# 2 FEM for two-dimensional elliptic PDEs

## 2.1 Basic idea

We consider a simple model problem, Poisson equation

$$\begin{array}{ll} -\Delta u = f(x,y), & (x,y) \in \Omega \\ u|_{\partial\Omega} = 0 & \end{array} \tag{2.1}$$

where $\partial\Omega$ denotes the boundary of $\Omega$ and

$$\Delta u \equiv \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\,.$$

Similarly to one-dimensional problem, we need a variational model which will be obtained by using Green formula instead of integration by part in the last section. Let $u = u(x,y)$ and $w = w(x,y)$ be smooth functions in $\Omega$. Green formula is given by

$$-\int_\Omega v\Delta w\, d\Omega = -\int_{\partial\Omega} v w_n\, ds + \int_\Omega \nabla w \cdot \nabla v\, d\Omega$$

*i.e.*,

$$-\int_\Omega v\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) dxdy = -\int_{\partial\Omega} v\frac{\partial w}{\partial n}\, ds + \int_\Omega \left(\frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y}\right) dxdy$$

where

$$\begin{array}{l} \frac{\partial w}{\partial n} = \frac{\partial w}{\partial x} n_x + \frac{\partial w}{\partial y} n_y \\ n_x = \cos < \overrightarrow{n}, \overrightarrow{x} >, \qquad n_y = \cos < \overrightarrow{n}, \overrightarrow{y} >, \\ \nabla v = \begin{pmatrix} \frac{\partial v}{\partial x} \\ \frac{\partial v}{\partial y} \end{pmatrix} \qquad \nabla w = \begin{pmatrix} \frac{\partial w}{\partial x} \\ \frac{\partial w}{\partial y} \end{pmatrix}. \end{array}$$

By using the Green formula for the model problem (2.1) and letting $v \in H_0^1(\Omega)$, we have

$$-\int_\Omega v\Delta w\, d\Omega = \int_\Omega \nabla u \cdot \nabla v\, d\Omega = \int_\Omega f v\, d\Omega\,.$$

Let

$$\begin{aligned} a(u,v) &= \int_\Omega \nabla u \cdot \nabla v\, d\Omega \\ (f,\, v) &= \int_\Omega f v\, d\Omega \\ F(v) &= \frac{1}{2} a(v,\, v) - (f,\, v)\,. \end{aligned}$$

We can obtain the equivalent (V) model and (M) model

$$\begin{array}{ll} \text{Find } u \in H_0^1(\Omega) \text{ such that for all } v \in H_0^1(\Omega), & \\ \quad a(u,\, v) = (f,\, v) & (V) \end{array}$$

and

$$\min_{v \in H_0^1(\Omega)} F(v) \qquad\qquad (M)$$

Similarly, the above models are to find a solution in a infinite dimensional space. In FEM, we need to approximate the infinite dimensional space by finite dimensional spaces $V_h$. Let $\phi_j(x, y)$, $j = 1, 2, \cdots, N$, be the basis functions of $V_h$. A function $u_h \in V_h$ can be expressed by

$$u_h(x, y) = \sum_{j=1}^{N} \alpha_j \phi_j(x, y) .$$

Letting $v = \phi_i$, $i = 1, 2, ..., N$, respectively, in the $(V_h)$-model, we have

$$a(u_h, \phi_i) = (f, \phi_i)$$

and

$$\sum_{j=1}^{N} \alpha_j \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, d\Omega = \int_\Omega f\phi_i \, d\Omega , \quad i = 1, 2, ..., N .$$

The linear system is given by

$$A\boldsymbol{\alpha} = b$$

where

$$a_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, d\Omega , \qquad b_i = \int_\Omega f\phi_i \, d\Omega .$$

The importance is the choice of finite dimensional space. Piecewise polynomial space is one of most popular choices. In this case, one has to construct a mesh, *i.e.*, divided the domain $\Omega$ into small pieces (element). Mesh generation in multi-dimensional space is one of important parts in study of FEM.

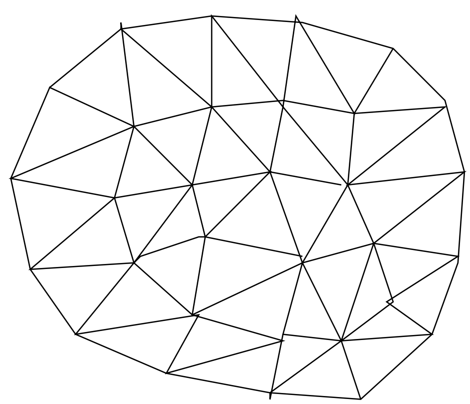

Figure 9: Finite element mesh in two-dimensional space.

**2.2 Basis functions**

Here we consider piecewise polynomial basis functions. Let $\Omega$ be a bounded domain which consists of the elements $\{K_i\}_{i=1}^{M}$ and nodal points $\{(x_j,\, y_j\}_{j=1}^{N}$. We first consider the piecewise linear space where

$$V_h = \{v : \ \text{continuous on } \Omega, \ \text{piecewise linear}, \ v = 0 \text{ on } \partial\Omega\} .$$

The corresponding basis functions $\phi_j(\boldsymbol{x}) \in V_h$, $j = 1, 2, ..., N$, satisfy

* $\phi_j(\boldsymbol{x})$ is piecewise linear; * $\phi_j(\boldsymbol{x}_i) = \delta_{ij}$ where $\boldsymbol{x} = (x, y)$.

