4 Further Applications

4.1 Linear Elasticity in two-dimensional space.

1. Basic equations: *Stress equations*:

$$\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + f_1(x,y) = 0$$
$$\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y} + f_2(x,y) = 0$$

Stress-stain relation:

$$\sigma_{11} = (2\mu + \lambda)\epsilon_{11} + \lambda\epsilon_{22}$$
$$\sigma_{12} = 2\mu\epsilon_{12}$$
$$\sigma_{22} = (2\mu + \lambda)\epsilon_{22} + \lambda\epsilon_{11}$$

where

$$\epsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \quad i,j = 1,2,$$

$x_1 = x$, $x_2 = y$, σ_{ij} and ϵ_{ij} are the stress and strain, respectively and u_i represents the displacement in x_i-direction.

General Green formula:

$$\int\int_\Omega \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dxdy = \int_{\partial\Omega} (Pdx + Qdy) \tag{4.1}$$

2. Variational model. Let $v_1, v_2 \in H_0^1$. We have

$$\int\int_\Omega \left(\frac{\partial \sigma_{11}}{\partial x} + \frac{\partial \sigma_{12}}{\partial y}\right) v_1 dxdy = -\int\int_\Omega f_1(x,y) v_1 dxdy$$
$$\int\int_\Omega \left(\frac{\partial \sigma_{12}}{\partial x} + \frac{\partial \sigma_{22}}{\partial y}\right) v_2 dxdy = -\int\int_\Omega f_2(x,y) v_2 dxdy$$

and moreover,

$$\int\int_\Omega \left(\frac{\partial(\sigma_{11}v_1)}{\partial x} + \frac{\partial(\sigma_{12}v_1)}{\partial y} - \sigma_{11}\frac{\partial v_1}{\partial x} - \sigma_{12}\frac{\partial v_1}{\partial y}\right) dxdy = -\int\int_\Omega f_1(x,y) v_1 dxdy$$
$$\int\int_\Omega \left(\frac{\partial(\sigma_{12}v_2)}{\partial x} + \frac{\partial(\sigma_{22}v_2)}{\partial y} - \sigma_{12}\frac{\partial v_2}{\partial x} - \sigma_{22}\frac{\partial v_2}{\partial y}\right) dxdy = -\int\int_\Omega f_2(x,y) v_2 dxdy$$

Using the general Green formula with $Q = \sigma_{11}v_1$ and $P = -\sigma_{12}v_1$ in the first equation gives

$$\int\int_\Omega \left(\sigma_{11}\frac{\partial v_1}{\partial x} + \sigma_{12}\frac{\partial v_1}{\partial y}\right) dxdy - \int_{\partial\Omega} v_1(-\sigma_{12}dx + \sigma_{11}dy) = \int\int_\Omega f_1(x,y) v_1 dxdy$$

and using the general Green formula with $Q = \sigma_{12}v_2$ and $P = -\sigma_{22}v_2$ in the second equation gives

$$\int\int_\Omega \left(\sigma_{12}\frac{\partial v_2}{\partial x} + \sigma_{22}\frac{\partial v_2}{\partial y}\right) dxdy - \int_{\partial\Omega} v_2(-\sigma_{22}dx + \sigma_{12}dy) = \int\int_\Omega f_2(x,y) v_2 dxdy$$

Let

$$\epsilon_{ij}(v) = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right)$$

Then

$$\int\int_\Omega (\sigma_{11}\epsilon_{11}(v) + 2\sigma_{12}\epsilon_{12}(v) + \sigma_{22}\epsilon_{22}(v))\, dxdy - \int_{\partial\Omega}(v_1T_1 + v_2T_2)ds = \int\int_\Omega (f_1v_1 + f_2v_2)dxdy \tag{4.2}$$

where T_1 and T_2 define the tractions on boundary.

If we have the boundary conditions

$$\begin{aligned} &u_1 = u_2 = 0, \quad \text{on } \Gamma_1 \\ &T_i = g_i, \quad \text{on } \Gamma_2 \end{aligned} \tag{4.3}$$

where $\partial\Omega = \Gamma_1 + \Gamma_2$.

