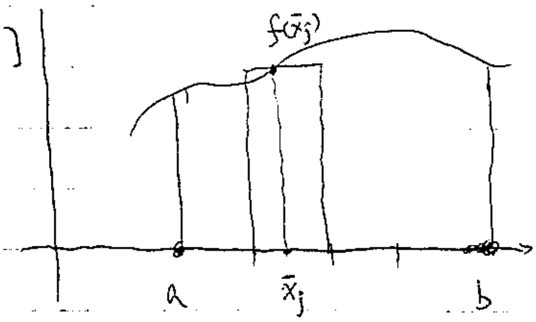
Chapter two Multiple Integrals

§1. Double integrals.

Single integrals: Let $f(x)$ be a function defined in $[a, b]$

A partition $\{x_j\}$ is defined in $[a, b]$



$$a = x_1 < x_2 < \cdots < x_{N+1} = b$$

Let $\bar{x}_j \in (x_j, x_{j+1})$

The definite integral of $f(x)$ from a to b is

$$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{j=1}^{N} f(\bar{x}_j)\,\delta x_j$$

where $\delta x_j = x_{j+1} - x_j$.

Geometrically, if $f(x)$ is positive, the integral corresponds to the area between the curve, the x-axis and lines $x=a$ and $x=b$.

Simply speaking, divide the domain (a, b) into small pieces, $f(\bar{x}_j)$ is the value at a point in j-th piece and obtain the sum

$$\sum_{j=1}^{N} f(\bar{x}_j)\,\delta x_j$$

Its limit is the integral.

The idea can be extended to two-dimensional problem, double integral.

Let $f(x,y)$ be a function of two variables defined in a domain $\Omega \subset R^2$

(i) Divide the domain into small pieces S_j

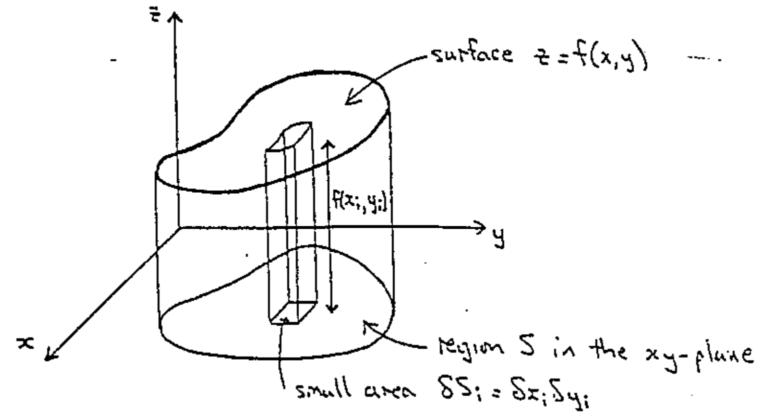
(ii) take any one point at each piece, $(\bar{x}_j, \bar{y}_j)$

(iii) obtain the sum

$$\sum_{j=1}^{N} f(\bar{x}_j, \bar{y}_j)\, \delta S_j$$

Then the double integral of $f(x,y)$ over the region Ω is defined by

$$\iint_{\Omega} f(x,y)\, ds = \lim_{N \to \infty} \sum_{j=1}^{N} f(\bar{x}_j, \bar{y}_j)\, \delta S_j$$



Such an integral is sometimes called a ~~surface~~ Double integral.

Geometrically, if $z = f(x,y)$ is positive (non-negative), this corresponds the valumn, as in Figure.

notation:

$$\iint_{\Omega} f(x,y)\, ds = \iint_{\Omega} f(x,y)\, dx\, dy$$

0.2. The calculation of 2-D integrals.

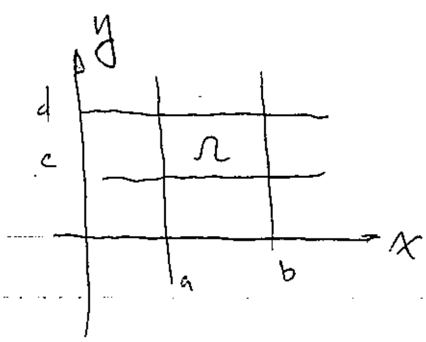
Ex. $f(x,y) = 1$

$$\iint_{\Omega} 1 \, ds = \lim_{N \to \infty} \sum_{j=1}^{N} 1 \cdot \delta s_j = \text{area of the region } S$$

Ex. (Rectangular regions)

If Ω is the region:

$a \le x \le b$, $c \le y \le d$

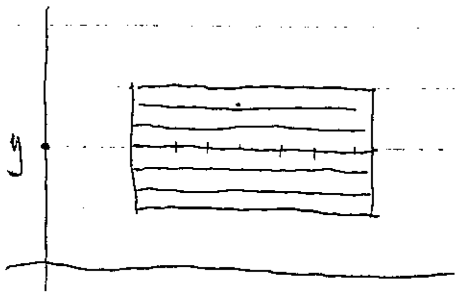


Then

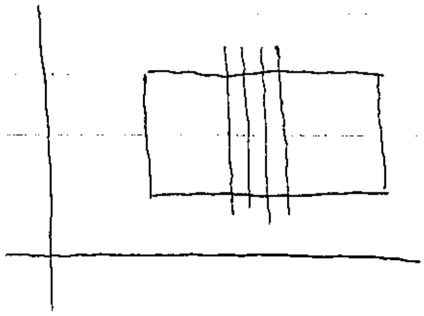
$$\iint_{\Omega} f(x,y) \, dx \, dy = \int_c^d \left(\int_a^b f(x,y) \, dx \right) dy \qquad (1)$$

$$= \int_a^b \left(\int_c^d f(x,y) \, dy \right) dx \qquad (2)$$

details



(1)