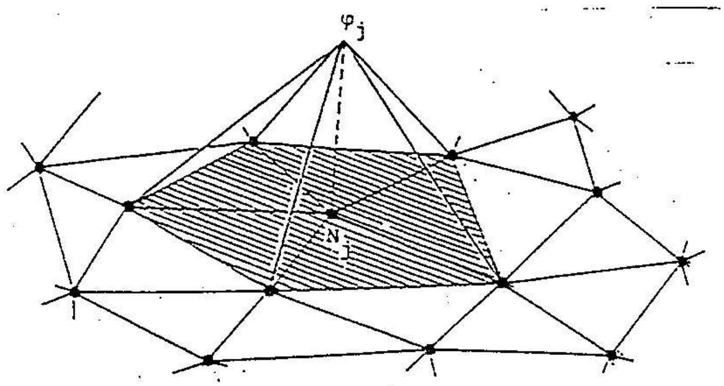


Figure 10: The basis function of linear element in two-dimensional space.

We see that the support of $\phi_j$ (the set of points $\boldsymbol{x}$ for which $\phi_j \neq 0$) consists of the triangles with the common node $\boldsymbol{x}_j$ (shaded area).

Now we need to formulate the piecewise linear function.

*Example.* Find a linear function $L(x, y)$ on $\Omega_1 = \{(x, y)|0 \leq x \leq a, 0 \leq y \leq b,\ x/a+y/b \leq 1\}$ satisfies

* linear;

* $L(a, 0) = 0,\ L(0, b) = 0;$

* $L(0, 0) = 1.$

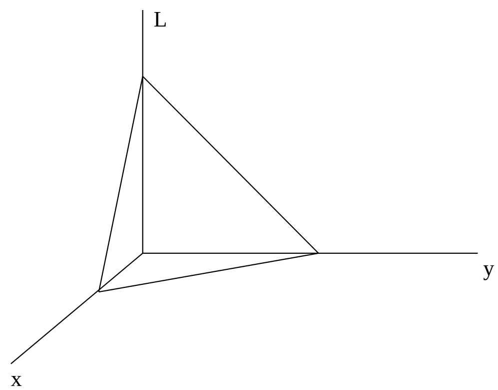


Figure 11: The function $L(x, y)$.

Since $L(x, y)$ is a linear function, it can be expressed by $L(x, y) = b_0 + b_1 x + b_2 y$. Then the solution of problem is unique. The equation of line passed through the points $(a, 0)$ and $(0, b)$ is

$$\frac{x}{a} + \frac{y}{b} = 1\,.$$

Let

$$l(x, y) = \frac{x}{a} + \frac{y}{b} - 1\,.$$

Then The solution is

$$L(x, y) = \frac{l(x, y)}{l(0, 0)} = 1 - \frac{x}{a} - \frac{y}{b}$$

which satisfies the all above conditions.

For a given triangle $\Omega_1$ with the three vertices $\boldsymbol{x}_1$, $\boldsymbol{x}_2$ and $\boldsymbol{x}_3$. We need to find three linear functions $L_j(\boldsymbol{x})$, $j = 1, 2, 3$, satisfying $L_j(\boldsymbol{x}_i) = \delta_{ij}$. Let $l_{ij}(\boldsymbol{x})$ denote the equation of line passed through the vertices $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. Then

$$L_1(\boldsymbol{x}) = \frac{l_{23}(\boldsymbol{x})}{l_{23}(\boldsymbol{x}_1)} \quad L_2(\boldsymbol{x}) = \frac{l_{31}(\boldsymbol{x})}{l_{31}(\boldsymbol{x}_2)} \quad L_3(\boldsymbol{x}) = \frac{l_{12}(\boldsymbol{x})}{l_{12}(\boldsymbol{x}_3)} .$$

Now we can formulate the linear basis functions, each of them corresponds to each interior node. $\phi_j(\boldsymbol{x})$ is nonzero only in the elements around the vertex $\boldsymbol{x}_j$. The formula in such each element can be obtained as above.

*Example (Quadratic element).* Find a function $Q_j(x, y)$ on a triangle $\Omega_1$ with its vertices $\boldsymbol{x}_i$, $i = 1, 2, 3$, satisfies

* quadratic;

* $Q_j(\boldsymbol{x}_i) = \delta_{ij}$, $i, j = 1, 2, 3$.

It is obvious that the solution is not unique. We need to add some more conditions. Here we consider a triangular element below.

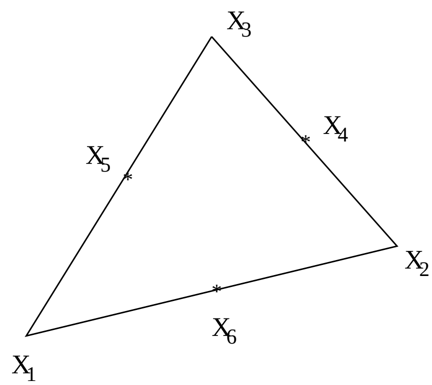


Figure 12: The basis function of quadratic element.

