$$

The graphs of these two functions are the upper and lower hemispheres, respectively.

Example 3 The graph of the function $f(x, y) = 0$ in three-dimensional space.

Example 4 Some other special geometries

$z = x^2 + y^2$	circular paraboloid
$z^2 = x^2 + y^2$	circular cone
$z = y^2 - x^2$	hyperbolic paraboloid
$x^2 + y^2 = 4$	circular cylinder

(3) *Level curves.* The level curves of a function $f(x, y)$ are the curves with equations

$$f(x, y) = k$$

where k is a constant.

Example 5 $f(x, y) = x^2 + y^2$.

(4) Inequalities and domain

Example 6 $x + y > 1$. Since $x + y = 1$ denotes a straight line geometrically, which divides the plane into two parts. $x + y > 1$ denotes the domain of one of two parts.

Example 7

$$S = \{(x, y) : x^2 + y^2 \leq 1\}$$

determines a bounded domain in xy-plane.

Example 8 Some other examples:

$$S_1 = \{(x, y) : \quad x^2 + y^2 \leq 1 \text{ and } x + y \geq 1\}$$
$$S_2 = \{(x, y, z) : 0 \leq x, y, z \leq 1\}$$

1.2 Partial Derivatives

(1) *Limit.*

$$\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0) \qquad \text{if } f \text{ is continuous}$$

Example 1

$$\lim_{(x,y)\to(0,0)} (\sin x + \cos y) = 1$$

It should be noted that the limit $\lim_{(x,y)\to(x_0,y_0)} f(x, y)$ exists only if $f(x, y) \to f(x_0, y_0)$ as $(x, y) \to (x_0, y_0)$ independent of the manner in which $(x, y) \to (x_0, y_0)$.

Example 2 Prove that

$$\lim_{(x,y)\to(0,0)} \frac{x^2}{x^2 + y^2}$$

does not exist.

Proof (i) As $(x, y) \to (0, 0)$ along the line $y = 0$,

$$f(x, 0) = \frac{x^2}{x^2} = 1\,.$$

(ii) As $(x, y) \to (0, 0)$ along the line $x = 0$,

$$f(0, y) = 0$$

Thus the limit does not exist.

(2) Definition of partial derivatives

The definition of derivatives of a single variable function $y = f(x)$ is defined by

$$\frac{df}{dx}|_{x=x_0} = \lim_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0} \qquad \text{(if this limit exist)}\,.$$

We have some different derivatives for functions of multiple variables.

Let $u = f(x, y)$ be a function of two variables. The partial derivative of $f(x, y)$ with respect to x is

$$\frac{\partial f}{\partial x}|_{x=x_0} = \lim_{x\to x_0} \frac{f(x,y)-f(x_0,y)}{x-x_0} \qquad \text{(if this limit exist for some fixed } y)\,.$$

The partial derivative of $f(x, y)$ with respect to y is

$$\frac{\partial f}{\partial y}|_{y=y_0} = \lim_{y\to y_0} \frac{f(x,y)-f(x,y_0)}{y-y_0} \qquad \text{(if this limit exist for some fixed } x)\,.$$

Notation: Let $z = f(x, y)$. Its partial derivatives are denoted by

$$\frac{\partial f}{\partial x} = \frac{\partial z}{\partial x} = f_x = z_x \qquad \frac{\partial f}{\partial y} = \frac{\partial z}{\partial y} = f_y = z_y\,.$$

The second order partial derivatives and higher-order partial derivatives can be defined similarly

$$\begin{aligned} \frac{\partial}{\partial x}\frac{\partial f}{\partial x} &= \frac{\partial^2 f}{\partial x^2} = f_{xx} = z_{xx} \\ \frac{\partial}{\partial y}\frac{\partial f}{\partial y} &= \frac{\partial^2 f}{\partial y^2} = f_{xx} = z_{yy} \\ \frac{\partial}{\partial x}\frac{\partial f}{\partial y} &= \frac{\partial^2 f}{\partial x \partial y} = f_{xy} = z_{xy} \end{aligned}$$

Usually (not always)

$$f_{xy} = f_{yx}$$

Important: We can calculate the partial derivatives by using those classical formulas in single variable calculus where other variables are assumed to be constants.