(2)

Ex Evaluate $\iint_{\Omega} (2xy + y^2) \, ds = I$

where Ω : $1 \le x \le 2$, $0 \le y \le 1$

Then $I = \int_0^1 \left(\int_1^2 (2xy + y^2)\,dx \right) dy$

$$= \int_0^1 (x^2 y + xy^2)\Big|_1^2\, dy \qquad (y \text{ is consided as a constant})$$

$$= \int_0^1 (4y + 2y^2 - y - y^2)\,dy$$

$$= \int_0^1 (3y + y^2)\,dy = \left(\tfrac{3}{2}y^2 + \tfrac{1}{3}y^3\right)\Big|_0^1 = \tfrac{3}{2} + \tfrac{1}{3} = 11/6$$

or

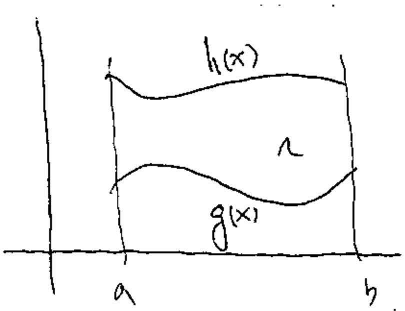
$$I = \int_1^2 \left(\int_0^1 (2xy + y^2)\,dy \right) dx$$

$$= \int_1^2 \left(xy^2 + \tfrac{1}{3}y^3\right)\Big|_0^1 dx = \int_1^2 \left(x + \tfrac{1}{3}\right) dx = 11/6$$

Ex (Non-rectangular region)

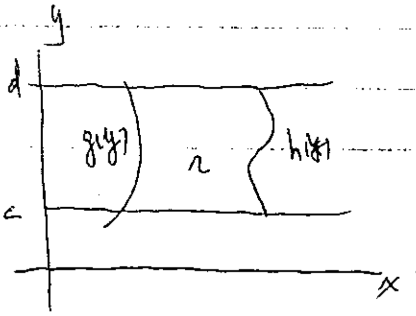
The main idea to evaluate a double integral is to transfer it to ~~sin~~ single integral. The main problem is to determine the upper and lower limits in these single integrals.

Some non-rectangular regions



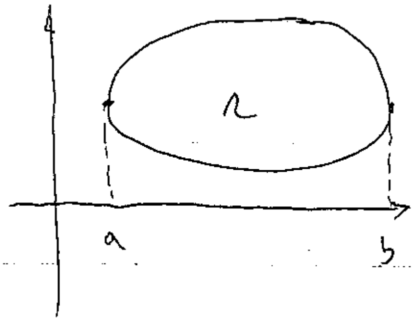
Ω: $a \le x \le b$, $g(x) \le y \le h(x)$

$$I = \iint_\Omega f(x,y)\,dx\,dy = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx$$



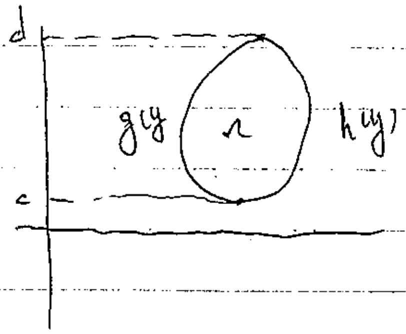
Ω: $c \le y \le d$, $g(y) \le x \le h(y)$

$$I = \iint_\Omega f(x,y)\,dx\,dy = \int_c^d \left(\int_{g(y)}^{h(y)} f(x,y)\,dx \right) dy$$



Ω: $a \le x \le b$, $g(x) \le y \le h(x)$

$$I = \int_a^b \left(\int_{g(x)}^{h(x)} f(x,y)\,dy \right) dx .$$



Similarly.

