MA4523 (Introduction of Finite Element Methods)

1 Variational formulation of an one-dimensional model.

We consider the following two-point boundary value problem

$$\begin{aligned} &-u''(x) = f(x), \qquad x \in (0,1) \\ &u(0) = u(1) = 0 \end{aligned} \tag{1.1}$$

which arises from many physical applications.

1.1 Physical models.

An elastic bar

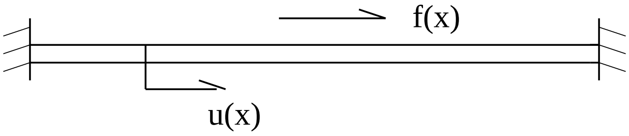

Figure 1: An elastic bar

$\sigma(x)$: traction at x;
$u(x)$: tangential displacement at x;
$f(x)$: load.

Hooke's law: $\qquad -\sigma = Eu'$,

Equilibrium equation: $\qquad \sigma' = f$

where E is a physical parameter dependent upon the physical material.

Then, we have

$$-(Eu')'(x) = f(x)$$

or

$$-u''(x) = f(x)/E$$

when E is a constant.

The boundary conditions are

$$u(0) = u(1) = 0 \qquad \text{(Dirichlet boundary conditions)}$$

since the two ending points are fixed. There are several different boundary conditions, such as

$$u(0) = 0 \quad u'(1) = 0\,.$$

An elastic cord

A heat-conduction model

$u(x)$: temperature at x;
$f(x)$: heat source of intensity;
q: heat flow.

Fourier's law: $\qquad -q = ku'$

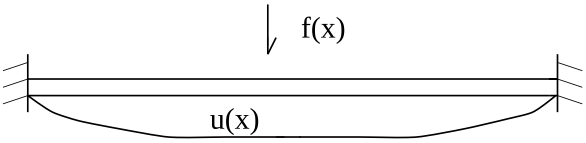

Figure 2: An elastic cord

Figure 3: A heat-conduction model

Conservation of energy: $q' = f$
where k is a physical parameter.

Then we have

$$-u''(x) = f(x)/k\,.$$

The boundary conditions depend upon physical models, such as Dirichlet boundary conditions

$$u(0) = u(1) = 0\,.$$

1.2 Finite difference method.

The basic idea in FD is to approximate derivatives by divided difference. Let $\{x_n\}_{n=0}^{N}$ be a partition (mesh) on $[0,1]$ where

$$0 = x_0 < x_1 < ... < x_{N-1} < x_N = 1\,.$$

For simplicity, we assume that the mesh is uniform, *i.e.*,

$$x_{i+1} - x_i = h, \qquad i = 0, 1, ..., N-1\,.$$

Then we have

$$u''(x_j) \approx \frac{u_{j+1} - 2u_j + u_{j-1}}{h^2}$$

where u_j is the numerical solution at $x = x_j$, *i.e.*, $u_j \approx u(x_j)$. Using the above formula in (1.1), we obtain

$$\begin{array}{l} -\frac{u_{j+1}-2u_j+u_{j-1}}{h^2} = f(x_j) \quad j = 0, 1, ..., N-1 \\ u_0 = u_N = 0\,. \end{array}$$

In a matrix form, the system can be given by

$$\frac{1}{h^2}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \end{bmatrix} = F$$

where $F = (f(x_1), f(x_2), \cdots, f(x_{N-1}))^T$.

Solve the above system to get the finite difference solution $\{u_j\}$.

Remarks:
- A is nonsingular matrix and the system has a unique solution.
- The error is $O(h^2)$ and it is called second-order method.

1.3 Finite element method (FEM).

Some notation: Let $v(x)$ and $u(x)$ be defined in $[0,1]$ and

$$\begin{aligned}&(v,\, u) = \textstyle\int_0^1 v(x)u(x)\,dx\\ &V_0 = \{v(x) | v \text{ piecewise smooth and } v(0) = v(1) = 0\}\,.\end{aligned}$$

It is obvious that if $u(x)$ is solution of (1.1), then $u \in V_0$.
For the model problem (1.1), we define the linear functional $F:\, V_0 \to \mathbb{R}$ (real number) by

$$F(v) = \frac{1}{2}(v',\, v') - (f,\, v)\,.$$

There are three equivalent problems:

Differential equation:

$$\begin{aligned}&-u''(x) = f(x), \qquad x \in (0,1)\\ &u(0) = u(1) = 0\,.\end{aligned} \qquad (D)$$

Minimization problem:

$$\min_{u \in V_0} \; F(u)\,. \qquad (M)$$

Variational problem:
Find $u \in V_0$ such that for all $v \in V_0$

$$(u',\, v') = (f,\, v)\,. \qquad (V)$$

The proof of equivalence
(i) $(D) \to (V)$
Assume that $u(x)$ is a solution of (D). Then

$$-u''(x) = f(x) \quad \text{ and } \quad u(0) = u(1) = 0\,.$$

For any $v \in V_0$,

$$-(u'',\, v) = (f,\, v)\,.$$

Using integration by part,

$$-(u'', v) = -\int_0^1 u''v\,dx = -u(x)v(x)\,\Big|_0^1 + \int_0^1 u'v'\,dx\,.$$

Since $v(0) = v(1) = 0$, we have

$$(u',\, v') = (f,\, v)\,,$$

i.e., $u(x)$ is a solution of (V).

