3 Applications (truss)

3.1 Introduction

An elastic bar

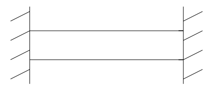

Figure 23: An elastic bar

$u(x)$: displacement at x;
$\sigma(x)$: stress at x;
$f(x)$: force per volume.
Hooke's law: $\sigma = Eu'$
Equilibrium: $\sigma' = f$
$-S\sigma' = f \cdot S$ for uniform bar
$-(S\sigma)' = F$ for non-uniform bar, where F : force per length and S is the area of cross section.

i.e.

$$-\frac{d}{dx}(AE\frac{du}{dx}) = F(x)$$

In engineering, one is interesting to find
the displacement : u
stress : $\sigma = E\frac{du}{dx}$
or strain : $\epsilon = \frac{du}{dx}$

This elastic bar problem can be solved easily by using finite element methods discussed in the previous sections.

Simple truss analysis

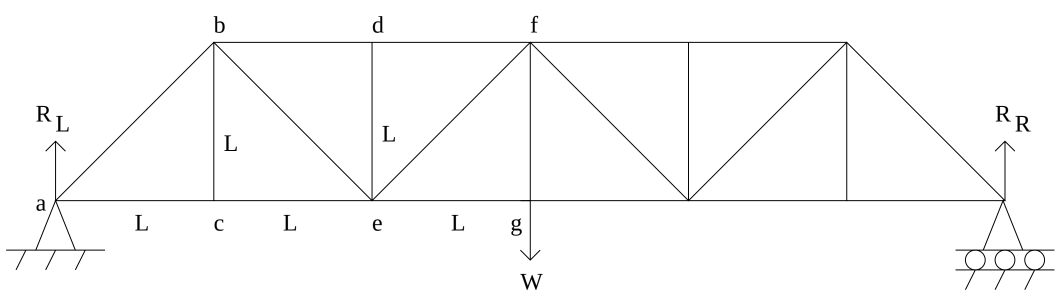

Figure 24: Truss

We need to find R_L, R_R and the tension in the members ab, ac, eg, ef, df.

Basic equations:

(1) By Newton's first law:

If a body is in equilibrium, the summation of all external forces acting on the body is zero, *i.e.*

$$\sum F_j = 0$$

(2) No rotation when a body in equilibrium

If a body is pivoted at a point O and is acted on by a force F_I, then the moment of this force about O that tends to cause the body to rotate about O is given by the vector product

$$r_I \times F_I$$

when a body is in equilibrium, the summation of the moments produced by external forces is zero, *i.e.*

$$\sum_j r_j \times F_j = 0$$

(3) Example (Truss analysis)

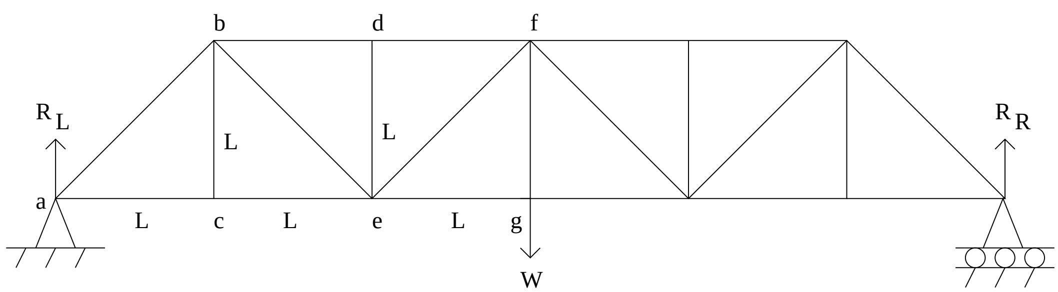

Figure 25: Truss

We wish to know R_L, R_R and the tension in the members ab, ac, eg, ef, df

(i) By the equilibrium principle, the summation of force in vertical direction is zero, *i.e.*

$$R_L + R_R - W = 0$$

(ii) Summation of moments about the point a is zero

$$W \cdot 3L - R_R \cdot 6L = 0$$

Solving the above two equations gives

$$R_L = R_R = \frac{W}{2}$$

(iii) Tension in the members ab and ac
Cut the truss in ab and ac and consider the special part as a body (matter).