We require the second condition to be satisfied for $j = 1, 2, 3, 4, 5, 6$. Let $l_{ij}(\boldsymbol{x})$ be the equation of line passed through $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$. Then the solution is

$$\begin{array}{ll} Q_1(\boldsymbol{x}) = \frac{l_{56}(\boldsymbol{x}) \cdot l_{23}(\boldsymbol{x})}{l_{56}(\boldsymbol{x}_1) \cdot l_{23}(\boldsymbol{x}_1)} & Q_2(\boldsymbol{x}) = \frac{l_{46}(\boldsymbol{x}) \cdot l_{13}(\boldsymbol{x})}{l_{46}(\boldsymbol{x}_2) \cdot l_{13}(\boldsymbol{x}_2)} \\ Q_3(\boldsymbol{x}) = \frac{l_{45}(\boldsymbol{x}) \cdot l_{12}(\boldsymbol{x})}{l_{45}(\boldsymbol{x}_3) \cdot l_{12}(\boldsymbol{x}_3)} & Q_4(\boldsymbol{x}) = \frac{l_{12}(\boldsymbol{x}) \cdot l_{13}(\boldsymbol{x})}{l_{12}(\boldsymbol{x}_4) \cdot l_{13}(\boldsymbol{x}_4)} \\ Q_5(\boldsymbol{x}) = \frac{l_{21}(\boldsymbol{x}) \cdot l_{23}(\boldsymbol{x})}{l_{21}(\boldsymbol{x}_5) \cdot l_{23}(\boldsymbol{x}_5)} & Q_6(\boldsymbol{x}) = \frac{l_{31}(\boldsymbol{x}) \cdot l_{32}(\boldsymbol{x})}{l_{31}(\boldsymbol{x}_6) \cdot l_{32}(\boldsymbol{x}_6)} . \end{array}$$

It is noted that $Q_j(x, y)$ defined above is quadratic for each component.

*Example (Quadrilateral element.* Four-point quadrilateral element

The linear basis functions $L_j(\boldsymbol{x})$ can be obtained analogously.

$$\begin{array}{ll} L_1(\boldsymbol{x}) = \frac{l_{23}(\boldsymbol{x}) \cdot l_{34}(\boldsymbol{x})}{l_{23}(\boldsymbol{x}_1) \cdot l_{34}(\boldsymbol{x}_1)} & L_2(\boldsymbol{x}) = \frac{l_{14}(\boldsymbol{x}) \cdot l_{34}(\boldsymbol{x})}{l_{14}(\boldsymbol{x}_2) \cdot l_{34}(\boldsymbol{x}_2)} \\ L_3(\boldsymbol{x}) = \frac{l_{12}(\boldsymbol{x}) \cdot l_{14}(\boldsymbol{x})}{l_{12}(\boldsymbol{x}_3) \cdot l_{14}(\boldsymbol{x}_3)} & L_4(\boldsymbol{x}) = \frac{l_{12}(\boldsymbol{x}) \cdot l_{23}(\boldsymbol{x})}{l_{12}(\boldsymbol{x}_4) \cdot l_{23}(\boldsymbol{x}_4)} \end{array}$$

which is not exact "linear" function in general.

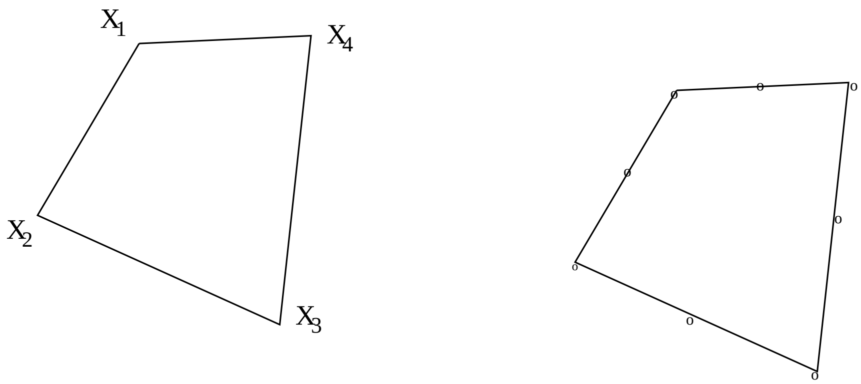


Figure 13: Some other elements.

Some other high-order elements are as follows.

**Stiffness matrix**

For the simple model problem (2.1), we have

$$a_{ij} = \int_\Omega \nabla\phi_i \cdot \nabla\phi_j \, d\Omega \,.$$

Here we only consider the linear element.

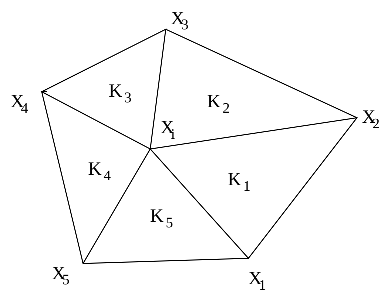


Figure 14: The elements around the node $X_i$

Let $K_i$ denote the five elements around the node $\boldsymbol{x}_j$. Then

$$\phi_j(x) = \frac{l_{12}(\boldsymbol{x})}{l_{12}(\boldsymbol{x}_j)} \,.$$

The equation of line passed through $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ is

$$\frac{x - x_1}{x_2 - x_1} - \frac{y - y_1}{y_2 - y_1} = 0$$

*i.e.*,

$$l_{12}(x, y) := (x_1 y_2 - x_2 y_1) - x(y_2 - y_1) + y(x_2 - x_1) = 0 \,.$$

We see that

$$l_{12}(x, y) = \begin{vmatrix} 1 & 1 & 1 \\ x & x_1 & x_2 \\ y & y_1 & y_2 \end{vmatrix}$$