(ii) $(V) \to (M)$

Assume that $u(x)$ is a solution of (V). Then for any given $v \in V_0$, we set $w = v - u$ so that $v = w + u$ and therefore,

$$\begin{aligned} F(v) = F(w+u) &= \tfrac{1}{2}(u' + w',\, u' + w') - (f,\, u + w) \\ &= \tfrac{1}{2}(u',\, u') - (f,\, u) + (u',\, w') - (f,\, w) + \tfrac{1}{2}(w',\, w')\,. \end{aligned}$$

Since u is a solution of (V), $(u',\, w') - (f,\, w) = 0$ and the last equation becomes

$$F(v) = F(u) + \frac{1}{2}(w',\, w') \geq F(u)$$

which implies that $u(x)$ is a solution of (M).

(iii) $(M) \to (D)$ (We follow $(M) \to (V) \to (D)$)

Assume that u is a solution of (M). Then for any $v \in V_0$ and real number ϵ, $u + \epsilon v \in V_0$ and

$$F(u) \leq F(u + \epsilon v)\,.$$

Let $g(\epsilon) := F(u + \epsilon v)$. Then

$$g(\epsilon) = F(u + \epsilon v) = \frac{1}{2}(u',\, u') + \epsilon(u',\, v') + \frac{\epsilon^2}{2}(v',\, v') - (f,\, u) - \epsilon(f,\, v)$$

and $g(0) \leq g(\epsilon)$, *i.e.*, $g(\epsilon)$ has the minimum at $\epsilon = 0$. The necessary condition $g'(0) = 0$ is satisfied and is written by

$$g'(0) = (u',\, v') - (f,\, v) = 0 \qquad \text{for any } v \in V_0\,.$$

Thus u is a solution of (V). To prove that $u(x)$ is a solution of (D), we need to assume that $u \in C^2[0, 1]$. By integration by part,

$$-(u'',\, v) - (f,\, v) = 0$$

i.e.,

$$\int_0^1 (u'' + f)v\, dx = 0 \qquad \text{for any } v \in V_0\,.$$

Hence,

$$-u''(x) = f(x) \qquad \text{and} \qquad u(0) = u(1) = 0$$

where we have noted that $C^2[0, 1] \subset \bar{V}_0$. So u is a solution of (D).

In general, we define $a(u, v)$ being a bi-linear form. In this simple case,

$$a(u, v) = (u',\, v')$$

which can be obtained by using integration by part for the original differential equation. For the bi-linear form, we can define

$$F(u) = \frac{1}{2}a(u, u) - (f,\, u)$$

and the variational form can be written by

$$a(u, v) = (f, v)\,.$$

Remarks

- The basic idea of finite difference method is

$$\text{Differential equation} \quad ======> \quad \text{finite difference system}$$

by replacing the derivatives with divided finite difference. The basic idea of finite element method is to solve the (M)-model or (V)-model numerically instead of the original differential equations.

- It should be noted that the corresponding minimization problem is different from those classical minimization problems in calculus. Here $F(u)$ is a functional defined on V_0, a infinitely dimensional space.

Example

$$min_{x \in R} f(x) := x^2 - 2x + 2\,.$$

Here we need to find the minimum for $x \in R^1$. The set R^1 is an one-dimensional (linear vector) space. By using classical results in calculus, the minimal point satisfies

$$\frac{df(x)}{dx} = 0\,.$$

Example

$$min_{(x,y) \in R^2} f(x,y) := x^2 + y^2 - 2x + 2$$

where R^2 is a two-dimensional (linear vector) space. The solution satisfies

$$\frac{\partial f(x,y)}{\partial x} = 0$$
$$\frac{\partial f(x,y)}{\partial y} = 0$$

Similarly, for a minimization problem in N-dimensional space R^N

$$\min_{\mathbf{x} \in \mathbb{R}^N} f(\mathbf{x})\,.$$

The necessary condition is

$$\frac{\partial f(\mathbf{x})}{\partial x_i} = 0 \qquad i = 1, 2, ..., N\,.$$

However, in the (M)-model, V_0 is a infinitely dimensional space. The above formulas are no longer true. In fact, it is more difficult to solve such a functional minimization problem (M). The basic idea of finite element method is to approximate the infinite dimensional space by using finite dimensional space V_N.