Example 3 $f(x, y) = \sin x + y$. Find f_x and f_y.

$$f_x = \cos x \qquad f_y = 1\,.$$

Example 4 Let $f(x, y) = e^{2x}\cos(xy)$. Find f_x, f_y and f_{yy}.

$$\begin{aligned} &f_x = 2e^{2x}\cos(xy) + e^{2x}(-\sin(xy))y = e^{2x}(2\cos(xy) - y\sin(xy)) \\ &f_y = e^{2x}(-\sin(xy))x = -xe^{2x}\sin(xy) \\ &f_{yy} = -xe^{2x}\cos((xy))x = -x^2e^{2x}\cos(xy)\,. \end{aligned}$$

Example 5 Calculate f_{xxyz} if $f(x,y) = \sin(3x + yz)$.

$$\begin{aligned} f_x &= 3\cos(3x + yz) \\ f_{xx} &= -9\sin(3x + yz) \\ f_{xxy} &= -9z\cos(3x + yz) \\ f_{xxyz} &= 9yz\sin(3x + yz) - 9\cos(3x + yz)\,. \end{aligned}$$

1.3 Two basic rules

Chain rule

In single variable calculus, if

$$y = f(x), \qquad x = x(t)$$

then $y = f(x(t))$ and we have

$$y_t = f_x \cdot x_t$$

In multivariable calculus,

(i) if $u = f(x,y)$, $x = x(t)$ and $y = y(t)$i, u is a functions of t defined by

$$u = f(x(t), y(t))\,.$$

Then

$$\frac{du}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

(ii) if

$$u = f(x,y), \quad x = x(t,s), \quad y = y(t,s)$$

u is a function of t and s and

$$\frac{\partial u}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \tag{1}$$

$$\frac{\partial u}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} \tag{2}$$

The case (ii) reduces to the case (i) when $x = x(t)$ and $y = y(t)$.

Example 6 Let $z = f(x,y) = x^2 + 2xy$ and $x = \sin(t)$, $y = \cos(t)$. Find z_t by using the chain rule.

We use the formula

$$z_t = f_x x_t + f_y y_t\,.$$

Since

$$\begin{aligned} &f_x = 2x + 2y, \quad f_y = 2x, \quad x_t = \cos(t), \quad y_t = -\sin(t) \\ &z_t = (2x + 2y)\cos(t) + 2x(-\sin(t)) = 2(\sin(t) + \cos(t))\cos(t) - 2\sin^2(t)\,. \end{aligned}$$

Example 7 Let $T = f(x, y) = x^3 - xy + y^3$ where $x = \rho\cos(\phi)$ and $y = \rho\sin(\phi)$. Calculate T_ρ and T_ϕ by using the Chain rule.

We use the formula

$$T_\rho = f_x x_\rho + f_y y_\rho, \quad T_\phi = f_x x_\phi + f_y y_\phi\,.$$

Since

$$f_x = (3x^2 - y), \quad f_y = (-x + 3y^2), \quad x_\rho = \cos(\phi), \quad y_\rho = \sin(\phi)$$

and

$$x_\phi = -\rho\sin(\phi), \quad y_\phi = \rho\cos(\phi)\,,$$

then

$$T_\rho = (3x^2 - y)\cos(\phi) + (-x + 3y^2)\sin(\phi)$$

and

$$T_\rho = (3x^2 - y)(-\rho\sin(\phi) + (-x + 3y^2)\rho\cos(\phi)$$

Implicit function rule

A function could be defined in different ways.

$$g(x, y) = 0$$

may define a function $y = f(x)$ implicitly. For example: $x^2 + y^2 = 1$ defines a function y of x implicitly. In this case, we may have the explicit form:

$$y = \pm\sqrt{1 - x^2}\,.$$

For many other implicit functions, we may not have their explicit forms. The question here is how to find the derivatives of such functions.