and

$$l_{12}(x_j, y_j) = \begin{vmatrix} 1 & 1 & 1 \\ x_j & x_1 & x_2 \\ y_j & y_1 & y_2 \end{vmatrix} = 2A(K_1) = (2\Delta_1)$$

where $A(K_1) = \Delta_1$ is the area of the element $K_1$. Since

$$\frac{\partial \phi_j}{\partial x} = -\frac{y_2 - y_1}{2\Delta_1} \qquad \frac{\partial \phi_j}{\partial y} = \frac{x_2 - x_1}{2\Delta_1}\,,$$

we have

$$a_{ij} = \sum_l \int_{K_l} \nabla \phi_i \cdot \nabla \phi_j \, d\Omega = \sum_l \int_{K_l} \left( \frac{\partial \phi_i}{\partial x} \frac{\partial \phi_j}{\partial x} \right) + \left( \frac{\partial \phi_i}{\partial y} \frac{\partial \phi_j}{\partial y} \right) d\Omega\,.$$

Obviously, if no segment between $\boldsymbol{x}_i$ and $\boldsymbol{x}_j$, $a_{ij} = 0$.

*Example.* We consider the following Poisson's equation

$$\begin{aligned} -\Delta u &= 1 \qquad (x, y) \in \Omega = [0,1] \times [0,1] \\ u|_{\partial\Omega} &= 0 \end{aligned}$$

By using linear triangular element and $h = 1/3$.

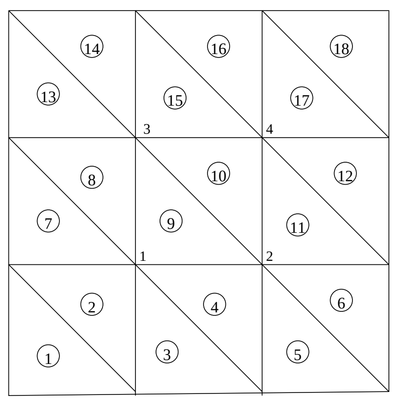


Figure 15: An example.

Node $i = 1$,

$$\begin{aligned} a_{11} &= \textstyle\int_{K_2+K_3+K_4+K_7+K_8+K_9} \nabla \phi_1 \cdot \nabla \phi_1 \, dxdy \\ &= \textstyle\int_{K_2} + \int_{K_3} + \int_{K_4} + \int_{K_7} + \int_{K_8} + \int_{K_9}\ . \end{aligned}$$

We have calculate the integrals one by one. Since on $K_2$, $\phi_1 = (x + y - h)/h$,

$$\int_{K_2} \nabla \phi_1 \cdot \nabla \phi_1 \, dxdy = \int_{K_2} 2 \left(\frac{1}{h}\right)^2 dxdy = 1\,.$$

Also we have

$$\int_{K_3} \nabla \phi_1 \cdot \nabla \phi_1 \, dxdy = \int_{K_3} \left(\frac{1}{h}\right)^2 dxdy = 1/2$$

and

$$\int_{K_4} = \int_{K_7} = \int_{K_8} = 1/2 \quad \int_{K_9} = 1\,.$$

Then

$$a_{11} = 4\,.$$

For $a_{12}$, we have

$$a_{12} = \int_{K_4+K_9} \nabla\phi_1 \cdot \nabla\phi_2 \, dxdy\,.$$

We can obtain

$$\int_{K_4} \nabla\phi_1 \cdot \nabla\phi_2 \, dxdy = \int_{K_9} \nabla\phi_1 \cdot \nabla\phi_2 \, dxdy = -1/2$$

and therefore,

$$a_{12} = -1\,.$$

Similarly,

$$a_{13} = -1 \qquad a_{14} = 0 \quad (\text{ no edge between } \boldsymbol{x}_1 \text{ and } \boldsymbol{x}_4)$$

and

$$\begin{aligned} &a_{22} = a_{33} = a_{44} = 4 \\ &a_{23} = 0 \quad a_{24} = a_{34} = -1\,. \end{aligned}$$

Then the stiffness matrix is

$$A = \frac{1}{h}\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix}$$

The calculation of the load vector $b$ is more complicated.

$$\begin{aligned} b_1 &= \textstyle\int_{K_2+K_3+K_4+K_7+K_8+K_9} \phi_1 \, dxdy \\ b_2 &= \textstyle\int_{K_4+K_5+K_6+K_9+K_10+K_{11}} \phi_2 \, dxdy \\ b_3 &= \textstyle\int_{K_8+K_9+K_{10}+K_{13}+K_{14}+K_{15}} \phi_3 \, dxdy \\ b_4 &= \textstyle\int_{K_{10}+K_{11}+K_{12}+K_{15}+K_{16}+K_{17}} \phi_4 \, dxdy\,. \end{aligned}$$

The finite element linear system is

$$A\boldsymbol{\alpha} = \boldsymbol{b}\,.$$

If we consider the case with $N \times N$ mesh. Then the stiffness matrix is block tridiagonal

$$A = \begin{bmatrix} B & -I & & \\ -I & B & -I & \\ & \ddots & \ddots & \ddots \\ & & -I & B \end{bmatrix}$$

where

$$B = \begin{bmatrix} 4 & -1 & & \\ -1 & 4 & -1 & \\ & \ddots & \ddots & \ddots \\ & & -1 & 4 \end{bmatrix}.$$

If we have used different partition, or different index, we will obtain different stiffness matrices. It will be very complicated for general geometry, equations and boundary conditions.