Example Let $u(x)$ be a smooth function in $[0,1]$. Its Taylor expansion is given by

$$\begin{aligned} u(x) &= u(0) + xu'(0) + ... + \frac{x^N}{N!}u^{(N)}(0) + R(u) \\ &= t(x) + R(u) \end{aligned}$$

where $R(u)$ is the residual of the Taylor expansion and $t(x)$ is a polynomial of degree N. $t(x)$ is a good approximation to $u(x)$ when N is large enough. Let V_N be a polynomial space defined by

$$V_N = \{v | v = \sum_{i=0}^{N-1} \alpha_i x^i\}$$

which is an N-dimensional space. We consider the following minimization problem in a finite dimensional space

$$\min_{u \in V_N} \; F(u) \tag{1.2}$$

instead of the original (M)-problem. We can expect the solution of the above minimization model is a good approximation to the solution of original (M)-problem (or differential equation problem) when N is large enough. This is the basic idea of FEM. In fact, the problem (1.2) is equivalent to

$$\min_{\alpha \in R_N} \; F(\sum_{i=0}^{N-1} \alpha_i x^i) \tag{1.3}$$

which can be solved by using classical methods in calculus.

By some basic results in linear algebra, for any N-dimensional space V_N, there are bases $\phi_i(x) \in V_N$, $i = 1, 2, ..., N$, such that for any $u \in V_N$,

$$u = \sum_{i=1}^{M} \alpha_i \phi_i \, .$$

The major problem is how to choose the finite dimensional space, *i.e.*, the basis functions ϕ_i, such as $\sin ix$, $\cos ix$ and polynomials. The corresponding minimization problem and variational problem in a finite dimensional space can be given as follows.

$$\min_{u \in V_N} \; F(u) \qquad\qquad\qquad (M_N)$$

which is called (M_N)-model (the method based on the model is called *Ritz*'s FEM method) and

The corresponding variational model is: to find $u \in V_N$ such that for all $v \in V_N$

$$(u', \, v') = (f, \, v) \qquad\qquad\qquad (V_N)$$

which is called (V_N)-model (the method based on the model is called Galerkin FEM method).

1.4 Piecewise linear polynomial approximations.

Most interesting basis functions are piecewise polynomials. We consider the model problem here. Let $\{x_i\}$ be a partition (mesh) on $[0, 1]$ where

$$0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = 1$$

which divides the interval $(0, 1)$ into N small pieces (element). For simplicity, we assume that the mesh used is uniform, *i.e.*, $x_{i+1} - x_i = h$. A piecewise polynomial is a polynomial at each piece.

The functions $\phi_i(x)$, $i = 1, 2, \cdots, N$, are the bases of piecewise linear polynomial space if $\phi_i(x)$ is continuous and piecewise linear and satisfies

$$\phi_j(x_i) = \begin{cases} 1 & \text{if } i = j \, , \\ 0 & \text{if } i \neq j \, , \end{cases} \quad .$$

Obviously the piecewise linear basis functions can be expressed by

$$\phi_j(x) = \begin{cases} \frac{x-x_{j-1}}{h} & x \in (x_{j-1},\, x_j)\,, \\ \frac{x_{j+1}-x}{h} & x \in (x_j,\, x_{j+1})\,, \\ 0 & \text{other}\,. \end{cases}$$

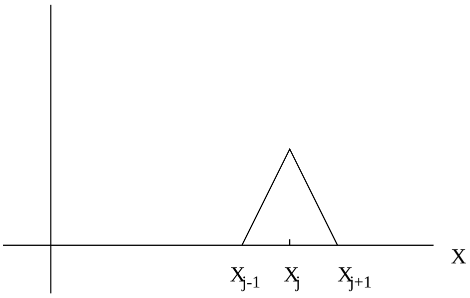

Figure 4: The basis function of linear element ($\phi_j(x)$).

The solution of piecewise linear FEM.
General formulas.
We consider the minimization model (M_N). Let

$$u(x) = \sum_{j=1}^{N} \alpha_j \phi_j(x)$$

where ϕ_j, $j = 1, 2, ..., N$, are the basis functions of piecewise linear polynomial space as defined above. Then

$$F(u) = \frac{1}{2} \int_0^1 \left(\sum_{j=1}^{N} \alpha_j \phi_j'(x) \right)^2 dx - \int_0^1 f(x) \left(\sum_{j=1}^{N} \alpha_j \phi_j(x) \right) dx$$

and

$$\begin{aligned} \frac{\partial F}{\partial \alpha_i} &= \int_0^1 \left(\sum_{j=1}^{N} \alpha_j \phi_j'(x) \right) \phi_i'(x)\, dx - \int_0^1 f(x)\phi_i(x)\, dx \\ &= \sum_{j=1}^{N} \int_0^1 \alpha_j \phi_i'(x)\phi_j'(x)\, dx - \int_0^1 f(x)\phi_i(x)\, dx\,. \end{aligned}$$

The minimal solution satisfies the necessary condition $\frac{\partial F}{\partial \alpha_i} = 0$. In a matrix form,

$$\sum_{j=1}^{N} a_{ij}\alpha_j = b_i$$

or

$$A\boldsymbol{\alpha} = \boldsymbol{b}$$

where $A = (a_{ij})$ be an $N \times N$ symmetric matrix, called *stiffness matrix* and $b = (b_i)$ be a vector, called *load vector* and

$$a_{ij} = \int_0^1 \phi_i'(x)\phi_j'(x)\,dx \qquad b_i = \int_0^1 f(x)\phi_i(x) \quad i,j = 1,2,...,N\,.$$