Generally, as to implicit function $g(x, y) = 0$, the derivatives $\frac{dy}{dx}$ can by expressed by

$$\frac{dy}{dx} = -\frac{g_x(x, y)}{g_y(x, y)}$$

Example 8 Let $\frac{x^2}{a^2} - \frac{y^2}{b^2} = c^2$, a, b, c are constants. To find $\frac{dy}{dx}$

We use above formula and obtain

$$\frac{dy}{dx} = -\frac{g_x(x, y)}{g_y(x, y)} = -\frac{\frac{2x}{a^2}}{-\frac{2y}{b^2}} = \frac{b^2 x}{a^2 y} \tag{3}$$

1.4 Directional derivatives and the gradient vector

Directional derivatives

For a given direction defined by a unit vector $\vec{u} = (a, b)$, the directional derivative of function $f(x, y)$ at (x_0, y_0) in the direction $\vec{u}$ is

$$D_u f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists. And in fact, $D_u f(x_0, y_0)$ represents the rate of change of $f(x, y)$ in the direction $\vec{u}$.

If $f(x, y)$ is a differentiable function of x and y, then f has a directional derivative in the direction of any unit vector $\vec{u} = (a, b)$ and

$$D_u f(x_0, y_0) = f_x(x, y)a + f_y(x, y)b$$

here, unit vector $\vec{u} = (a, b)$ means that $a^2 + b^2 = 1$

Example 9 Find the directional derivative $D_u f(x, y)$ if

$$f(x, y) = x^3 - 3xy + 4y^2$$

and $\vec{u}$ is the unit vector given by $(\frac{1}{2}, \frac{\sqrt{3}}{2})$, what is $D_u f(1, 2)$?

Following the above formula, $f_x = 3x^2 - 3y$, $f_y = -3x + 8y$ and

$$D_u f(x, y) = \frac{1}{2}(3x^2 - 3y) + \frac{\sqrt{3}}{2}(-3x + 8y)\,.$$

Since $f_x(1, 2) = -3$, $f_y(1, 2) = 13$,

$$D_u f(1, 2) = \frac{1}{2} \cdot (-3) + \frac{\sqrt{3}}{2} \cdot 13$$

Gradient

Let $z = f(x, y)$. The gradient of f is the vector function defined by

$$\nabla f(x, y) = \begin{pmatrix} f_x(x, y) \\ f_y(x, y) \end{pmatrix}$$

Example 10 Let $f(x, y) = \sin x + e^{xy}$. Calculate the gradient at (0,1).
Since $f_x = \cos x + ye^{xy}$, $f_x(0, 1) = 2$, $f_y = xe^{xy}$, $f_y(0, 1) = 0$

$$\nabla f(0, 1) = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$$

In a vector notation,

$$D_u f(x, y) = \nabla f \cdot \vec{u}$$

where $\cdot$ denotes the dot product. By some basic properties in vector analysis,

$$D_u f = |\nabla f||\vec{u}| \cos\theta = |\nabla f| \cos\theta$$

where θ denotes an angle between ∇f and $\vec{u}$. It is obvious that the maximum value of the directional derivative $D_u f$ is $|\nabla f(x)|$ and it occurs when $\vec{u}$ is the same direction as the gradient $\nabla f(x)$

Tangent plane

Generally, a curved surface can be expressed by

$$F(x, y, z) = 0$$

Let $P_0(x_0, y_0, z_0)$ be a point on this surface , then the tangent plane pass through P_0 can be expressed by

$$F_x(P_0)(x - x_0) + F_y(P_0)(y - y_0) + F_z(P_0)(z - z_0) = 0\,.$$

Here, $F_x(P_0), F_y(P_0), F_z(P_0))$ is a tangent vector passing through P_0

On the other hand, if we consider such a function $z = f(x, y)$, then the tangent plane pass through P_0 can be expressed by

$$z - z0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Example 11 Find the tangent plane to the elliptic paraboloid $z = 2x^2 + y^2$ at the point $(1, 1, 3)$

