MA6612 (Numerical PDEs) Assignment 1

Q1. Present a finite difference scheme for solving the following two-point boundary value problem

$$\begin{aligned} &Lu(x) \equiv u''(x) + p(x)u'(x) + q(x)u(x) = f(x) \quad x \in (0,1) \\ &u(0) = a \qquad u(1) = b \end{aligned} \tag{1}$$

with a general mesh (non-uniform). Write it in matrix form and prove the system has a unique solution when $\max h_i$ is small enough where $h_i = x_i - x_{i-1}$ and

$$|p(x)| \le p^* \qquad 0 < q_* \le q(x)\,.$$

Q2. Consider the 2-point boundary value problem (BVP)

$$Lu(x) \equiv -u'' + q(x)u' = f(x), \quad 0 < x < 1, \qquad u(0) = g_0, \;\; u(1) = g_1,$$

where $|q(x)| \le \alpha$ and $\alpha > 0$. Let $\{x_j\}_{j=0}^N$ define a uniform mesh on $[0,1]$ and the finite difference solution be defined by

$$L_h u_j \equiv \frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} + q(x_j)\frac{u_{j+1} - u_j}{h^2} = f(x_j), \qquad u_0 = g_0, \;\; u_N = g_1$$

Present an error estimate (convergence) for the finite difference method.

Q3. Let $A = (a_{ij})_{n\times n}$ be strictly diagonally dominant, *i.e.*,

$$a_{ii} > \sum_{j\ne i} |a_{ij}|, \qquad i = 1, 2, ..., n\,.$$

Show that the matrix A is nonsingular.

Q4. Let

$$A = \begin{bmatrix} 2 & -1 & 0 & & \\ -1 & 2 & -1 & 0 & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix}$$

be an $n \times n$ tridiagonal matrix. Prove that there exists a constant $c > 0$ (independent of n) such that

$$\frac{c}{n^2} x^T x \le x^T A x \qquad x \in R^n\,.$$

Q5. Consider the 2-point boundary value problem (BVP)

$$Lu(x) \equiv -(a(x)u')' + q(x)u = f(x), \quad 0 < x < 1, \qquad u(0) = g_0, \;\; u(1) = g_1,$$

where

$$0 < a_* \le a(x), \quad 0 < q_* \le q(x), \quad 0 \le x \le 1.$$

Let $\{x_j\}_{j=0}^{N+1}$ be a uniform partition of $[0,1]$ such that $x_j = jh$, $j = 0, \ldots, N+1$, $h = 1/(N+1)$. Suppose that the BVP is approximated by the finite difference (FD) scheme

$$L_h u_j \equiv -\frac{1}{h}\left\{ \frac{a_{j+1} + a_j}{2}\frac{u_{j+1} - u_j}{h} - \frac{a_j + a_{j-1}}{2}\frac{u_j - u_{j-1}}{h} \right\} + q_j u_j = f_j, \quad j = 1, \ldots, N,$$

$$u_0 = g_0, \quad u_{N+1} = g_1,$$

where $a_j = a(x_j)$, $q_j = q(x_j)$, and $f_j = f(x_j)$.

(a) Show that if the coefficient $a \in C^3[0,1]$ and $u \in C^4[0,1]$ is a solution of the BVP, then there is a positive constant C_1 independent of h such that

$$\max_{1 \le j \le N} |L_h u(x_j) - f_j| \le C_1 h^2.$$

(b) Show that L_h is stable with no restriction on h. Specifically, show that there exists a positive constant C_2 independent of h such that

$$\max_{0 \le j \le N+1} |v_j| \le C_2 \max_{1 \le j \le N} |L_h v_j|$$

for any mesh function $\{v_j\}_{j=0}^{N+1}$ with $v_0 = v_{N+1} = 0$.