The formulas for linear FEM. For the piecewise linear basis functions,

$$\phi_j'(x) = \begin{cases} 1/h & x \in (x_{j-1},\, x_j)\,, \\ -1/h & x \in (x_j,\, x_{j+1})\,, \\ 0 & \text{other}\,. \end{cases}$$

and

$$\begin{array}{l} a_{ii} = \int_0^1 \phi_i'(x)\phi_i'(x)\,dx = 2/h \\ a_{i,i+1} = a_{i-1,i} = \int_0^1 \phi_i'(x)\phi_{i+1}'(x)\,dx = -1/h \\ a_{i,j} = 0 \qquad |i-j| > 1 \end{array}$$

$$b_i = \int_{x_{i-1}}^{x_i} f(x)\frac{x - x_i}{h}\,dx + \int_{x_i}^{x_{i+1}} f(x)\frac{x_{i+1} - x}{h}\,dx\,.$$

Hence, the FEM linear system is given by

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 2 \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}. \tag{1.4}$$

The finite element solution can be obtained by solving the above linear system in which the coefficient matrix is the same as in finite difference system.

Now we consider the variational model (V_N) (or V_h). We need to find $u \in V_N$ such that for any $v \in V_N$,

$$(u',\, v') = (f,\, v)\,.$$

We choose $v = \phi_i(x)$, $i = 1, 2, \cdots, N$, respectively. Since

$$u = \sum_{j=1}^N \alpha_j \phi_j(x)$$

we have

$$\left(\sum_{j=1}^N \alpha_j \phi_j',\, \phi_i' \right) = (f,\, \phi_i) \quad i = 1, 2, \cdots, N$$

i.e.,

$$\sum_{j=1}^N \alpha_j (\phi_j',\, \phi_i') = (f,\, \phi_i)$$

which is the same as the linear system (1.4).

A (simple) error estimate.

Let $u(x)$ be a solution of (D)-model and $u_h(x)$ be the solution of the linear FEM method. We need to give an upper bound for

$$\text{error } = \|u(x) - u_h(x)\|$$

where

$$\|w(x)\| = \|w(x)\|_{L^2} = (w, w)^{1/2} = \sqrt{\int_0^1 w^2\, dx}\,.$$

Theorem 1.1. Let V_h (or V_N) be the piecewise linear space defined above. For any $v \in V_h$,

$$\|(u - u_h)'\| \le \|(u - v)'\|\,. \tag{1.5}$$

Proof. Let $v \in V_h$ be arbitrary. We have

$$\begin{aligned}(u_h',\, v') - (f,\, v) = 0\\ (u',\, v') - (f,\, v) = 0\end{aligned}$$

and

$$((u - u_h)',\, v') = 0\,.$$

We set $w = u_h - v$. Then $w \in V_h$ and

$$\begin{aligned}\|(u - u_h)'\|^2 &= ((u - u_h)',\, (u - u_h)') + ((u - u_h)',\, w')\\ &= ((u - u_h)',\, (u - u_h + w)')\\ &= ((u - u_h)',\, (u - v)')\,.\end{aligned}$$

By using Cauchy's inequality $|(v,\, w)| \le \|v\| \cdot \|w\|$, we obtain

$$\|(u - u_h)'\|^2 \le \|(u - u_h)'\| \cdot \|(u - v)'\|\,.$$

(1.5) follows immediately. The proof is complete. ∎

In order to get an error estimate from (1.5), we need to choose a special $v(x)$. Let $\bar{u}_h$ be the linear piecewise interpolation of the exact solution $u(x)$, *i.e.*,

$$\bar{u}_h = \sum_{j=1}^{N} u(x_j)\phi_j(x) \in V_h$$

which satisfies $\bar{u}_h(x_j) = u(x_j)$.

By Theorem 1.1,

$$\|(u - u_h)'\| \le \|(u - \bar{u}_h)'\|\,.$$

By some results in elementary numerical analysis (numerical differentiation),

$$\|(u - \bar{u}_h)'\| \le h \cdot \max_{x \in (0,1)} |u''(x)| = hM_2$$

A simple error estimate for the first-order derivative is

$$\|(u - \bar{u}_h)'\| \le hM_2\,.$$

Since $u(0) - u_h(0) = 0$,

$$u(x) - u_h(x) = \int_0^x (u(s) - u_h(s))'\, ds$$

and therefore,

$$\begin{array}{l} |u(x) - u_h(x)| = \int_0^x |(u(s) - u_h(s))'|\, ds \\ \qquad \leq \|(u - u_h)'\| \leq hM_2 \end{array}$$

which means that

$$u_h(x) \to u(x) \qquad\qquad \text{as } h \to 0(\text{ or } N \to \infty)\,.$$

The above upper bound is not best one. The optimal error bound is

$$\|u - u_h\| \leq O(h^2)\,.$$

But the proof is a little complicated.