Let $f(x, y) = 2x^2 + y^2$. Then

$$f_x(x, y) = 4x, \quad f_y(x, y) = 2y$$

and

$$f_x(1, 1) = 4, \quad f_y(1, 1) = 2$$

The equation of this tangent plane is

$$z - 3 = 4(x - 1) + 2(y - 1)\,.$$

1.5 Taylor's series

For a function $f(x)$ of one variable has infinite continuous derivative $f'(x), f''(x), \cdots, f^(n)(x), \cdots$ in the region around x_0. Then, the Taylor's series of $f(x)$ at x_0 can be expressed by

$$f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots\,.$$

Under certain conditions, the series may converge to the function $f(x)$. In this case, we can write down

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots\,.$$

If we take first $N+1$ (N is finite integer) terms,

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2 + \cdots + \frac{f^{(N)}(x_0)}{N!}(x-x_0)^N$$

is called Taylor's polynomial of degree N.

For a function $f(x,y)$ of two variables, its Taylor's series at (x_0, y_0) is defined in

$$f(x_0,y_0)+[f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)]+\frac{1}{2!}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^2 f(x_0,y_0)$$

$$+\cdots+\frac{1}{n!}\left[(x-x_0)\frac{\partial}{\partial x}+(y-y_0)\frac{\partial}{\partial y}\right]^n f(x_0,y_0)+\cdots$$

Similarly under certain conditions, the series may converge to $f(x,y)$. The Taylor's polynomial can be defined similarly.

Example 12 Find Taylor's series of $f(x) = e^x$

$$f(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots,$$

Example 13 Find Taylor's series of $f(x) = \sin x$

$$f(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + \cdots.$$

Example 14. Find Taylor's series of $f(x) = \cos x$

$$f(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots + (-1)^n \frac{x^{2n}}{(2n)!} + \cdots.$$

Example 15 Let $f(x,y) = (x^2+y^2)^{1/2}$, calculate $f(0.9, 0.1)$ by using Taylor's polynomial with N=2 and N=3, respectively.

We have

$$f_x = \frac{x}{(x^2+y^2)^{1/2}}, f_y = \frac{y}{(x^2+y^2)^{1/2}} \tag{4}$$

$$f_{xx} = \frac{y^2}{(x^2+y^2)^{3/2}}, f_{xy} = \frac{-xy}{(x^2+y^2)^{3/2}}, f_{yy} = \frac{x^2}{(x^2+y^2)^{3/2}} \tag{5}$$

Then, when N=2, $x - x_0 = -0.1, y - y_0 = 0.1$

$$f(0.9, 0.1) \approx f(1,0) + [-0.1 \cdot f_x(1,0) + 0.1 \cdot f_y(1,0)] = 1 + (-0.1 \cdot 1 + 0) = 0.9$$

when N=3

$$f(0.9,0.1) \approx f(1,0) + [-0.1 \cdot f_x(1,0) + 0.1 \cdot f_y(1,0)] +$$
$$\frac{1}{2}[(-0.1)^2]f_{xx}(1,0) + 2(-0.1)(0.1)f_{xy}(1,0) + (0.1)^2 f_{yy}(1,0)]$$
$$= 0.9 + 0.5 \cdot (0.01 \cdot 0 + 0 + 0.01 \cdot 1) = 0.905$$

The exact value $f(0.9, 0.1) = 0.9055385$.

1.6 Maximum and minimum values

For single variable, the extremal points, minimum or maximum points, satisfy the equation

$$f_x = 0$$

The solutions of the above equation and points at which the derivative does not exist are called critical points. The global minimum or maximum values of the function $y = f(x)$ are achieved at these critical points or boundary points.