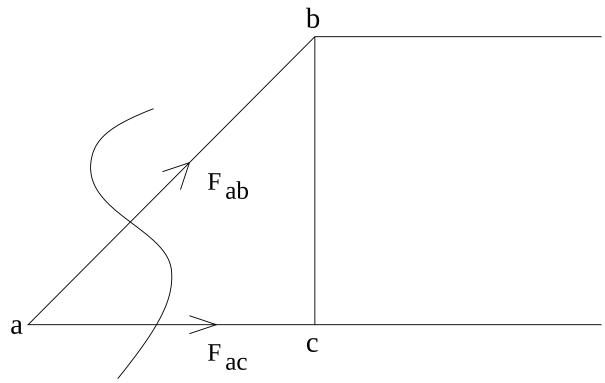

Figure 26: The special part of the truss

We use the equilibrium principle on this body. The summation of forces in the vertical direction is zero

$$\frac{W}{2} + F_{ab} sin\theta = 0, \qquad \theta = \frac{\pi}{4}$$
$$then \quad F_{ab} = -\frac{W}{\sqrt{2}}$$

The summation of forces in the horizontal direction is zero

$$F_{ab} cos\theta + F_{ac} = 0$$
$$then \quad F_{ac} = \frac{W}{2}$$

(iv) We consider a special part as in Fig.

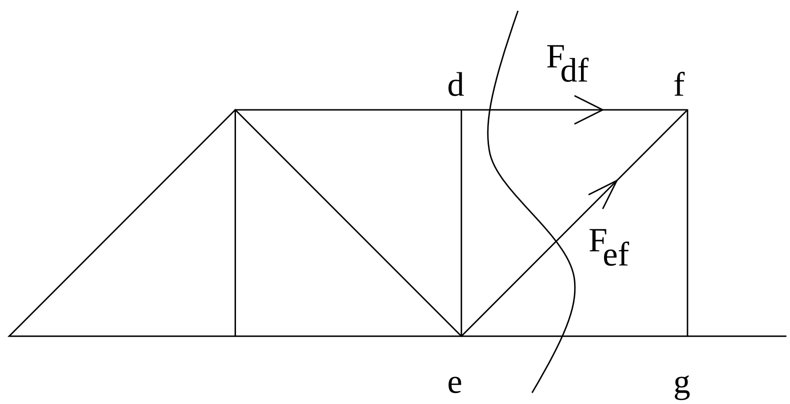

Figure 27: The special part of the truss

The summation of horizontal forces

$$F_{df} + F_{ef} cos\theta + F_{eg} = 0$$

The summation of vertical forces

$$\frac{W}{2} + F_{ef} sin\theta = 0$$

The moments about the point f

$$\frac{W}{2} \cdot 3L - F_{eg} L = 0$$

Solving above three equations gives

$$F_{df} = -W, \quad F_{eg} = \frac{3W}{2}, \quad F_{ef} = -\frac{W}{\sqrt{2}}$$

It should be noted that the method mentioned above can only be applied for solving some simple truss problems in which the displacement is not involved.

Finite element methods can be applied for solving more complicated and general problems.

3.2 Plane Trusses

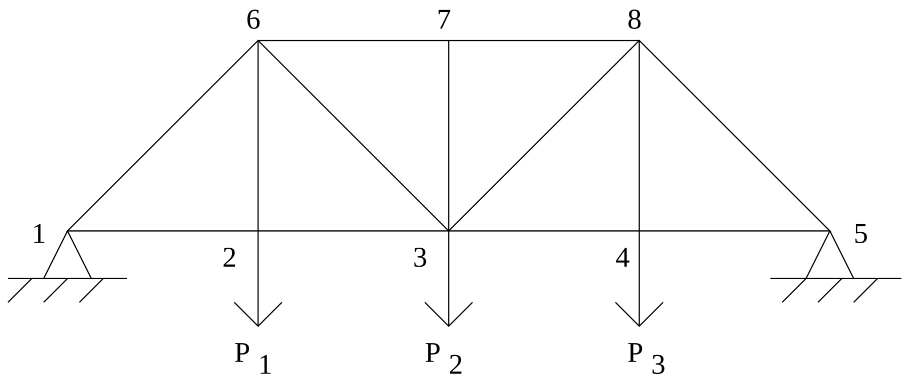

Figure 28: The plane trusses

In section One, we present some simple methods for solving plane truss problems with statically determinate structures or with simple statically undeterminate structures. The finite element method is applicable to general truss problems. Here, we assume that a truss structure consists only of two-force members. That is, every truss element, is in direct tension or compression. In a truss, it is required that all loads and reactions are applied only at the joints, and that all members are connected together at their ends by frictionless pin joints. Then $F(x) = 0$ (no body force)

(1) Local and global coordinate systems

Each element of truss can be considered as an elastic bar in the 1-D local coordinate system.