(c) Use (a) and (b) to show that if $a \in C^3[0,1]$, $u \in C^4[0,1]$ is a solution of the BVP, and $\{u_j\}_{j=0}^{N+1}$ is a solution of the FD scheme, then there is a positive constant C independent of h such that

$$\max_{0 \le j \le N+1} |u(x_j) - u_j| \le C h^2.$$

(d) Rewrite the FD scheme in the matrix-vector form $A\vec{u} = \vec{b}$, $\vec{u} = [u_1, \dots, u_N]^T$, $\vec{b} = [b_1, \dots, b_N]^T$. Show that A is strictly diagonally dominant, symmetric and has positive elements on diagonal. (**Remark:** These properties imply that A is also positive definite.)

Q6. Consider the 2-point boundary value problem (BVP)

$$-u'' + qu = f(x), \qquad 0 \le x \le 1, \qquad u(0) - u'(0) = \alpha, \qquad u(1) + u'(1) = \beta, \tag{2}$$

where q, α, β are given constants and $q > 0$. Let $\{x_j\}_{j=0}^{N+1}$ be a uniform partition of $[0,1]$ such that $x_j = jh$, $j = 0, \dots, N+1$, $h = 1/(N+1)$. Suppose that the BVP (2) is approximated by the finite difference (FD) scheme

$$L_h u_j = f_j, \quad j = 0, \dots, N+1, \tag{3}$$

where

$$L_h u_j = \begin{cases} [1 + (h/2)q]\, u_0 - (u_1 - u_0)/h, & j = 0 \\ (-u_{j-1} + 2u_j - u_{j+1})/h^2 + q u_j, & j = 1, \dots, N, \\ [1 + (h/2)q]\, u_{N+1} + (u_{N+1} - u_N)/h, & j = N+1, \end{cases} \tag{4}$$

$$f_j = \begin{cases} \alpha + (h/2) f(0), & j = 0 \\ f(x_j), & j = 1, \dots, N, \\ \beta + (h/2) f(1), & j = N+1. \end{cases} \tag{5}$$

(a) Show that if $u \in C^4[0,1]$ is a solution of the BVP (2), then

$$\max_{0 \le j \le N+1} |L_h u(x_j) - f_j| = O(h^2),$$

where L_h and f_j are given by (4) and (5), respectively. (**Hint:** For $j = 0$ and $j = N+1$ use the fact that the ODE in (2) is also satisfied at $x = 0$ and $x = 1$.)

(b) Show that L_h defined by (4) is stable with no restriction on h. Specifically, show that

$$\max_{0 \le j \le N+1} |v_j| \le \max(1, 1/q) \max_{0 \le j \le N+1} |L_h v_j|$$

for any mesh function $\{v_j\}_{j=0}^{N+1}$.

(c) Use (a) and (b) to show that if $u \in C^4[0,1]$ is a solution of the BVP (2) and $\{u_j\}_{j=0}^{N+1}$ is a solution of the FD scheme (3)-(5), then

$$\max_{0 \le j \le N+1} |u(x_j) - u_j| = O(h^2).$$

(d) Rewrite (3)-(4) in the matrix-vector form $A\vec{u} = \vec{b}$, where $\vec{u} = [u_0, \ldots, u_{N+1}]^T$ and $\vec{b} = [f_0, \ldots, f_{N+1}]^T$. Show that A is strictly diagonally dominant. Modify the first and the last equations in $A\vec{u} = \vec{b}$ so that the resulting matrix becomes symmetric.

Q7. Write a computer code for solving the two-point boundary value problem (1) with uniform mesh and use this code for solving the following two-point BVP

$$\begin{gathered} -u'' + 10(\sin x)u' + u = f(x), \qquad 0 < x < 1, \\ u(0) = 1, \quad u(1) = e, \end{gathered}$$

with the exact solution $u(x) = e^x$. Print the error

$$e_h = \max_{0 \le j \le N+1} |u(x_j) - u_j|$$

with $h = 0.1, 0.05, 0.025$, respectively. Estimate the order of convergence based on the numerical results (**Do not** print u_j for $j = 0, \ldots, N+1$).

Q8. Let A be an $n \times n$ matrix defined in Q4. Prove that the inverse of the matrix A is nonnegative, *i.e.*, all entries of A^{-1} is nonnegative.