For a function of two variables, $z = f(x, y)$, the extremal points satisfy the system of the equations

$$f_x(x,y) = 0 \qquad f_y(x,y) = 0\,.$$

Besides, if $f(x, y)$ has continuous second-order partial derivatives, let

$$D(a,b) = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

(1) if $D > 0$, and $f_{xx}(a, b) > 0$, then $f(a, b)$ is a local minimum.
(2) if $D > 0$, and $f_{xx}(a, b) < 0$, then $f(a, b)$ is a local maximum.
(3) if $D < 0$, then $f(a, b)$ is a saddle point.

Example 16 Find the local maximum or minimum of $f(xy) = x^3 - 3xy + y^3$.

Since $f_x = 3x^2 - 3y$, $f_y = -3x + 3^2$, solving the system $f_x = f_y = 0$, we get two critical points $(0, 0)$ and $(1, 1)$. Also, since $f_{xx} = 6x$, $f_{yy} = 6y$, and $f_{xy} = -3$,

$$D = 36xy - 9, \qquad D(0,0) = -9 \quad D(1,1) = 27$$

and therefore, $(0, 0)$ is a saddle point and $(1, 1)$ is a local minimum.

1.7 Lagrange multipliers

Now we consider finding the maximum and minimum values of the function $f(x, y, z)$ subject to the constraint $g(x, y, z) = 0$

$$\begin{aligned} &\min(\text{ or } \max) f(x,y,z) \\ &s.t.\ g(x,y,z) = 0 \end{aligned}$$

Critical points satisfy the following equations

$$\begin{cases} \nabla f(x,y,z) = \lambda \nabla g(x,y,z) \\ g(x,y,z) = 0 \end{cases}$$

By solving above equations, we get all possible extremal points. The largest one is the maximum of $f(x, y, z)$ and the smallest is the minimum.

Example 17

$$\begin{aligned} &\min f(x,y) = x^2 + y^2 \\ &\text{s.t.} i \ x + y - 1 = 0 \end{aligned}$$

We need to solve the following system

$$\begin{cases} 2x = \lambda \\ 2y = \lambda \\ x + y - 1 = 0 \end{cases}$$

The solution is $\lambda = 1, x = 0.5, y = 0.5$. $f(0.5, 0.5) = 0.5$ is the minimum.

Example 18 Find the points on the sphere $x^2 + y^2 + z^2 = 4$ that are closest to and farthest from the point $(3, 1, -1)$.

The distance between a point (x, y, z) and the point $(3, 1, -1)$ is

$$d = \sqrt{(x-3)^2 + (y-1)^2 + (z+1)^2}\,.$$

Let

$$f(x, y) = d^2 = (x-3)^2 + (y-1)^2 + (z+1)^2\,.$$

The problem can be described by

$$\begin{array}{l} \min f(x,y) \\ x^2 + y^2 + z^2 - 4 = 0 \end{array} \quad \text{and} \quad \begin{array}{l} \max f(x,y) \\ x^2 + y^2 + z^2 - 4 = 0 \end{array}$$

The system to be solved is

$$\begin{aligned} 2(x-3) &= 2\lambda x \\ 2(y-1) &= 2\lambda y \\ 2(z+1) &= 2\lambda z \\ x^2 + y^2 + z^2 - 4 &= 0 \end{aligned}$$

Obviously from the first three equations,

$$x = \frac{3}{1-\lambda}, \quad y = \frac{1}{1-\lambda}, \quad z = \frac{-1}{1-\lambda}$$

Substituting them into the last equation of the system gives

$$\frac{11}{(1-\lambda)^2} = 4$$

and the solution is $\lambda = 1 \pm \sqrt{11}/2$ and the critical points are

$$(6/\sqrt{11}, 2/\sqrt{11}, -2/\sqrt{11}), \quad (-6/\sqrt{11}, -2/\sqrt{11}, 2/\sqrt{11})$$