Basic Idea

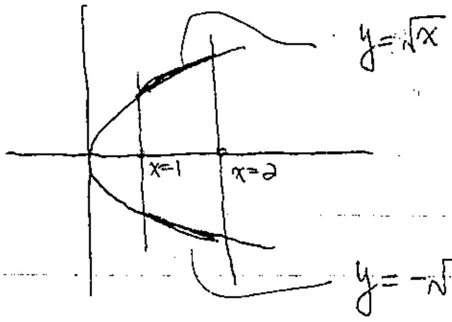
$$\int_a^b \quad \int_{g(x)}^{h(x)}$$

constant limit   variable limits (constant limits are included)

Ex. Calculate

$$\iint_\Omega x y^2\,dx\,dy$$

where Ω: region by $y^2 = x$, $x=1$, $x=2$

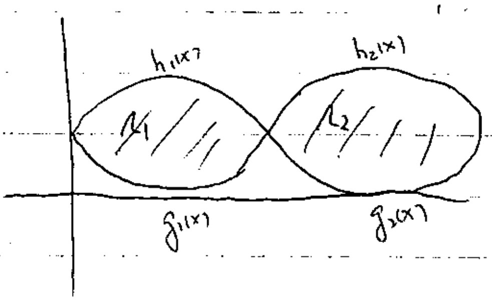


$$\iint_{\Omega} xy\,dx\,dy = \int_1^2 \left(\int_{-\sqrt{x}}^{\sqrt{x}} x y^2\,dy \right) dx$$

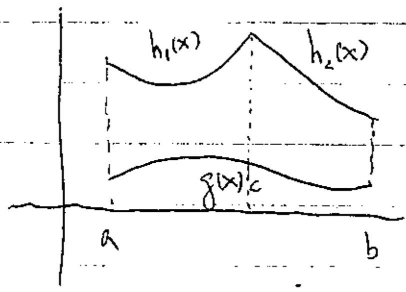
$$= \int_1^2 \left(\frac{x}{3} y^3 \Big|_{-\sqrt{x}}^{\sqrt{x}} \right) dx = \int_1^2 \frac{2}{3} x^{5/2}\,dx$$

$$= \frac{2}{3} \cdot \frac{2}{7} x^{7/2} \Big|_1^2 = \frac{4}{21}\left(2^{7/2} - 1 \right). \#$$

Some other regions :



$$I = \iint_{\Omega_1} f(x,y)\,dx\,dy + \iint_{\Omega_2} f(x,y)\,dx\,dy$$

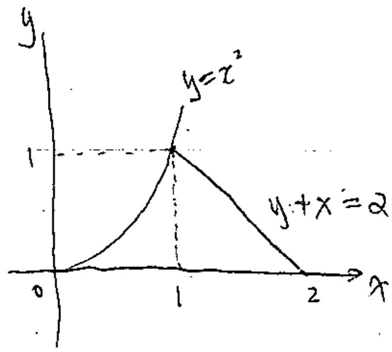


$$I = \int_a^c \left(\int_{g(x)}^{h_1(x)} f(x,y)\,dy \right) dx + \int_c^b \left(\int_{g(x)}^{h_2(x)} f(x,y)\,dy \right) dx .$$

Ex. Calculate $I = \iint_{\Omega} xy\,dx\,dy$ where Ω is the region by $y = x^2$, $y = 2-x$, $y = 0$

$$I = \int_0^1 \left(\int_0^{x^2} x y\,dy \right) dx + \int_1^2 \left(\int_0^{2-x} x y\,dy \right) dx$$

$$= \int_0^1 \frac{x^4}{2}\,dx + \int_1^2 2x^3\,dx = 37/120$$



Also

$$I = \int_0^1 \left(\int_{\sqrt{y}}^{2-y} x y\,dx \right) dy = \int_0^1 \left(\frac{(y-2)^2 y}{2} - \frac{y^2}{2} \right) dy$$

Change of variable in double integrals:

For single integrals, we have

$$I = \int_a^b f(x)\,dx \;\underset{=\!=\!=}{\overset{x=x(u)}{}}\; \int_\alpha^\beta \left(f(x(u))\,\frac{dx}{du}\right)du$$

where $x(\alpha) = a$, $x(\beta) = b$.

Ex. $\displaystyle\int_0^{\sqrt{\pi}} x \sin(x^2)\,dx \;\underset{=\!=\!=}{\overset{\text{Let } x^2=u,\ \frac{dx}{du}=1/2x}{}}\; \int_0^{\pi} x\cdot\sin(u)\cdot\frac{1}{2x}\,du$

$$= \int_0^{\pi} \frac{1}{2}\sin(u)\,du = -\frac{1}{2}\cos u\Big|_0^{\pi} = 1$$

For the double integral $I = \iint_R f(x,y)\,dx\,dy$

if we use the change of variable $x = x(u,v)$, $y = y(u,v)$. Then

$$I = \iint_{R^*} f(x(u,v), y(u,v))\,|J|\,du\,dv$$

where

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

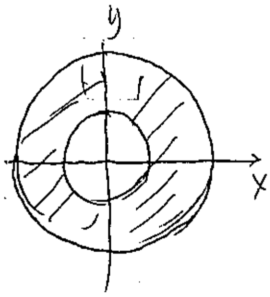
i.e. $dx\,dy = |J|\,du\,dv$

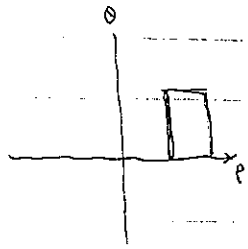
is called the Jacobian of the transformation.

