Higher order ODE

Ex. Solve $y''' - 6y'' + 11y' - 6y = 0$

Auxilliary equation $m^3 - 6m^2 + 11m - 6 = 0$

$m_1 = 1, \quad m_2 = 2, \quad m_3 = 3$

the general solution: $y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

Initial problem & boundary value problem :

Ex. Simple Harmonic Oscillator

According to Newton's law

$$F = m \frac{d^2x}{dt^2}$$

force-equilibrium equation

$$F = -kx$$

k ↓ spring constant

position $x = 0$

∴ we have

$$\frac{d^2x}{dt^2} + \frac{k}{m} x = 0$$

① $x(0) = a, \quad x'(0) = 0$

② $x(0) = 0, \quad x'(0) = b$

} Initial value problem

③ $x(0) = a, \quad x(1) = c \Rightarrow$ boundary value problem

the general solution:

$$x(t) = A \cos\sqrt{\frac{k}{m}}\,t + B \sin\sqrt{\frac{k}{m}}\,t$$

Using the initial conditions or boundary conditions we can determine the constants A and B.

② Nonhomogeneous problem

$$a_0 y'' + a_1 y' + a_2 y = f(x)$$

The general solution of the associated homogeneous equation is called the complementary solution : y_c

It has been proved that if y_p is a particular solution of the nonhomogeneous equation, then the general solution is

$$y = y_c + y_p .$$

Verify: substituting $y = y_c + y_p$ into the above equation

$$a_0(y_c'' + y_p'') + a_1(y_c' + y_p') + a_2(y_c + y_p)$$

$$= \underbrace{(a_0 y_c'' + a_1 y_c' + a_2 y_c)}_{=0} + \underbrace{(a_0 y_p'' + a_1 y_p' + a_2 y_p)}_{=f(x)} = f(x).$$

How can we find this particular solution ? Try !!

Ex Solve $y'' + 4y = 4e^{2x}$ Ex $y'' - 4y = 4e^{2x}$

Solution try. $y_p = Ae^{2x}$

$$4Ae^{2x} + 4Ae^{2x} = 4e^{2x}$$

$$8A = 4 \qquad A = \cancel{\tfrac{4}{3}}\ \tfrac{1}{2}$$

$y_p = \frac{4}{3}e^{2x}$. and $y_c = C_1 \cos 2x + C_2 \sin 2x$

the general sol. $y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{2}e^{2x}$ #

Ex. Solve $y'' + 4y' + 4y = 6 \sin 3x$

Solution: $y_c = (c_1 + c_2 x)e^{-2x}$

$y_p = A \sin 3x$?

For the particular solution y_p, we try $y_p = A\cos 3x + B \sin 3x$

Then

$$(-9A\cos 3x - 9B\sin 3x) + 4(-3A\sin 3x + 3B\cos 3x)$$

$$+ 4(A\cos 3x + B\sin 3x) = 6\sin 3x$$

$$(-9A + 12B + 4A)\cos 3x + (-9B - 12A + 4B)\sin 3x = 6\sin 3x$$

We choose A and B such that

$$\begin{cases} -5A + 12B = 0 \\ +12A - 5B = 6 \end{cases} \Rightarrow \begin{aligned} A &= -72/169 \\ B &= -30/169 \end{aligned}$$

Particular solution

$$y_p = \frac{-72}{169}\cos 3x - \frac{30}{169}\sin 3x$$

The general solution

$$y = (c_1 + c_2 x)e^{-2x} - \frac{30}{169}\sin 3x - \frac{72}{169}\cos 3x.$$

Method of Variation of parameters :

We consider

$$a_0 y'' + a_1 y' + a_2 y = f(x)$$

Complementary solution

$$y_c = C_1 y_1(x) + C_2 y_2(x).$$

We try

$$\underbrace{y_p = \gamma_1(x)\, y_1(x) + \gamma_2(x)\, y_2(x)}$$

We have :

$$\therefore y_p' = \gamma_1 y_1' + \gamma_2 y_2' + \gamma_1' y_1 + \gamma_2' y_2$$

If we choose γ_1 and γ_2 such that

$$\boxed{\gamma_1' y_1 + \gamma_2' y_2 = 0} \qquad (*)$$

then $\quad y_p' = \gamma_1 y_1' + \gamma_2 y_2'$

and

$$y_p'' = \gamma_1 y_1'' + \gamma_2 y_2'' + \gamma_1' y_1' + \gamma_2' y_2'$$

Substituting into equation :

$$a_0(\gamma_1 y_1'' + \gamma_2 y_2'' + \gamma_1' y_1' + \gamma_2' y_2') + a_1(\gamma_1 y_1' + \gamma_2 y_2') + a_2(\gamma_1 y_1 + \gamma_2 y_2)$$

$$= \gamma_1 \underbrace{(a_0 y_1'' + a_1 y_1' + a_0)}_{= 0} + \gamma_2 \underbrace{(a_0 y_2'' + a_1 y_2' + a_2)}_{= 0} + a_0(\gamma_1' y_1' + \gamma_2' y_2')$$

Then y_p is a solution if

$$\boxed{r_1' y_1' + r_2' y_2' = \frac{f(x)}{a_0}} \qquad (**)$$

We may have the particular solution by solving (*) and (**) for $r_1(x)$ and $r_2(x)$.

This is "method of variation of parameters"

Ex. Solve $y'' + y = \tan x$

Auxiliary equation for H-E.

$$m^2 + 1 = 0, \qquad m_1 = i \quad m_2 = -i$$

complementary solution

$$y_c = A\cos x + B\sin x \qquad (\text{i.e. } y_1(x) = \cos x \quad y_2 = \sin x$$

In order to obtain a particular solution, we solve

$$\begin{cases} r_1' \cos x + r_2' \sin x = 0 \\ -r_1' \sin x + r_2' \cos x = \tan x \end{cases}$$

linear system for r_1' and r_2'. we have

$$r_1' = -\frac{\sin^2 x}{\cos x}, \qquad r_2' = \sin x.$$

Integrating these

$$r_1 = -\int \frac{\sin^2 x}{\cos x}\,dx = \sin x - \ln|\sec x + \tan x|$$

$$r_2 = \int \sin x\,dx = -\cos x$$

Then

$$y_p(x) = \left(\sin x - \ln|\sec x + \tan x|\right)\cos x + (-\cos x)\sin x$$

$$= -\cos x\left(\ln|\sec x + \tan x|\right)$$

The general sol.

$$y(x) = y_c + y_p$$

$$= A\cos x + B\sin x - \cos x \ln|\sec x + \tan x|$$

Existence of solution of

linear system