**2.3 Element stiffness matrix.**

The above approach is very complicated to implement in computer. As we discussed in the section one, we consider an alternative approach, element stiffness matrix.

We assume that we have had a triangular mesh on the domain $\Omega$, which consists of the elements $K_k$, $k = 1, 2, ..., M$ and $\phi_j$, $j = 1, 2, ..., N$, be the linear basis functions.

For the element $K_k$, we denote its element stiffness matrix by

$$A_k = (a_{ij}^{(k)})_{3\times 3}$$

where for the simple model (2.1),

$$a_{ij}^{(k)} = \bar{a}(\phi_i,\, \phi_j) = \int_{K_k} \nabla\phi_i \cdot \nabla\phi_j \, dxdy \, .$$

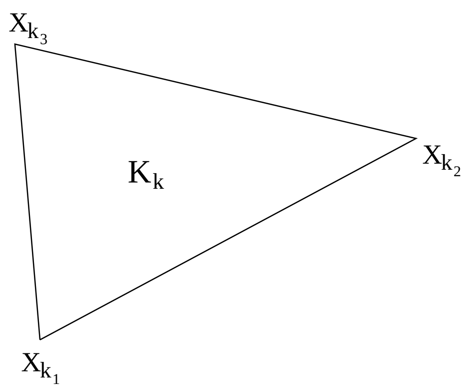


Figure 16: A typical element.

Let $(\bar{x}_i,\, \bar{y}_i)$, $i = 1, 2, 3$, be the coordinates of three vertices of $K_k$. Then

$$a_{ij}^{(k)} = \int_{K_k} \left( \frac{\partial \phi_{k_i}}{\partial x}\frac{\partial \phi_{k_j}}{\partial x} + \frac{\partial \phi_{k_i}}{\partial y}\frac{\partial \phi_{k_j}}{\partial y} \right) dxdy \quad i, j = 1, 2, 3$$

and

$$\phi_{k_1} = \frac{1}{2\Delta_k} \begin{vmatrix} 1 & 1 & 1 \\ x & \bar{x}_2 & \bar{x}_3 \\ y & \bar{y}_2 & \bar{y}_3 \end{vmatrix} \quad \phi_{k_2} = \frac{1}{2\Delta_k} \begin{vmatrix} 1 & 1 & 1 \\ x & \bar{x}_3 & \bar{x}_1 \\ y & \bar{y}_3 & \bar{y}_1 \end{vmatrix} \quad \phi_{k_3} = \frac{1}{2\Delta_k} \begin{vmatrix} 1 & 1 & 1 \\ x & \bar{x}_1 & \bar{x}_2 \\ y & \bar{y}_1 & \bar{y}_2 \end{vmatrix} .$$

In general,

$$\phi_{k_i} = \frac{1}{2\Delta_k} \begin{vmatrix} 1 & 1 & 1 \\ x & \bar{x}_{i+1} & \bar{x}_{i+2} \\ y & \bar{y}_{i+1} & \bar{y}_{i+2} \end{vmatrix}$$

where $\boldsymbol{x}_4 = \boldsymbol{x}_1$ and $\boldsymbol{x}_5 = \boldsymbol{x}_2$. Hence,

$$\frac{\partial \phi_{k_i}}{\partial x} = -\frac{1}{2\Delta_k} \begin{vmatrix} 1 & 1 \\ \bar{y}_{i+1} & \bar{y}_{i+2} \end{vmatrix} \qquad \frac{\partial \phi_{k_i}}{\partial y} = \frac{1}{2\Delta_k} \begin{vmatrix} 1 & 1 \\ \bar{x}_{i+1} & \bar{x}_{i+2} \end{vmatrix} .$$

We have the following formulas:

$$a_{ij}^{(k)} = \frac{1}{4\Delta_k}\left[(\bar{y}_{i+2}-\bar{y}_{i+1})(\bar{y}_{j+2}-\bar{y}_{j+1})+(\bar{x}_{i+2}-\bar{x}_{i+1})(\bar{x}_{j+2}-\bar{x}_{j+1})\right] \tag{2.2}$$

and

$$b_i^{(k)} = \int_{K_k} f(x,y)\phi_i \, dxdy \, .$$

Since $f(x, y)$ could be an arbitrary function, we need to use Gauss-Quadrature technique to calculate the above integrals

$$b_i^{(k)} = \sum_{l=1}^{p} w_l f(x_l^*, y_l^*)\phi_i(x_l^*, y_l^*)$$

where $(x_l^*, y_l^*)$ and $w_l$ define the Gaussian points and the corresponding weight.

*Remarks.* For some complicated elliptic PDEs, $a(u, v)$ could be complicated and we cannot obtain the above general formula. We have to use Gauss-Quadrature to calculate both $a_{ij}^{(k)}$ and $b_i^{(k)}$.

In most computer program, the stiffness matrix is obtained by using the following approach. Let $A = (a_{ij})$ and $A_k = a_{ij}^{(k)}$ be the (global) stiffness matrix and element stiffness matrix, respectively. Then for each element $K_k$, we need the following loop

$$a_{i_p,j_q} = a_{i_p,j_q} + a_{pq}^{(k)} \qquad p, q = 1, 2, 3 \tag{2.3}$$

and

$$b_{i_p} = b_{i_p} + b_p^{(k)} \qquad p, q = 1, 2, 3 \, .$$

The relationship between $(i_p, j_q)$ and $(p, q)$ will be discussed later. Finally we use the boundary conditions to reduce the size of the matrix.