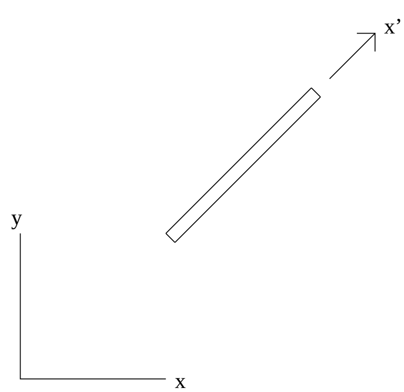

Figure 29: An elastic bar in the 1-D local coordinate system.

In practice, we know only the coordinate of each joint of the truss in (x, y) - coordinate system (global coordinate system) and also, we need to know the displacement vector in (x, y) coordinate system. (global coordinate system)

Let u_1 and v_1 denote the displacements at the joint 1 in x- and y-direction, respectively. u_2 and v_2 can be defined similarly.

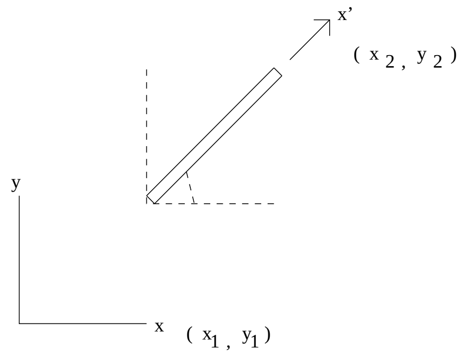

Figure 30: The displacement vector in (x, y) global coordinate system.

Let u_1' and u_2' be the displacements in x'-direction at the joint 1 and 2 respectively. Then,

$$u_1' = u_1 cos\theta + v_1 sin\theta$$
$$u_2' = u_2 cos\theta + v_2 sin\theta$$

In a vector-matrix form

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix}$$

where $l = cos\theta$ and $m = sin\theta$

Let l_e be the length of this element. Then $l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ and obviously

$$l = cos\theta = \frac{x_2 - x_1}{l_e} \quad and \quad m = \frac{y_2 - y_1}{l_e}$$

Usually, we use the notation

$$L = \begin{bmatrix} l & m & 0 & 0 \\ 0 & 0 & l & m \end{bmatrix}$$

for so-called transformation matrix. Then,

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = L \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} = L\beta$$

Noting our notation in 1-D,

$$\alpha_1 = u_1' \quad and \quad \alpha_2 = u_2'$$

(2) FEM

First we consider the equation

$$-(SEu_x)_x = F(x)$$

for one member defined by (x_1, x_2).

To get a variational model, we choose $v \in H^1$ and by the integration by part

$$\int_{x_1}^{x_2} SEu_x v_x dx - SEu_x v|_{x_1}^{x_2} = \int_{x_1}^{x_2} F(x)v(x)dx$$

Physically, we have

$$\sigma = Eu_x : \text{traction} \qquad \text{and} \qquad S\sigma : \text{force}$$

If we choose one linear element to approximate the solution,

$$u = u_1\phi_1(x) + u_2\phi_2(x)$$

and choosing $v = \phi_i$, $i = 1, 2$, we have

$$\frac{SE}{h} A\vec{u} - \begin{pmatrix} -S\sigma_1 \\ S\sigma_2 \end{pmatrix} = \vec{b}$$

where

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$

In a local coordinate system,

$$\vec{u} = L\beta$$

and then,

$$\frac{SE}{h} AL\beta - \begin{pmatrix} -S\sigma_1 \\ S\sigma_2 \end{pmatrix} = \vec{b}$$

or

$$\frac{SE}{h} B\beta - \vec{f} = L^T\vec{b}$$

where

$$B = L^TAL = \begin{bmatrix} l^2 & lm & -l^2 & -lm \\ lm & m^2 & -lm & -m^2 \\ -l^2 & -lm & l^2 & lm \\ -lm & -m^2 & lm & m^2 \end{bmatrix} \qquad \vec{f} = L^T \begin{pmatrix} -S\sigma_1 \\ S\sigma_2 \end{pmatrix} = \begin{pmatrix} -lS\sigma_1 \\ -mS\sigma_1 \\ lS\sigma_2 \\ mS\sigma_2 \end{pmatrix}$$

In the above system of linear equation, there are six unknowns: β and σ_1, σ_2. If we consider a general truss problem, for each member, we have a subsystem. We can eliminate σ_1, σ_2 by using Newton's first law.