1.5 Other two-point BVPs and boundary conditions.

Notation: Let Ω be a bounded domain, in this simple case $\Omega = [0, 1]$.

$$\begin{array}{l} L_2(\Omega) = \{v | v \text{ is defined on } \Omega \text{ and } \int_\Omega v^2 dx \text{ bounded }\}\,, \\ H^1(\Omega) = \{v | v \in L_2(\Omega) \text{ and } v' \in L_2(\Omega)\}\,, \\ H_0^1(\Omega) = \{v | v \in H^1(\Omega) \text{ and } v = 0 \text{ on } \partial\Omega\}\,. \end{array}$$

The basic procedure of formulating a FEM is:

(i) Boundary value problem (D).

(ii) Choose a suitable bi-linear form $a(u,\, v)$ and formulate the corresponding (V)-model or (M)-model.

(iii) Choose a finite dimensional space V_N and the corresponding basis functions $\{\phi_j\}_{j=1}^N$.

(iv) Present a formula to calculate coefficient matrix and load vector in the FEM system.

(v) Present an algorithm for solving the FEM system.

Example 1 We consider a more general two-point boundary value problem

$$\begin{array}{l} -\frac{d}{dx}\left(p(x)\frac{du}{dx}\right) + q(x)u = f(x), \qquad x \in \Omega = [0, 1]\,, \\ u(0) = u(1) = 0\,. \end{array} \tag{1.6}$$

In order to get a suitable bilinear form, we use integration by part. Let $v \in V_0$ (or $H_0^1(\Omega)$). We have

$$-\int_0^1 v(x)\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)\, dx + \int_0^1 v(x)q(x)u\, dx = \int_0^1 v(x)f\, dx$$

and therefore,

$$\int_0^1 p(x)v'u'\, dx - pu'v|_0^1 + \int_0^1 qvu\, dx = \int_0^1 v(x)f\, dx\,.$$

Then we choose

$$a(u,\, v) = \int_0^1 p(x)v'u'\, dx + \int_0^1 qvu\, dx\,.$$

Since $v \in H_0^1(\Omega)$, $v(0) = v(1) = 0$ and we have

$$a(u,\, v) = (f,\, v)\,.$$

The corresponding (M)-model and (V)-model are given below.

(M)-model:

$$\min_{u\in H_0^1(\Omega)} F(u) := \frac{1}{2}a(u,\, v) - (f,\, v)$$

(V)-model: Find $u \in H_0^1(\Omega)$ such that for all $v \in H_0^1(\Omega)$

$$a(u,\, v) = (f,\, v)\,.$$

Example 2

$$\begin{array}{ll} -u'' = f(x), & \qquad x \in (0,1) \\ u'(0) = u'(1) = 0 & \end{array}$$

We have

$$-\int_0^1 u''v\, dx = \int_0^1 fv\, dx\,.$$

Using integration by part

$$\int_0^1 u'v'\, dx - \left(u'(1)v(1) - u'(0)v(0)\right) = \int_0^1 fv\, dx\,.$$

since no boundary conditions are imposed in V. We choose $V = \{v|smooth\}$ and the mesh $\{x_j\}_{j=0}^{N+1}$. ϕ_j, $j = 0, 1, ..., N$, denote the piecewise basis functions.

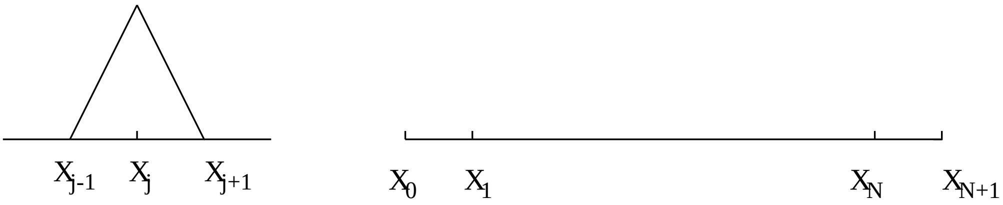

Figure 5: Linear elements

Figure 6: Linear elements

For any $u \in V_h$, $u = \sum_{j=0} \alpha_j \phi_j(x)$. Let $v = \phi_i(x)$. Then

$$\sum_{j=0}^{N} \int_0^1 \alpha_j \phi_j' \phi_i'\, dx = \int_0^1 f(x)\phi_i\, dx$$

The corresponding linear system is

$$\frac{1}{h}\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} \boldsymbol{\alpha} = \boldsymbol{b}\,.$$

1.6 Other approximation.

FEM methods depend upon the choice of basis functions ϕ_j. There are many different choices.

polynomial spline (piecewise polynomial approximation);
global polynomial approximation;
orthogonal functions, $\sin j\pi x$ and $\cos j\pi x$.