*Example.* We consider the same problem as in the last example. Here $\Delta_k = 1/18$.

(i) Element stiffness matrices. By using the above formulas,

$$\begin{aligned}
a_{11}^{(1)} &= \tfrac{1}{4\Delta_1}\left[\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)+\left(\tfrac{-1}{3}\right)\left(\tfrac{-1}{3}\right)\right] = 1\\
a_{12}^{(1)} &= \tfrac{1}{4\Delta_1}\left[\left(\tfrac{1}{3}\right)\left(-\tfrac{1}{3}\right)+\left(\tfrac{-1}{3}\right)\cdot 0\right] = -1/2\\
a_{13}^{(1)} &= \tfrac{1}{4\Delta_1}\left[\left(\tfrac{1}{3}\right)\cdot 0+\left(\tfrac{-1}{3}\right)\left(\tfrac{1}{3}\right)\right] = -1/2\\
a_{22}^{(1)} &= \tfrac{1}{4\Delta_1}\left[\left(\tfrac{-1}{3}\right)\left(\tfrac{-1}{3}\right)+0\right] = 1/2\\
a_{23}^{(1)} &= \tfrac{1}{4\Delta_1}\left[\left(\tfrac{-1}{3}\right)\cdot 0+0\cdot\left(\tfrac{1}{3}\right)\right] = 0\\
a_{33}^{(1)} &= \tfrac{1}{4\Delta_1}\left[0+\left(\tfrac{1}{3}\right)\left(\tfrac{1}{3}\right)\right] = 1/2\, .
\end{aligned}$$

Then we have

$$A_1 = \begin{bmatrix} 1 & -1/2 & -1/2 \\ -1/2 & 1/2 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}$$

and similarly

$$A_2 = \begin{bmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 1 & -1/2 \\ 0 & -1/2 & 1/2 \end{bmatrix} .$$

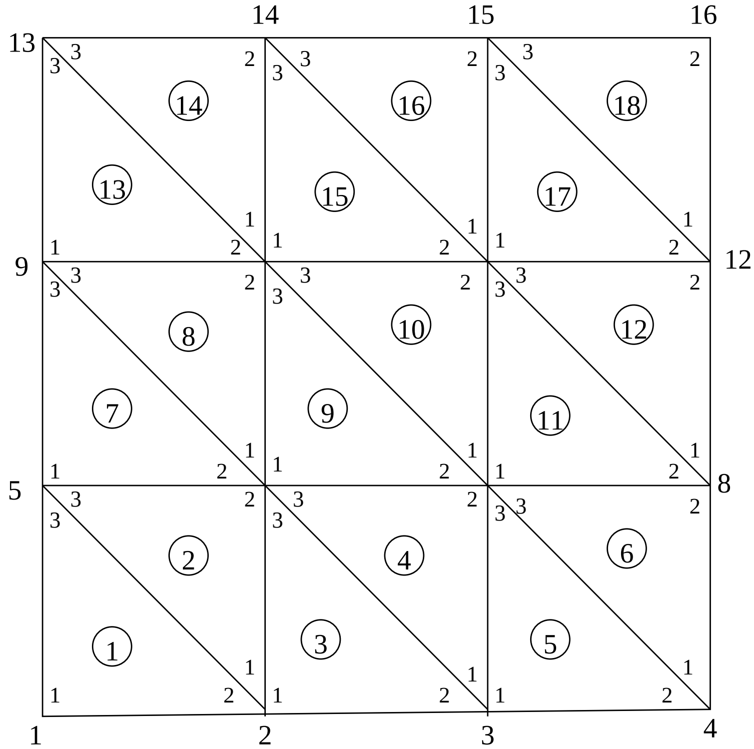

Figure 17: An example.

Moreover, we have

$$A_1 = A_3 = A_5 \cdots = A_{15} = A_{17} \qquad A_2 = A_4 = \cdots = A_{16} = A_{18}\,.$$

(ii) Global stiffness matrix. Now we assemble them into a global stiffness matrix. Key point for the assembling is the local index and global index. In our formulas, $p, q$ represent the local indexes and $j_p, j_q$ are the global indexes. The relationship is as follows.

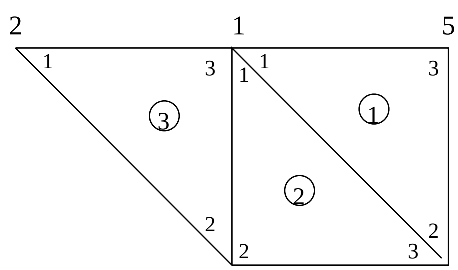

Figure 18: The connection.

$$A_1: \quad \begin{array}{ccc} \textit{local} & & \textit{global} \\ 1 & \rightarrow & 1 \\ 2 & \rightarrow & 2 \\ 3 & \rightarrow & 5 \end{array} \qquad A_2: \quad \begin{array}{ccc} \textit{local} & & \textit{global} \\ 1 & \rightarrow & 2 \\ 2 & \rightarrow & 6 \\ 3 & \rightarrow & 5 \end{array}$$

Following the rule, we can put the element matrix into the global matrix. For example,

$$\begin{aligned} & a_{13}^{(1)} \text{ should be added in } a_{15} \\ & a_{15} = a_{15} + a_{13}^{(1)} \end{aligned}$$

and

$$\begin{aligned} & a_{12}^{(2)} \text{ should be added in } a_{26} \\ & a_{26} = a_{26} + a_{12}^{(2)} \,. \end{aligned}$$

