MA4523 Introduction fo Finite Elment Methods, Assignment 2

1. Formulate a finite element method for the problem

$$-u'' = x$$
$$u(0) = u(1) = 0$$

using a piecewise linear approximation, determine the corresponding linear system of equations in the case of the general mesh $\{x_j\}_{j=0}^{N+1}$ and use the formulation to solve the problem with the following mesh

$$\{0, 0.1, 0.3, 0.5, 0.8, 0.9, 1.0\}.$$

2. Solve the problem in Q1 using a quadratic finite element method with a uniform mesh and two elements.

3. Solve the problem in Q1 with the basis functions:

$$\phi_i = \sin i\pi x, \qquad i = 1, 2, 3 \quad V_N = \{u : u = \sum \alpha_i \phi_i(x)\}.$$

4. Consider the following problem

$$u''''(x) = f(x), \qquad 0 < x < 1$$
$$u(0) = u(1) = u'(0) = u'(1) = 0.$$

(i) present an equivalent minimization problem and variational problem.

(ii) Can we use linear element method to solve this problem?

5. Let P_3 a vector space which consists of all cubic polynomials. Find the basis functions $\{\phi_i\}_{i=1}^4$ such that for any $v(x) \in P_3$

$$v(x) = v(0)\phi_1(x) + v'(0)\phi_1(x) + v(1)\phi_3(x) + v'(1)\phi_4(x).$$

6. (a). Show that the problem in Q4 can be given by the following variational formulation: Find $u \in V$ such that for all $v \in V$

$$(u'', v'') = (f, v)$$

where

$$V = \{v : v \text{ and } v' \text{ are continuous on } [0,1], v'' \text{ is piecewise continuous and } v(0) = v(1) = v'(0) = v'(1)\}$$

(b) Let

$$V_h = \{v(x) \in C^1 : v(x) \in P_3 \text{ at each element}\}$$

where P_3 denotes the set of polynomials of degree≤ 3. Formulate a finite element method for the problem in Q4 based on the space V_h. Find the corresponding linear system of equations in the case of uniform mesh (partition). Determine the solution in the case of two elements and f being constant. Compare with the exact solution.

7. Consider the equation

$$u'' + 2u' - u = x$$

with the boundary conditions (i) $u(0) = 1$, $u(3) = 4$, OR (ii) $u'(0) = 2$, $u'(1) + 3u(1) = 5$. Derive a variational formulation of the problem for each case.