R^* is the region in uv-plane corresponding to the region R in the xy-plan

Ex Calculate

$$I = \iint_{\Omega} (x^2+y^2)\,dx\,dy \qquad \Omega: \quad x^2+y^2 \le b, \quad x^2+y^2 \ge a \qquad b > a.$$





Change to polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$

$$\frac{\partial x}{\partial r} = \cos\theta, \quad \frac{\partial y}{\partial r} = \sin\theta, \quad \frac{\partial x}{\partial \theta} = -r\sin\theta, \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

Then

$$I = \iint_{\Omega} (x^2+y^2)\,dx\,dy = \iint_{\Omega^*} \left((r\cos\theta)^2 + (r\sin\theta)^2\right) \cdot r \cdot dr\,d\theta$$

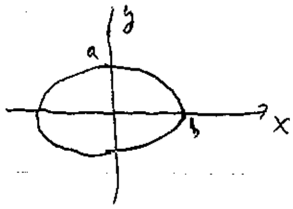
$$= \iint_{\Omega^*} r^3\,dr\,d\theta$$

$$\Omega^*: \quad a \le r \le b, \quad 0 \le \theta \le 2\pi$$

$$I = \iint_{\Omega^*} r^3\,dr\,d\theta = \int_a^b \left(\int_0^{2\pi} r^3\,d\theta\right) dr = \int_a^b r^3 \cdot 2\pi\,dr = \frac{2\pi}{4}(b^4 - a^4)$$

Ex. Calculate $I = \iint_{\Omega} \sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}}\,dx\,dy$

where $\Omega: \left\{(x,y) \,\middle|\, \frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1\right\}$



Sol: Change of variable, $x = ar\cos\theta$

$y = br\sin\theta$

$\frac{\partial x}{\partial r} = a\cos\theta \quad \frac{\partial y}{\partial r} = b\sin\theta, \quad \frac{\partial x}{\partial \theta} = -ar\cos\theta, \quad \frac{\partial y}{\partial \theta} = br\sin\theta$

$$J = \begin{vmatrix} a\cos\theta & -ar\cos\theta \\ b\sin\theta & br\sin\theta \end{vmatrix} = abr$$

$$\sqrt{1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}} = \sqrt{1 - (r\cos\theta)^2 - (r\sin\theta)^2} = \sqrt{1 - r^2}$$

$$I = \iint_{\Omega^*} \sqrt{1 - r^2}\, abr\, dr\, d\theta$$

$\Omega^*: \quad 0 \le r \le 1 \quad 0 \le \theta \le 2\pi$

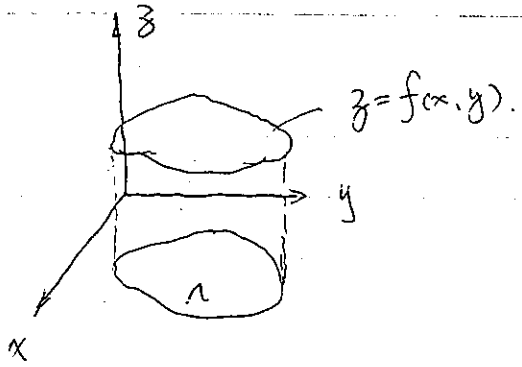
$$I = \int_0^1 \left(\int_0^{2\pi} \sqrt{1-r^2}\, abr\, d\theta \right) dr = \int_0^1 ab \cdot 2\pi \sqrt{1-r^2}\, r\, dr$$

$$= -ab\pi \frac{1}{3}(1-r^2)^{3/2} \Big|_0^1 = \frac{\pi}{3} ab \quad \#$$

The calculation of volume.

If $f(x,y) \geq 0$,

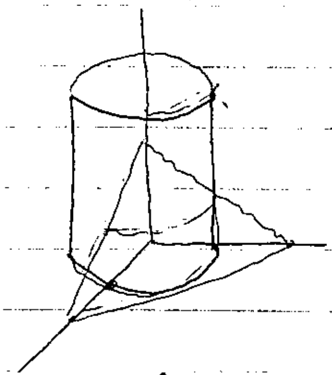
$$\text{Volume} = \iint_{\Omega} f(x,y)\,dx\,dy.$$



Ex Calculate the volume bounded by

$$x+y+z=a,\quad x^2+y^2=R^2,\quad x=0,\quad y=0 \text{ and } z=0$$

where $a \geq R\sqrt{2}$.

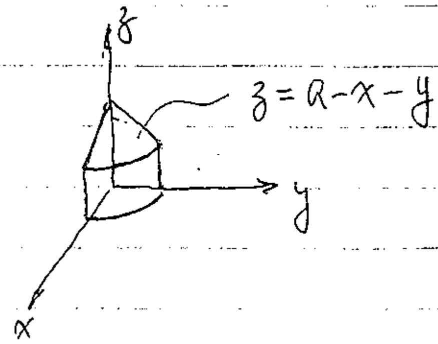


Sol.

$$V = \iint_{\Omega} (a-x-y)\,dx\,dy$$

Use the polar coordinates.