The variational model is to find $u_1, u_2 \in H_0^1\{u|u = 0 \text{ on } \Gamma_1\}$ such that

$$a(u, v) = (f, v), \qquad v \in H_0^1 \tag{4.4}$$

where

$$a(u, v) = \int\int_\Omega (\sigma_{11}\epsilon_{11}(v) + 2\sigma_{12}\epsilon_{12}(v) + \sigma_{22}\epsilon_{22}(v))\, dxdy - \int_{\partial\Gamma_2}(v_1g_1 + v_2g_2)ds$$

and

$$(f, v) = \int\int_\Omega (f_1v_1 + f_2v_2)dxdy\,.$$

The minimization model is:

$$\min_{u\in H_0^1} F(u)$$

where

$$F(u) = \frac{1}{2}a(u, v) - (f, v)\,.$$

3. FEM for elasticity. Let V_N be the finite dimensional FEM space and $\{\phi_j\}_{j=1}^N$ be the base of the space. Then the finite element method is to find $u_1, u_2 \in V_N$ such that

$$a(u, v) = (f, v), \qquad v \in V_N \tag{4.5}$$

Let $u_1 = \sum \alpha_{1j}\phi_j$ and $u_2 = \sum \alpha_{2j}\phi_j$. Then

$$\begin{aligned} \sigma_{11} &= (2\mu + \lambda)\frac{\partial u_1}{x_1} + \lambda\frac{\partial u_2}{x_2} \\ \sigma_{12} &= \mu\left(\frac{\partial u_1}{x_2} + \frac{\partial u_2}{x_1}\right) \\ \sigma_{22} &= (2\mu + \lambda)\frac{\partial u_2}{x_2} + \lambda\frac{\partial u_1}{x_1} \end{aligned}$$

Choose

$$v = v_{1i} = \begin{pmatrix} \phi_i \\ 0 \end{pmatrix} \quad \text{and} \quad v = v_{2i} = \begin{pmatrix} 0 \\ \phi_i \end{pmatrix},$$

respectively. We have the system

$$
\begin{aligned}
a(u, v_{1i}) &= 0 \\
a(u, v_{2i}) &= 0,
\end{aligned}
$$

4.2 Parabolic PDEs

Here we consider numerical solutions of a simple model problem

$$u_t = u_{xx}, \qquad x \in (0,1),\ t > 0 \tag{4.6}$$

$$u(0,x) = u_0(x), \qquad x \in (0,1)$$

$$u(t,0) = u(t,1) = 0, \qquad t > 0 \tag{4.7}$$

where $u_0(x)$ is given.

1 Semi-discrete FE for parabolic PDEs. To formulate a finite element method, we need a variational model for the parabolic PDE.

Let $v \in H_0^1$ and

$$\begin{aligned}(u_t, v) &= (u_{xx}, v)\\ (u(0,x), v) &= (u_0, v)\end{aligned}$$

Using integration by part,

$$\begin{aligned}(u_t, v) + a(u,v) &= 0\\ (u(0,x), v) &= (u_0, v)\end{aligned} \tag{4.8}$$

where $a(u,v) = (u_x, v_x)$.

The corresponding variational model is: Find $u \in H_0^1$ such that for all $v \in H_0^1$, the above equation holds. Theoretically, we can prove the equivalence of the parabolic PDE and the variational model.

The variational model for FEM is to find $u \in V_n$ such that for all $v \in V_N$, the equation (4.8) holds.