We obtain (assemble the first 8 elements)

$$A = \left[ \begin{array}{cccccccccc} 1 & -1/2 & & & -1/2 & & & & & \\ & 1/2+1/2+1 & -1/2 & & 0 & -1/2-1/2 & & & & \\ & & 1/2+1/2+1 & -1/2 & & -1/2 & -1/2 & & & \\ & & & 1/2+1/2 & & & & -1/2 & & \\ & & & & 1/2+1/2+1 & -1/2-1/2 & & & -1/2 & \\ & & & & & 1+1/2+1/2+1/2+1/2 & -1/2 & & & -1/2 \\ & & & & & & 1+1/2+1/2 & & & \\ & & & & & & & 1 & & \\ & & & & & & & & 1/2+1/2 & -1/2 \\ & & & & & & & & & 1 \\ \cdots & & & & & & & & & \\ \cdots & & & & & & & & & \end{array} \right]$$

The right hand side can be assembled similarly.

(iii) Condensation. By using the boundary conditions

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha_5 = \alpha_8 = \alpha_9 = \alpha_{12} = \alpha_{13} = \alpha_{14} = \alpha_{15} = \alpha_{16} = 0\,,$$

we obtain the final linear system

$$\begin{bmatrix} 4 & -1 & -1 & 0 \\ -1 & 4 & 0 & -1 \\ -1 & 0 & 4 & -1 \\ 0 & -1 & -1 & 4 \end{bmatrix} \begin{bmatrix} \alpha_6 \\ \alpha_7 \\ \alpha_{10} \\ \alpha_{11} \end{bmatrix} = \begin{bmatrix} b_2^{(2)} + b_3^{(3)} + b_3^{(4)} + b_2^{(7)} + b_1^{(8)} + b_1^{(9)} \\ b_2^{(4)} + b_3^{(5)} + b_3^{(6)} + b_2^{(9)} + b_1^{(10)} + b_1^{(11)} \\ b_2^{(8)} + b_3^{(9)} + b_3^{(10)} + b_2^{(13)} + b_1^{(14)} + b_1^{(15)} \\ b_2^{(10)} + b_3^{(11)} + b_3^{(12)} + b_2^{(15)} + b_1^{(16)} + b_1^{(17)} \end{bmatrix}.$$

The finite element solution can be obtained by solving the above system.

**2.4 Implementation Computer program structure.**

(i) Data input

* functions (such as $f(x, y)$) and parameters in the PDE;
* boundary conditions;
* Geometry $\Omega$.

(ii) Mesh generation

* generate the coordinates of all nodal points;
* generate the connection among the local indexes (of points) and global indexes (of points).

(iii) Stiffness matrix and load vector

* generate element stiffness matrices;
* generate element load vector;

* Gaussian quadrature;
* assembling.
(iv) Linear solver (iterative methods or direct methods)
(v) Post-process.

**Gauss quadrature**

We consider the integral

$$\int_0^1 f(x)\,dx \quad or \quad \int_{-1}^1 f(x)\,dx$$

In general,

$$\int\int_\Delta f(x,y)\,dx\,dy$$

(i) Integration via polynomial interpolation

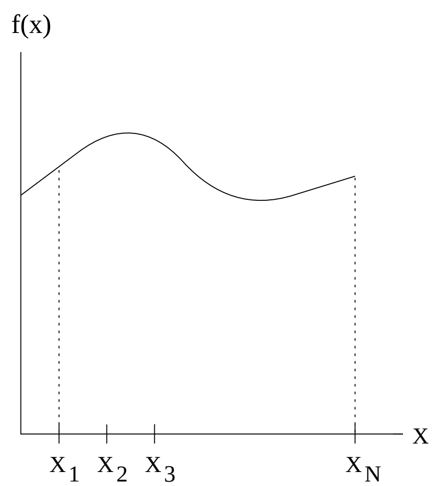


Figure 19: Polynomial $P_n(x)$

We can select nodes $x_0 < x_1 < x_2 < \cdots < x_n$ in [0,1] and set up a Lagrange interpolation, *i.e.* find a polynomial $p_n(x)$ of degree n such that

$$P_n(x_j) = f(x_j)$$

It is obvious that there exists an unique polynomial $P_n(x)$ satisfying these conditions and

$$P_n(x) = \sum_{i=0}^{n} f(x_i) l_i(x)$$

where

$$l_i(x) = \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}$$

The integration formula via polynomial interpolation is

$$\int_0^1 f(x)\,dx \approx \int_0^1 P_n(x)\,dx = \sum_{i=0}^{n} f(x_i) \int_0^1 l_i(x)\,dx$$

However, when n is large, the interpolation approximation is very good. We often use lower-order interpolation polynomial in calculating integrals numerically.