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$



$$V = \int_0^{\pi/2}\left(\int_0^R (a - r\cos\theta - r\sin\theta)\,r\,dr\right)d\theta$$

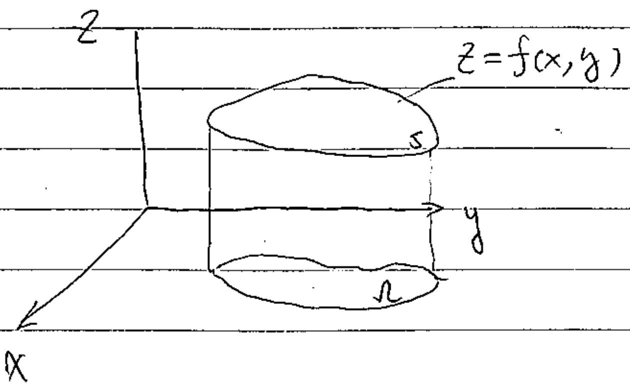
$$= \int_0^{\pi/2}\left(\frac{R^2}{2}a - \frac{R^3}{3}\cos\theta - \frac{R^3}{3}\sin\theta\right)d\theta$$

$$= \frac{R^2}{4}a\pi - \frac{R^3}{3}\left(\sin\theta - \cos\theta\right)\Big|_0^{\pi/2} = \underline{\frac{a\pi R^2}{4} - \frac{2R^3}{3}}$$

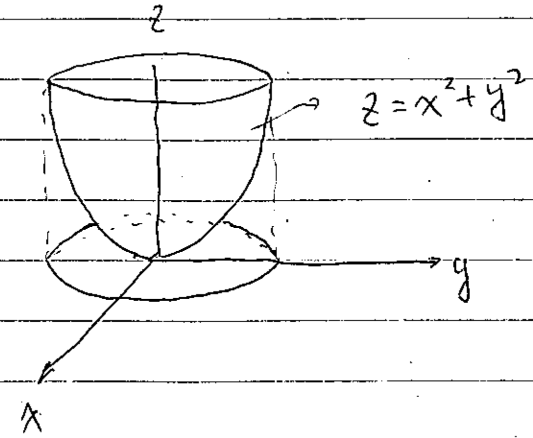
Surface area:

The surface area

$$A(S) = \iint_{\Omega} \sqrt{f_x^2 + f_y^2 + 1}\, d\Omega$$



Ex. Find the area of the part of paraboloid $z = x^2 + y^2$ that lies under the plane $z = 9$.



Solution: $z_x = 2x \quad z_y = 2y$

$$A(S) = \iint_{\Omega} \sqrt{4x^2 + 4y^2 + 1}\, d\Omega$$

where $\Omega = \{(x, y) \mid x^2 + y^2 \le 9\}$

Converting to polar coordinates. $x = r\cos\theta$, $y = r\sin\theta$

$$A(S) = \int_0^{2\pi} \int_0^3 \sqrt{4(r^2\sin^2\theta + r^2\cos^2\theta) + 1} \cdot r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left(\int_0^3 \sqrt{4r^2+1}\; r\, dr \right) d\theta$$

$$= \int_0^{2\pi} \frac{1}{12} (4r^2+1)^{3/2} \Big|_0^3 d\theta = \int_0^{2\pi} \frac{1}{12} \left[37^{3/2} - 1 \right]$$

$$= \frac{\pi}{6} \left(37^{3/2} - 1 \right)$$

$\#$

§3. Triple integrals:

Definition:

1-D $$\int_a^b f(x)\,dx = \lim_{N\to\infty} \sum_{j=1}^{N} f(\bar{x}_j)\,\delta x_j$$

2-D $$\iint_{\Omega} f(x,y)\,dx\,dy = \lim_{N\to\infty} \sum_{j=1}^{N} f(\bar{x}_j, \bar{y}_j)\,\delta S_j$$

3-D

$$\iiint_V f(x,y,z)\,dv = \iiint_V f(x,y,z)\,dx\,dy\,dz = \lim_{N\to\infty} \sum_{j=1}^{N} f(\bar{x}_j, \bar{y}_j, \bar{z}_j)\cdot\delta v_j$$

which is called a volume integral on the region V.

Examples:

(i) $f(x,y,z) = 1$

$$\iiint_V dv = \lim \sum_{j=1}^{N} \delta v_j = V = \text{volume of the region } V$$

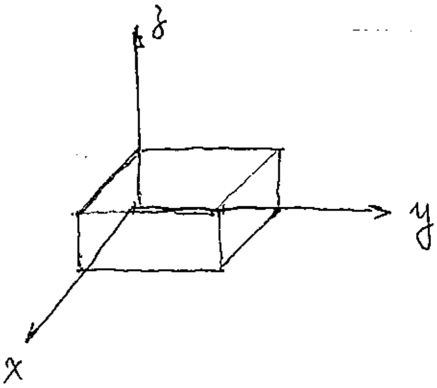
(ii) $f = \rho$ is the density

$$\iiint_V \rho\,dv = \text{mass of the region } V$$

$$\rho(\bar{x}_j, \bar{y}_j, \bar{z}_j)\cdot\delta v_j = \text{mass of } \delta v_j$$

Evaluation: Similarly to double integral, the methods used are dependent upon the region V.