Now we consider the linear FEM. Let $\{\phi_j\}_{j=0}^N$ be the basis functions of linear FEM. Then

$$u(t,x) = \sum \alpha_j(t)\phi_j(x), \qquad u_t(t,x) = \sum \alpha_j'(t)\phi_j(x)\,.$$

Let $v = \phi_i(x)$, $i = 1, 2, ..., N$,

$$\begin{aligned}&(\sum \alpha_j'(t)\phi_j, \phi_i) + a(\sum \alpha_j(t)\phi_j, \phi_i) = 0\\ &(\sum \alpha_j(0)\phi_j, \phi_i) = (u_0, \phi_i)\end{aligned} \tag{4.9}$$

equivalently

$$\begin{aligned}&\sum \alpha_j'(t)(\phi_j, \phi_i) + \sum \alpha_j(t)a(\phi_j, \phi_i) = 0\\ &\sum \alpha_j(0)(\phi_j, \phi_i) = (u_0, \phi_i)\,.\end{aligned} \tag{4.10}$$

In matrix form,

$$\begin{aligned}C\boldsymbol{\alpha}' + A\boldsymbol{\alpha} &= 0\\ C\boldsymbol{\alpha}(0) &= b\end{aligned} \tag{4.11}$$

where

$$a_{ij} = a(\phi_j, \phi_i), \quad c_{ij} = (\phi_j, \phi_i) \quad b_i = (u_0, \phi_i)$$

or

$$A = \frac{1}{h}A_1, \quad C = tridiag(1/6, 2/3, 1/6), \quad A_1 = tridiag(-1, 2, -1)$$

$$\boldsymbol{\alpha} = (\alpha_1(t), \alpha_2(t), \cdots, \alpha_N(t))^T \qquad \boldsymbol{\alpha}' = (\alpha_1'(t), \alpha_2'(t), \cdots, \alpha_N'(t))^T\,.$$

(4.11) is a system of ODEs. In fact, in terms of FEM approximation in x direction, we reduce the parabolic PDE to a system of ODEs. One can use classical ODE solvers for solving the resulting ODE system. Thus such a scheme is called the semi-discrete method.

2 Fully discrete FE for parabolic PDEs. There are many different fully-discrete schemes. Here we only consider some FD schemes for t direction. If we use the approximation in (4.8), the variational model is to find $u \in V_N$ such that for all $v \in V_N$,

$$\begin{aligned}&\left(\tfrac{u^{n+1}(x)-u^n(x)}{\tau}, v\right) + a(u^n(x), v) = 0\\&(u^0(x), v) = (u_0, v)\end{aligned} \tag{4.12}$$

The variational model for FEM is to find $u \in V_n$ such that for all $v \in V_N$, the equation (4.10) holds.

Similarly, we consider the linear FEM. Let $\{\phi_j\}_{j=0}^N$ be the basis functions of linear FEM. Then

$$u^n(x) = \sum \alpha_j^n \phi_j(x), \,.$$

Let $v = \phi_i(x)$, $i = 1, 2, ..., N$,

$$\begin{aligned}&\textstyle\sum(\alpha_j^{n+1} - \alpha_j^n)(\phi_j, \phi_i) + \tau \sum \alpha_j^n a(\phi_j, \phi_i) = 0\\&\textstyle\sum \alpha_j^0(\phi_j, \phi_i) = (u_0, \phi_i)\end{aligned} \tag{4.13}$$

In matrix form

$$\begin{aligned}&C\boldsymbol{\alpha}^{n+1} = (C - \tau A)\boldsymbol{\alpha}^n\\&C\boldsymbol{\alpha}^0 = b\end{aligned} \tag{4.14}$$

where

$$\boldsymbol{\alpha}^n = (\alpha_1^n(t), \alpha_2^n(t), \cdots, \alpha_N^n(t))^T \,.$$

This is the so-called the explicit Euler scheme with linear FEM.

The variational model for the implicit Euler scheme with linear FEM is to find $u \in V_N$ such that for all $v \in V_N$,

$$\begin{aligned}&\left(\tfrac{u^n(x)-u^{n-1}(x)}{\tau}, v\right) + a(u^n(x), v) = 0\\&(u^0(x), v) = (u_0, v)\end{aligned} \tag{4.15}$$

and in matrix form, we have

$$\begin{aligned}&(C + \tau A)\boldsymbol{\alpha}^n = C\boldsymbol{\alpha}^{n-1}\\&C\boldsymbol{\alpha}^0 = b\,.\end{aligned} \tag{4.16}$$