Example. Quadratic FEM. Let

$$0 = \bar{x}_0 < \bar{x}_1 \cdots < \bar{x}_{N-1} < \bar{x}_N = 1$$

be a partition on $[0, 1]$ and each subinterval $[\bar{x}_j, \bar{x}_{j+1}]$ defines an element. To form a quadratic interpolation, let $x_{2j} = \bar{x}_j$ and $x_{2j+1} = (\bar{x}_j + \bar{x}_{j+1})/2$. We can define piecewise quadratic Lagrange basis function ϕ_j satisfying

* quadratic polynomial in each element $(x_{2j-2},\, x_{2j})$;
* $\phi_j(x_i) = \delta_{ij}$.

Let $0 = x_0 < x_1 \cdots < x_{2N-1} < x_{2N} = 1$ be a uniform mesh and ϕ_j be the quadratic basis functions. Then

$$\phi_{2j}(x) = \begin{cases} \frac{(x-x_{2j-1})(x-x_{2j-2})}{2h^2} & x \in (x_{2j-2},\, x_{2j})\,, \\ \frac{(x-x_{2j+1})(x-x_{2j+2})}{2h^2} & x \in (x_{2j},\, x_{2j+2})\,, \\ 0 \qquad\qquad\quad \mathit{other} & \end{cases}$$

$$\phi_{2j-1}(x) = \begin{cases} \frac{(x-x_{2j-2})(x-x_{2j})}{-h^2} & x \in (x_{2j-2},\, x_{2j})\,, \\ 0 \qquad\qquad\quad \mathit{other} & \end{cases}$$

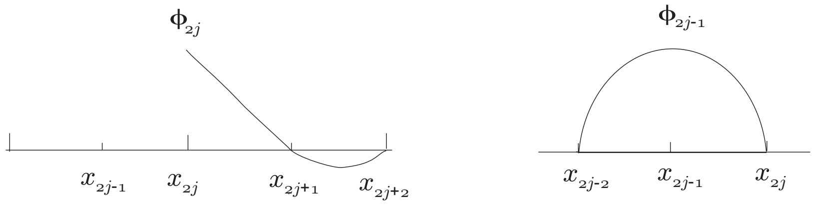

Figure 7: The basis functions of quadratic element.

1.7 Element stiffness matrix.

For an one-dimensional problem (simple model (1.1) or general model (1.6)), the stiffness matrix can be obtained by

$$a_{ij} = a(\phi_i, \phi_j)$$

where $a(\ ,\ )$ denotes the corresponding bilinear form. However, formulas become much more complicated in multi-dimensional space. In most computer program, one uses a so-called element stiffness matrix technique, in which one first generates an element stiffness matrix for each element and second, assemble them into the (global) stiffness matrix.

In order to present the formulas of element stiffness matrix and the procedure of algorithm, we consider the simple model problem

$$\begin{array}{ll} -u'' = f(x) & x \in (0,1) \\ u(0) = a_0, u(1) = a_1 & \end{array}$$

with four linear elements denoted by K_k, $k = 1, 2, ..., 4$, where the parition is defined by the mesh points, x_k, $l = 0, 1, , , , 4$.

Multiplying $v \in H_0^1(K_k)$ in the equation and using integration by part,

$$\int_{x_{k-1}}^{x_k} u_x v_x dx - u_x v \Big|_{x_{k-1}}^{x_k} = \int_{x_{k-1}}^{x_k} f v dx, \quad k = 1, 2, 3, 4\,.$$

Let

$$u = \alpha_{k-1}\phi_{k-1}(x) + \alpha_k \phi_k(x), \qquad x \in [x_{k-1}, x_k]$$

and choose $v = \phi_{k-1}(x), \phi_k(x)$, respectively. Then

$$\begin{aligned} \int_{x_{k-1}}^{x_k} (\alpha_{k-1}\phi'_{k-1} + \alpha_k \phi'_k)\phi'_{k-1} dx + u_x(x_{k-1}) &= \int_{x_{k-1}}^{x_k} f\phi_{k-1} dx, \\ \int_{x_{k-1}}^{x_k} (\alpha_{k-1}\phi'_{k-1} + \alpha_k \phi'_k)\phi'_k dx - u_x(x_k) &= \int_{x_{k-1}}^{x_k} f\phi_k dx \end{aligned}$$

Let

$$a_{ij}^{(k)} = \int_{x_{k-1}}^{x_k} \phi'_{k+i-2}(x)\phi'_{k+j-2}(x)\, dx \qquad b_i^{(k)} = \int_{x_{k-1}}^{x_k} f(x)\phi_{k+i-1}(x)\, dx \quad i, j = 1, 2;$$

We have the linear system for the elment K_k,

$$\begin{bmatrix} a_{11}^{(k)} & a_{12}^{(k)} \\ a_{21}^{(k)} & a_{22}^{(k)} \end{bmatrix} \begin{bmatrix} \alpha_{k-1} \\ \alpha_k \end{bmatrix} + \begin{bmatrix} u'(x_{k-1}) \\ -u'(x_k) \end{bmatrix} = \begin{bmatrix} b_1^{(k)} \\ b_2^{(k)} \end{bmatrix}$$

or

$$A^{(k)}\alpha^{(k)} + \sigma^{(k)} = b^{(k)}$$

where $A^{(k)}$ is the element stiffness matrix and $b^{(k)}$ the element loading vector for the k-th element.