(3) Example

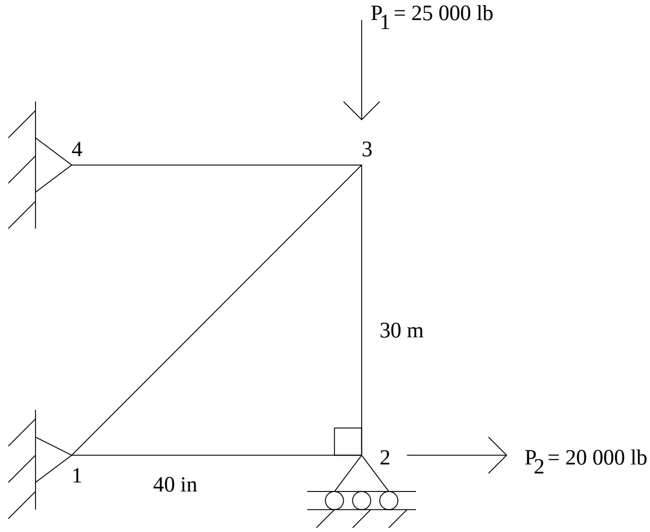

Figure 31: The four bar truss.

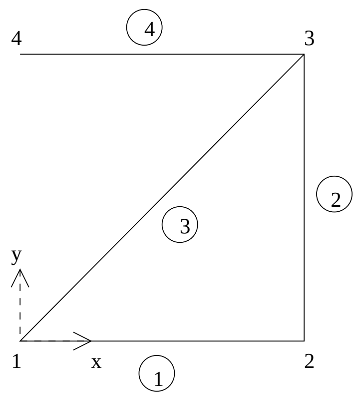

Figure 32: The structural of the entire truss.

We consider the four-bar truss shown as below. It is given that $E = 29.5 \times 10^6 psi$ and the area of cross section $S_e = 1 in^2$ for all elements.

(i) Determine the element stiffness matrix for each element

(ii) Assemble the structural (global) stiffness matrix for the entire truss.

(iii) Reduce to global stiffness matrix.

(iv) Solve for nodal displacement.

(v) Calculate the stresses at each element.

<u>Solution</u>

Coordinate data:

Node	x	y
1	0	0
2	40	0
3	40	30
4	0	30

Connectivity:

	Local nodal number	
Element	1	2
1	1	2
2	3	2
3	1	3
4	4	3

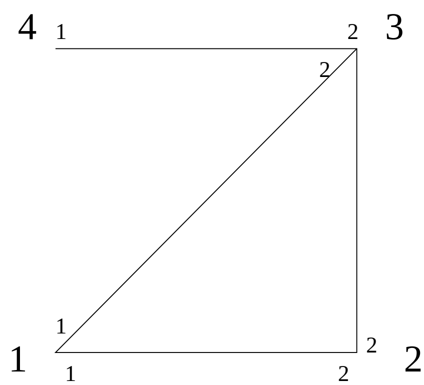

Figure 33: The structural of the entire truss.

Directional cosine:

Element	l_e	l	m
1	40	1	0
2	30	0	-1
3	50	0.8	0.6
4	40	1	0

Then,

(i)

$$B^{(1)} = \frac{29.5 \times 10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$B^{(2)} = \frac{29.5 \times 10^6}{30} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

$$B^{(3)} = \frac{29.5 \times 10^6}{50} \begin{bmatrix} 0.64 & 0.48 & -0.64 & -0.48 \\ 0.48 & 0.36 & -0.48 & -0.36 \\ -0.64 & -0.48 & 0.64 & 0.48 \\ -0.48 & -0.36 & 0.48 & 0.36 \end{bmatrix}$$

$$B^{(4)} = \frac{29.5 \times 10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(ii) The structural (global) stiffness matrix is

$$B = \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ & & 15.0 & 0 & 0 & 0 & 0 & 0 \\ & & & 20.0 & 0 & 20.0 & 0 & 0 \\ & & & & 22.68 & 5.76 & -15.0 & 0 \\ & & & & & 24.32 & 0 & 0 \\ & & & & & & 15.0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

and the global force vector is

$$\vec{f} = \begin{pmatrix} -l_1 S_1 \sigma_1^1 - l_3 S_3 \sigma_3^1 \\ -m_1 S_1 \sigma_1^1 - m_3 S_3 \sigma_3^1 \\ l_1 S_1 \sigma_1^2 + l_2 S_2 \sigma_2^2 \\ m_1 S_1 \sigma_1^2 + m_2 S_2 \sigma_2^2 \\ -l_2 S_2 \sigma_2^1 + l_3 S_3 \sigma_3^2 + l_4 S_4 \sigma_4^2 \\ -m_2 S_2 \sigma_2^1 + m_3 S_3 \sigma_3^2 + m_4 S_4 \sigma_4^2 \\ -l_4 S_4 \sigma_4^1 \\ -m_4 S_4 \sigma_4^1 \end{pmatrix}$$