Ex. (Rectangular block)



V is a solid bounded by

$x = a, \quad x = b. \qquad b > a$

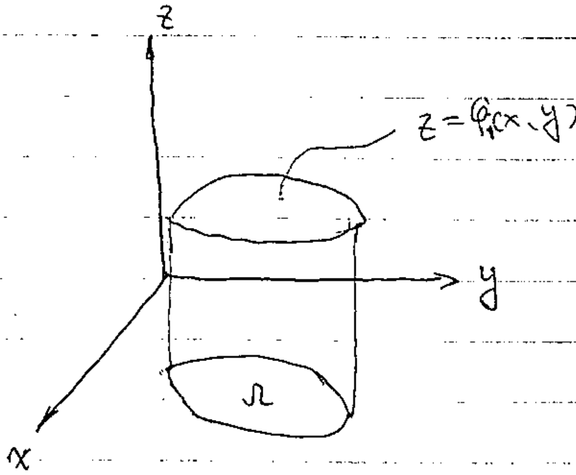
$y = c, \quad y = d \qquad d > c$

$z = e, \quad z = h. \qquad h > e.$

Then

$$\iiint_V f(x, y, z)\,dx\,dy\,dz = \int_e^h \left(\int_c^d \left(\int_a^b f(x, y, z)\,dx \right) dy \right) dz$$

Ex. (non-rectangular regions)
(cylinder)



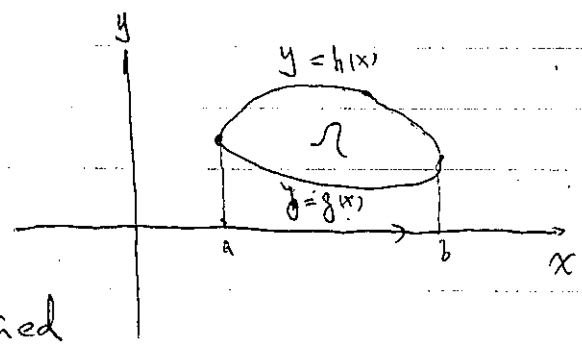
V is a cylinder as in Figure.

Then

$$\iiint_V f(x, y, z)\,dx\,dy\,dz$$

$$= \iint_\Omega \left(\int_0^{\varphi_1(x,y)} f(x, y, z)\,dz \right) dx\,dy$$

$$= \int_a^b \left(\int_{g(x)}^{h(x)} \left(\int_0^{\varphi_1(x,y)} f(x, y, z)\,dz \right) dy \right) dx$$



More general. if the bottom is a surface defined by $z = \varphi_2(x, y)$ then

$$\iiint_V f\,dv = \int_a^b \left(\int_{g(x)}^{h(x)} \left(\int_{\varphi_2(x,y)}^{\varphi_1(x,y)} f\,dz \right) dy \right) dx$$

constant limit (a, b), variable limit ($g(x)$, $h(x)$), variable limit (φ_1, φ_2).

Some complicated geometries can be considered as several cylinders.

Ex. A solid is given by $x=1$, $x=2$, $y=0$, $y=3$, $z=-1$, $z=0$.
The density is $\rho(x,y,z) = x(y+1)-z$. Calculate the mass of the block.

$$\text{Mass} = \iiint_V \rho\,dv = \int_1^2 \left(\int_0^3 \left(\int_{-1}^0 \left(x(y+1)-z\right) dz \right) dy \right) dx$$

$$= \int_1^2 \left(\int_0^3 \left(x(y+1) + \frac{1}{2} \right) dy \right) dx$$

$$= \int_1^2 \left(x\frac{(y+1)^2}{2} + \frac{1}{2}y \right)\Big|_0^3 dx$$

$$= \int_1^2 \left(\frac{15}{2}x + \frac{3}{2} \right) dx$$

$$= \left(\frac{15}{4}x^2 + \frac{3}{2}x \right)\Big|_1^2$$

$$= \frac{45}{4} + \frac{3}{2} = \frac{51}{4}$$

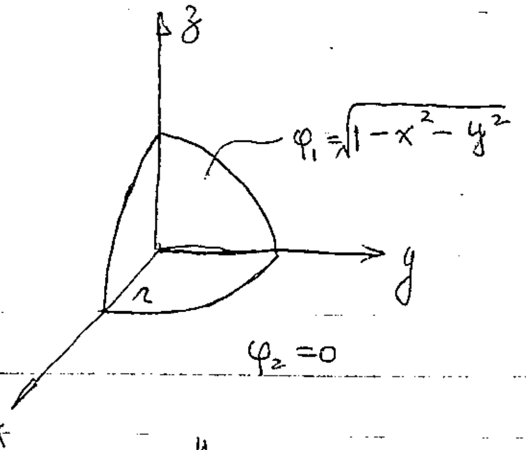
(2) moment

(3) center of mass

Ex. Evaluate $\iiint_V xyz\,dv$

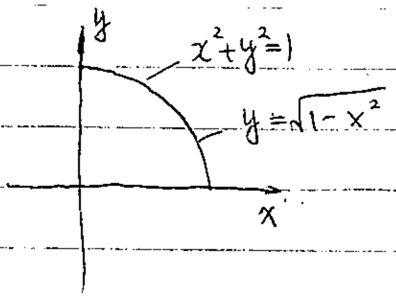
where V is a solid bounded by $x^2+y^2+z^2 \le 1$, $x \ge 0$, $y \ge 0$, $z \ge 0$