Adding the second equation in k-th element system to the first equation of $k+1$-th element system, we obtain

$$\begin{bmatrix} a_{11}^{(k)} & a_{12}^{(k)} & 0 \\ a_{21}^{(k)} & a_{22}^{(k)} + a_{11}^{(k+1)} & a_{12}^{(k+1)} \\ 0 & a_{21}^{(k+1)} & a_{22}^{(k+1)} \end{bmatrix} \begin{bmatrix} \alpha_{k-1} \\ \alpha_k \\ \alpha_{k+1} \end{bmatrix} + \begin{bmatrix} u'(x_{k-1}) \\ 0 \\ -u'(x_{k+1}) \end{bmatrix} = \begin{bmatrix} b_1^{(k)} \\ b_2^{(k)} + b_1^{(k+1)} \\ b_2^{(k+1)} \end{bmatrix}$$

and moreover, we have the global system

$$\begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & 0 & & \\ a_{21}^{(1)} & a_{22}^{(1)}+a_{11}^{(2)} & a_{12}^{(2)} & & \\ 0 & a_{21}^{(2)} & a_{22}^{(2)}+a_{11}^{(3)} & a_{12}^{(3)} & \\ 0 & 0 & a_{21}^{(3)} & a_{22}^{(3)}+a_{11}^{(4)} & a_{12}^{(4)} \\ 0 & 0 & 0 & a_{21}^{(4)} & a_{22}^{(4)} \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{bmatrix} + \begin{bmatrix} u'(x_0) \\ 0 \\ -u'(x_4) \end{bmatrix} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)}+b_1^{(2)} \\ b_2^{(2)}+b_1^{(3)} \\ b_2^{(3)}+b_1^{(4)} \\ b_2^{(4)} \end{bmatrix}.$$

Using the boundary conditions $u(0) = a_0, u(1) = a_1$, finally we obtain a system of three equations,

$$\begin{bmatrix} a_{22}^{(1)}+a_{11}^{(2)} & a_{12}^{(2)} & \\ a_{21}^{(2)} & a_{22}^{(2)}+a_{11}^{(3)} & a_{12}^{(3)} \\ 0 & a_{21}^{(3)} & a_{22}^{(3)}+a_{11}^{(4)} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} b_2^{(1)}+b_1^{(2)}-a_{21}^{(1)}a_0 \\ b_2^{(2)}+b_1^{(3)} \\ b_2^{(3)}+b_1^{(4)}-a_{21}^{(4)}a_1 \end{bmatrix}.$$

For the computation of element stiffness matrices and element loading vectors, we introduce the change of variable $x = x_{k-1} + th$, by which,

$$a_{ij}^{(k)} = \frac{1}{h_k}\int_0^1 \bar{\phi}_i'(t)\bar{\phi}_j'(t)\,dt$$
$$b_i^{(k)} = h\int_0^1 f(x_{k-1}+th)\bar{\phi}_i(t)dt \qquad i,j = 1,2$$

where $\bar{\phi}_1(t)$ and $\bar{\phi}_2(t)$ are two standard basis functions defined in $(0, 1)$.

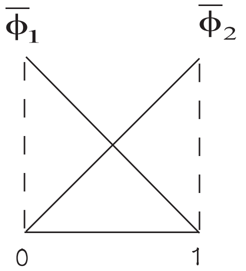

Figure 8: Two basis functions, ϕ_1 and $\phi_2(x)$, for the element K_1.

Step I. Generate element stiffness matrices and element loading vectors. On K_k, we have

$$\bar{\phi}_1'(t) = -1 \qquad \bar{\phi}_2'(t) = 1\,.$$

Then

$$a_{11}^{(k)} = a_{22}^{(k)} = \frac{1}{h}\int_0^1 (\bar{\phi}_1'(t))^2\,dt = \frac{1}{h_k}$$
$$a_{12}^{(k)} = a_{21}^{(k)} = \frac{1}{h}\int_0^1 \bar{\phi}_1'(t)\bar{\phi}_2'(t)\,dt = -\frac{1}{h_k}$$
$$b_1^{(k)} = h\int_0^1 f(th)\bar{\phi}_1(t)\,dt \qquad b_2^{(1)} = h\int_0^1 f(th)\bar{\phi}_2(t)\,dt$$

and

$$A_k = \frac{1}{h_k}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad b_1 = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} \end{bmatrix}.$$

Step II. Assemble the global stiffness matrix.

$$\bar{A} = \begin{bmatrix} a_{11}^{(1)} & a_{12}^{(1)} & & & \\ a_{21}^{(1)} & a_{22}^{(1)} + a_{11}^{(2)} & a_{12}^{(2)} & & \\ & a_{21}^{(2)} & a_{22}^{(2)} + a_{11}^{(3)} & a_{12}^{(3)} & \\ & & a_{21}^{(3)} & a_{22}^{(3)} + a_{11}^{(4)} & a_{12}^{(4)} \\ & & & a_{21}^{(4)} & a_{22}^{(4)} \end{bmatrix} \qquad \bar{b} = \begin{bmatrix} b_1^{(1)} \\ b_2^{(1)} + b_1^{(2)} \\ b_2^{(2)} + b_1^{(3)} \\ b_2^{(3)} + b_1^{(4)} \\ b_2^{(4)} \end{bmatrix}.$$

Step III. Condensation. By the boundary conditions $\alpha_0 = a_0$ and $\alpha_{N+1} = a_1$, we have the FEM linear system $A\alpha = b$. In the simple case with uniform mesh h,

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & \\ -1 & 2 & -1 \\ & -1 & 2 \end{bmatrix}$$

Step IV. Solve the linear system to obtain the finite element solution.