The variational model for the Crank-Nicolson scheme with linear FEM is to find $u \in V_N$ such that for all $v \in V_N$,

$$\begin{aligned}&\left(\tfrac{u^{n+1}(x)-u^n(x)}{\tau}, v\right) + \tfrac{1}{2}a(u^{n+1}(x), v) + \tfrac{1}{2}a(u^n(x), v) = 0\\&(u^0(x), v) = (u_0, v)\end{aligned} \tag{4.17}$$

and in matrix form, we have

$$\begin{aligned}&(C + \tfrac{1}{2}\tau A)\boldsymbol{\alpha}^{n+1} = (C - \tfrac{1}{2}\tau A)\boldsymbol{\alpha}^{n-1}\\&C\boldsymbol{\alpha}^0 = b\,.\end{aligned} \tag{4.18}$$

3 Stability.

1. Stability of parabolic PDEs

We consider model problem (4.6) and (4.7). A perturbed model problem is given by

$$v_t = v_{xx}, \qquad x \in (0,1),\ t > 0 \tag{4.19}$$
$$v(0,x) = u_0(x) + \delta(x), \qquad x \in (0,1)$$
$$v(t,0) = v(t,1) = 0, \qquad t > 0 \tag{4.20}$$

where $u_0(x)$ is given.

Let $e(t,x) = u(t,x) - v(t,x)$. In terms of (4.6)-(4.7) and (4.19)-(4.20), we have

$$e_t = e_{xx}, \qquad x \in (0,1),\ t > 0 \tag{4.21}$$
$$e(0,x) = \delta(x), \qquad x \in (0,1)$$
$$e(t,0) = e(t,1) = 0, \qquad t > 0 \tag{4.22}$$

The model problem (4.6)-(4.7) is said to be stable with respect to the norm $\|\cdot\|$, if there exist positive constants K such that

$$\|e\| \le K\|\delta\|\,.$$

2. Stability of discrete methods

First we consider a general scheme

$$u^{n+1} = Mu^n + f$$
$$u^0 = b(given) \tag{4.23}$$

where M is an $N \times N$ matrix. A perturbed system is defined by

$$v^{n+1} = Mv^n + f$$
$$v^0 = b + \delta \tag{4.24}$$

Let $\epsilon^n = v^n - u^n$. Then

$$\epsilon^{n+1} = M\epsilon^n$$
$$\epsilon^0 = \delta \tag{4.25}$$

and therefore,

$$\epsilon^{n+1} = M\epsilon^n = M^2\epsilon^{n-1} = ... = M^{n+1}\epsilon^0 = M^{n+1}\delta\,.$$

It follows that

$$\|\epsilon^{n+1}\| \le \|M^{n+1}\|\|\delta\|\,.$$

By theorem in Linear Algebra, if

$$\rho(M) := \max_j |\lambda_j(M)| < 1 \tag{4.26}$$

$$\|M^{n+1}\| \to 0 \qquad \text{as} \quad n \to \infty$$

In this case, we call the discrete method is (absolutely) stable.

For FEM, we have

$$A = \frac{1}{h}A_1, \qquad C = h(I - \frac{1}{6}A_1)$$

where

$$A_1 = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

Theoretically, we can prove that the eigenvalues of A_1 are given by

$$\lambda_j(A_1) = 4\sin^2 \frac{j\pi}{2(N+1)}\,.$$

For linear FEM with explicit Euler scheme,

$$M = C^{-1}(C - \tau A) = I - \tau C^{-1}A = I - \frac{\tau}{h^2}(I - \frac{1}{6}A_1)^{-1}A_1\,.$$

Then

$$\lambda_j(M) = 1 - \frac{\tau}{h^2}\frac{4\sin^2\left(\frac{j\pi}{2(N+1)}\right)}{1 - \frac{2}{3}\sin^2\left(\frac{j\pi}{2(N+1)}\right)}$$

The scheme is absolutely stable if

$$\frac{\tau}{h^2} < \frac{1}{6}\,.$$

We can study the stability of linear FEM methods with implicity and Crank-Nicolson schemes similarly.