Composite trapezoid rule:

**Trapezoid rule**

$$\int_0^1 f(x)\,dx = (f(0) + f(1))/2$$

**Composite trapezoid rule**
Let $0 = x_0 < x_1 < x_2 < \cdots < x_n = 1$

$$\begin{aligned} \textstyle\int_0^1 f(x)\,dx &= \textstyle\sum_{j=1}^{n} \int_{x_{j-1}}^{x_j} f(x)\,dx \\ (use \quad trapezoid \quad rule) &= \textstyle\sum_{j=1}^{n} (f(x_j) + f(x_{j-1}) \frac{(x_j - x_{j-1})}{2} \end{aligned}$$

For uniform mesh

$$\int_0^1 f(x)\,dx \approx \sum_{j=1}^{n} \frac{(f(x_j) + f(x_{j-1}))h}{2}$$

We can prove that

$$\int_0^1 f(x)\,dx = \sum_{j=1}^{n} \frac{(f(x_j) + f(x_{j-1}))h}{2} - \frac{1}{12} h^2 f''(\xi)$$

where

$$\frac{1}{12} h^2 f''(\xi) = O(h^2) \quad second - order$$

(ii) Gauss-quadrature in 1-D

We need to construct a quadrature formula of the type

$$\int_0^1 f(x)\,dx \approx \sum_{i=1}^{n} A_i f(x_i)$$

We want the formula to be exact for polynomial of degree $\leq n-1$ by choosing the suitable $A_j$ and $x_j, j = 0, 1, \cdots, n$

It has been proved that these $x_j, j = 1, 2, \cdots, n$, are the roots of some special polynomials. For some small n, we can obtain $x_j$ and $A_j$ by solving the following equations.

$$\begin{gathered} \textstyle\int_0^1 1\,dx = \sum_{i=1}^{n} A_i \\ \textstyle\int_0^1 x\,dx = \sum_{i=1}^{n} A_i x_i \\ \vdots \\ \textstyle\int_0^1 x^{2n-1}\,dx = \sum_{i=1}^{n} A_i x_i^{2n-1} \end{gathered}$$

*Example n=1*

$$\int_0^1 1\,dx = A_1$$
$$\int_0^1 x\,dx = A_1 x_1 ===> x_1 = \tfrac{1}{2}$$

The Gauss-quadrature formula with $n = 1$ is

$$\int_0^1 f(x)\,dx = f(\frac{1}{2})$$

*Example n=2*

$$\int_0^1 1\,dx = 1 = A_1 + A_2$$
$$\int_0^1 x\,dx = \tfrac{1}{2} = A_1 x_1 + A_2 x_2$$
$$\int_0^1 x\,dx = \tfrac{1}{3} = A_1 x_1^2 + A_2 x_2^2$$
$$\int_0^1 x\,dx = \tfrac{1}{4} = A_1 x_1^3 + A_2 x_2^3$$

Solving the system gives

$$\overline{x_1} = \tfrac{1}{2}(1 - \tfrac{1}{\sqrt{3}}) \qquad \overline{x_2} = 1 - x_1$$
$$A_1 = \tfrac{1}{2} \qquad A_2 = \tfrac{1}{2}$$

The formula of Gauss-quadrature with $n = 2$ is

$$\int_0^1 f(x)\,dx = \frac{1}{2}(f(\overline{x_1}) + f(\overline{x_2}))$$

(iii) Gauss-quadrature in 2-D

$$I = \int_0^1 \int_0^1 f(x, y)\,dx\,dy$$

1-node formula: $I \approx f(x_1, y_1)$

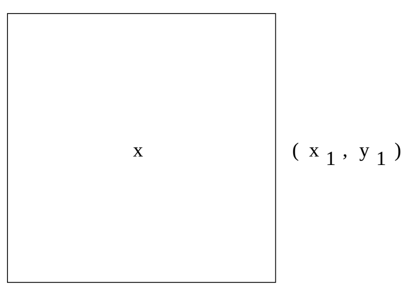


Figure 20: 1-node

4-node formula: $I \approx \sum_{j=1}^{4} w_j f(x_j, y_j)$

$$I = \int \int_\delta f(x, y) \partial x \partial y \approx \sum_{j=1}^{n} w_j f(x_j)$$

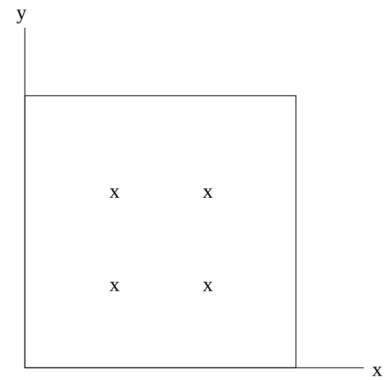


Figure 21: 4-node

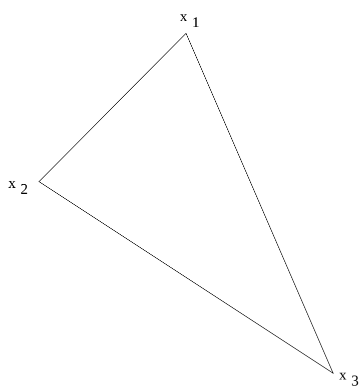


Figure 22: A triangle element

**Linear Solver**

(i) Direct method

Gauss elimination

(ii) Iterative algorithm

$$Ax = b$$

Let $A = D - L - U$
where D, L, U, are diagonal, lower triangle and upper triangle.
Then we have some iterative algorithms as follows.

**Jacobi iteration**

$$\begin{cases} \text{Given} \quad x^0 \\ Dx^{n+1} = (L+U)x^n + b \end{cases}$$

**Gauss-Seidel**

$$\begin{cases} \text{Given} \quad x^0 \\ (D-L)x^{n+1} = Ux^n + b \end{cases}$$

**SOR (Successive over-relaxation)**

$$\begin{cases} \text{Given} \quad x^0 \\ (D - wL)x^{n+1} = [(1-w)D + wU]x^n + wb \end{cases}$$