Remarks. One can obtain the same linear system as before by using the above technique. In the computing $a_{ij}^{(k)}$, one only needs to calculate the integrals at each subinterval and the formulas are easy to extend to more general problems and complicated boundary conditions. For a given problem, the bilinear form $a(u,v)$ is an integral in the interval $(0,1)$. If we define a local bilinear form $a_k(u,v)$ in the same form as $a(u,v)$ in the kth subinterval. Then

$$a_{ij}^{(k)} = a_k(\phi_i, \phi_j).$$

Now we use a quadratic finite element method for solving the above model with two elements, K_1 and K_2. In this case, $h = 0.5$. In each element, there are three basis functions

$$\bar{\phi}_1(t) = \frac{(t-0.5)(t-1)}{0.5}, \quad \bar{\phi}_2(t) = \frac{t(t-1)}{-0.5^2}, \quad \bar{\phi}_3(t) = \frac{t(t-0.5)}{0.5}$$

and

$$\bar{\phi}_1'(t) = \frac{2t-1.5}{0.5}, \quad \bar{\phi}_2'(t) = \frac{2t-1}{-0.5^2}, \quad \bar{\phi}_3'(t) = \frac{2t-0.5}{0.5}.$$

Step I. On K_1, we have

$$\begin{aligned} a_{11}^{(1)} &= \frac{1}{h}\int_0^1 (2t-1.5)^2 dt = \frac{7}{6} \\ a_{22}^{(1)} &= \frac{1}{h}\int_0^1 4(2t-1)^2 dt = \frac{8}{3} \\ a_{33}^{(1)} &= \frac{1}{h}\int_0^1 (2t-0.5)^2 dt = \frac{7}{6} \\ a_{12}^{(1)} &= \frac{-1}{h}\int_0^1 2(2t-1.5)(2t-1) dt = -\frac{4}{3} \\ a_{13}^{(1)} &= \frac{1}{h}\int_0^1 (2t-1.5)(2t-0.5) dt = \frac{1}{6} \\ a_{23}^{(1)} &= \frac{-1}{h}\int_0^1 2(2t-1)(2t-0.5) dt = -\frac{4}{3} \end{aligned}$$

and

$$
\begin{aligned}
b_1^{(1)} &= h \int_0^1 f(x_0 + th) \frac{(t-0.5)(t-1)}{0.5} dt, \\
b_2^{(1)} &= h \int_0^1 f(x_0 + th) \frac{t(t-1)}{-0.5^2} dt, \\
b_3^{(1)} &= h \int_0^1 f(x_0 + th) \frac{t(t-0.5)}{0.5} dt
\end{aligned}
$$

For K_2, we have

$$A_2 = A_1$$

and similalry we can calculate $b^{(2)}$.

Step II. Assemble the symmetric global matrix

$$
\bar{A} = \begin{bmatrix}
a_{11}^{(1)} & a_{12}^{(1)} & a_{13}^{(1)} & & \\
 & a_{22}^{(1)} & a_{23}^{(1)} & & \\
 & & a_{33}^{(1)} + a_{11}^{(2)} & a_{12}^{(2)} & a_{13}^{(2)} \\
 & & & a_{22}^{(2)} & a_{23}^{(2)} \\
 & & & & a_{33}^{(2)}
\end{bmatrix}
\qquad
\bar{b} = \begin{bmatrix}
b_1^{(1)} \\
b_2^{(1)} \\
b_3^{(1)} + b_1^{(2)} \\
b_2^{(2)} \\
b_3^{(2)}
\end{bmatrix}
$$

Step III. By the boundary conditions $\alpha_0 = a_0$ and $\alpha_4 = a_1$, the final global matrix and global loading vector become

$$
A = \begin{bmatrix}
a_{22}^{(1)} & a_{23}^{(1)} & \\
 & a_{33}^{(1)} + a_{11}^{(2)} & a_{12}^{(2)} \\
 & & a_{22}^{(2)}
\end{bmatrix}
\qquad
b = \begin{bmatrix}
b_2^{(1)} - a_{12}^{(1)} a_0 \\
b_3^{(1)} + b_1^{(2)} - a_{13}^{(1)} a_0 - a_{13}^{(2)} a_1 \\
b_2^{(2)} - a_{23}^{(2)} a_1
\end{bmatrix}
$$