Sol: This is a more general case.



$$\iiint_V xyz\,dv = \iint_\Omega \left(\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\right)dx\,dy$$

$$= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} \left(\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\right)dy\right)dx$$



$$= \int_0^1 \left(\int_0^{\sqrt{1-x^2}} xy\,\frac{1}{2}(1-x^2-y^2)\,dy\right)dx$$

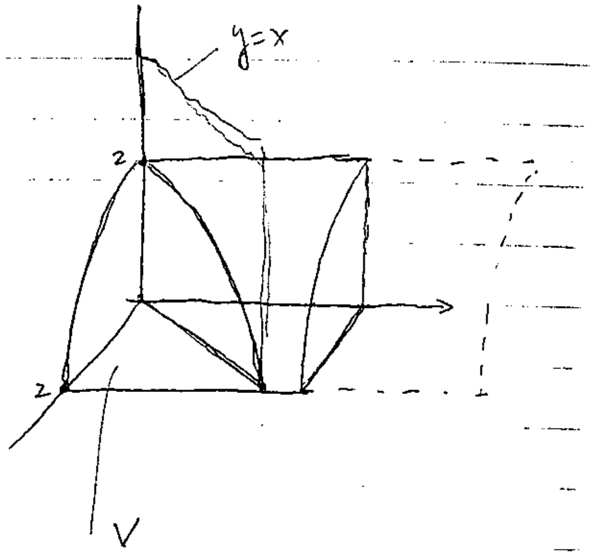
$$= \int_0^1 -\frac{1}{8}x(1-x^2-y^2)^2\Big|_0^{\sqrt{1-x^2}}\,dx$$

$$= \int_0^1 \frac{1}{8}x(1-x^2)^2\,dx$$

$$= -\frac{1}{48}(1-x^2)^3\Big|_0^1 = \frac{1}{48}$$

Importance is the "Geometry"

Ex. Evaluate $\iiint_V 2xyz\,dv$ where V is the region bounded by the parabolic cylinder $z = 2 - \frac{1}{2}x^2$ and the planes $z=0$, $y=x$ and $y=0$.



Using the previous formula

$$\varphi_1 = 2-\tfrac{1}{2}x^2 , \quad \varphi_2 = 0$$

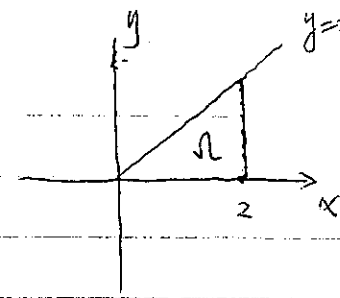
$$\iiint_V 2xyz\,dv = \iint_\Omega \left(\int_0^{2-\frac{1}{2}x^2} 2xyz\,dz \right) dx\,dy$$

$$= \int_0^2 \left(\int_0^x \left(\int_0^{2-\frac{1}{2}x^2} 2xyz\,dz \right) dy \right) dx$$

$$= \int_0^2 \left(\int_0^x xy\left(2-\tfrac{1}{2}x^2\right)^2 dy \right) dx$$

$$= \int_0^2 x\left(2-\tfrac{1}{2}x^2\right)^2 \tfrac{1}{2}x^2\,dx$$

$$= 4/3$$



Change of variable:

Let $\quad x = x(u, v, w)$

$\qquad y = y(u, v, w)$

$\qquad z = z(u, v, w)$

We have

$$\iiint_V f(x, y, z)\,dv = \iiint_{V^*} f(x(u,v,w), y(u,v,w), z(u,v,w))\,|J| \cdot dv^*$$

where

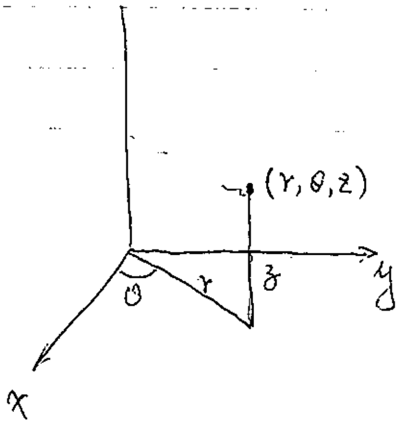
$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}$$

is called Jacobian of the transformation

V^* is the corresponding region in u-v-w space

Cylindrical Polar coordinates

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

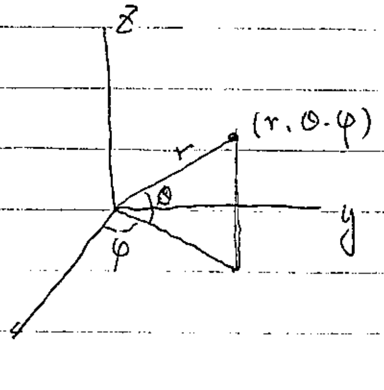


In this case

$$J = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & -r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r$$