Noting the unknown,

$$\vec{\beta} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

and the global system is

$$B\vec{\beta} - \vec{f} = 0\,.$$

(iii) By boundary conditions $u_1 = v_1 = 0$, $v_2 = 0 u_4 = v_4 = 0$. Then, only u_2, u_3 and v_3 are unknown. Keeping the rows and columns corresponding to $u_2, u_3 and v_3$, we have the final system

$$\frac{29.5 \times 10^6}{600} \begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix} \begin{bmatrix} u_2 \\ u_3 \\ v_3 \end{bmatrix} = \begin{pmatrix} m_1 S_1 \sigma_1^2 + m_2 S_2 \sigma_2^2 \\ -l_2 S_2 \sigma_2^1 + l_3 S_3 \sigma_3^2 + l_4 S_4 \sigma_4^2 \\ -m_2 S_2 \sigma_2^1 + m_3 S_3 \sigma_3^2 + m_4 S_4 \sigma_4^2 \end{pmatrix} = \begin{bmatrix} 20000 \\ 0 \\ -25000 \end{bmatrix}$$

where we have used the Newton's first law at the points 2 and 3.

(iv) Solving the system gives

$$u_2 = 27.12 \times 10^{-3},\ u_3 = 5.65 \times 10^{-3},\ v_3 = -22.25 \times 10^{-3}$$

Stress calculation

Since we assume that the displacements at each element are linear function, the strain at each element is a constant.

$$\begin{aligned} \sigma = E_e \epsilon = E_e \frac{u_2' - u_1'}{l_e} \\ = \frac{E_e}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} \\ = \frac{E_e}{l_e} \begin{bmatrix} -1 & 1 \end{bmatrix} L \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \\ = \frac{E_e}{l_e} \begin{bmatrix} -l & -m & l & m \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{bmatrix} \end{aligned}$$

(v) $\sigma_1 = 20000 psi$ $\quad \sigma_3 = 5208 psi$

$\sigma_2 = 21880 psi$ $\quad \sigma_4 = 4167 psi$

(3) Three-dimensional trusses

u, v, w: displacements in x, y, z-direction respectively.

Then,

$$u_1' = l u_1 + m v_1 + n w_1$$
$$u_2' = l u_2 + m v_2 + n w_2$$

where

$$l = cos\alpha, \quad m = cos\beta \quad and \quad n = cos\gamma$$
$$cos\alpha = \frac{x_2 - x_1}{l_e} \quad cos\beta = \frac{y_2 - y_1}{l_e} \quad cos\gamma = \frac{z_2 - z_1}{l_e}$$
$$l_e = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

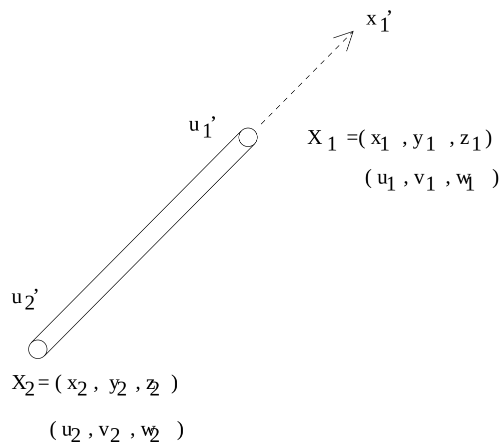

Figure 34: Three dimensional trusses.

Moreover, in a matrix form

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} l & m & n & 0 & 0 & 0 \\ 0 & 0 & 0 & l & m & n \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ u_2 \\ v_2 \\ w_2 \end{bmatrix}$$

In the local-coordinate, the element stiffness matrix for the element j is

$$B^{(j)} = L^T A^{(j)} L = \begin{bmatrix} l^2 & lm & ln & -l^2 & -lm & -ln \\ lm & m^2 & mn & -lm & -m^2 & -mn \\ ln & mn & n^2 & -lm & -mn & -n^2 \\ -l^2 & -lm & -ln & l^2 & lm & ln \\ -lm & m^2 & -mn & lm & m^2 & mn \\ -ln & -mn & -n^2 & ln & mn & n^2 \end{bmatrix} \frac{E_e A_e}{h}$$