Spherical Polar coordinates

$$\begin{cases} x = r\sin\theta\cos\varphi \\ y = r\sin\theta\sin\varphi \\ z = r\cos\theta \end{cases} \qquad J = r^2\sin\theta$$



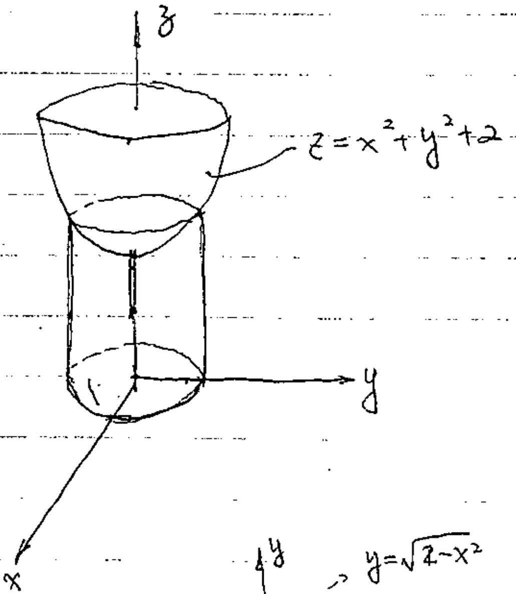
$r \geq 0$, $0 \leq \theta \leq \pi$, $0 \leq \varphi < 2\pi$

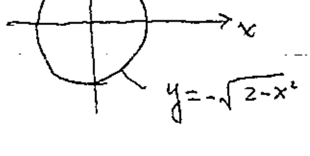
√ Ex. Find the volume between the surfaces $x^2+y^2=2$, $z=x^2+y^2+2$ and the plane $z=0$.

Sol In Cartesian coordinates.

$$\iiint_V dv = \iint_\Omega \left(\int_0^{x^2+y^2+2} dz \right) dx\,dy$$

$$= \int_{-\sqrt{2}}^{\sqrt{2}} \left(\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \left(\int_0^{x^2+y^2+2} dz \right) dy \right) dx$$





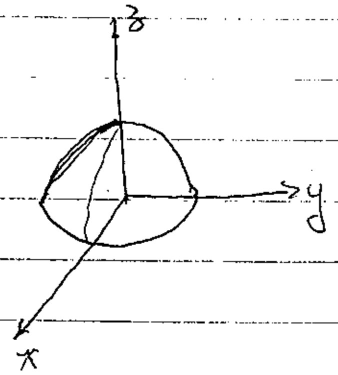
In cylindrical polar coordinates

$$\text{Volume} = \int_0^{\sqrt{2}} \int_0^{2\pi} \int_0^{r^2+2} r\,dz\,d\theta\,dr$$

$$\left(z = x^2+y^2+2 \;\Rightarrow\; z = r^2+2 \right)$$

Ex. Calculate the volume of a ball with radius R.

Using spherical coordinates $\qquad x^2+y^2+z^2 = R$



$$\text{Volume} = \iiint_V dv$$

$$= \int_0^{2\pi} \left(\int_0^{\pi} \left(\int_0^{R} r^2 \sin\theta \, dr \right) d\theta \right) d\varphi$$

$$= \int_0^{2\pi} \left(\int_0^{\pi} \frac{1}{3} R^3 \sin\theta \, d\theta \right) d\varphi$$

$$= \int_0^{2\pi} \frac{1}{3} R^3 \cdot 2 \, d\varphi = \frac{4}{3} R^3$$

More general Cylindrical and spherical polar coordinates

$$\begin{cases} x = a r \cos\theta \\ y = b r \sin\theta \\ z = z \end{cases} \qquad\qquad \begin{aligned} x &= a r \sin\theta \cos\varphi \\ y &= b r \sin\theta \sin\varphi \\ z &= c r \cos\theta \end{aligned}$$

where a, b, and c are constant

$$\Downarrow \qquad\qquad\qquad \Downarrow$$

$$J = abr \qquad\qquad\qquad J = r^2 \sin\theta \cdot abc.$$

Ex. Find the volume of a ellipsoid. $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = R^2$

Using
$x = ar\sin\theta\cos\varphi$
$y = br\sin\theta\sin\varphi$
$z = cr\cos\theta$

$$\text{Volume} = \int_0^{2\pi}\left(\int_0^{\pi}\left(\int_0^{R} r^2\sin\theta \cdot abc\, dr\right)d\theta\right)d\varphi$$

$$= \frac{4}{3}R^3 \cdot abc.$$

Applications.

① Mass of solid. V

$$\iiint_V \rho(x,y,z)\, dv$$

② moments

$$M_{yz} = \iiint_V x\rho(x,y,z)\,dv, \quad M_{xz} = \iiint_V y\rho\, dv \quad M_{xy} = \iiint_V z\rho\, dv$$

③ center of mass

$$\bar{x} = \frac{M_{yz}}{m}, \quad \bar{y} = \frac{M_{xz}}{m}, \quad \bar{z} = \frac{M_{xy}}{m}$$

Ex Find the center of mass of a solid of constant density that is bounded by the parabolic cylinder $x = y^2$ and the planes $x = z$, $z = 0$ and $x = 1$.