MA6612 (Numerical Analysis PDEs) Assignment 2

Q1. Use the standard second-order finite difference method to solve the nonlinear two-point boundary value problem

$$-u'' + u^3 = g(x), \quad 0 < x < \pi,$$
$$u(0) = u(\pi) = 0,$$

with the exact solution $u = \sin x$. Use Newton's method with no more than 20 iterations to solve the resulting system of nonlinear equations. As a starting vector $\mathbf{u}^{(0)}$, take the corresponding finite difference approximation to the linear problem

$$-u'' = g(x), \quad 0 < x < \pi,$$
$$u(0) = u(\pi) = 0.$$

Print the error

$$e_h = \max_{0 \le j \le N+1} |u(x_j) - u_j|,$$

and estimates of the order of convergence for a sequence of values of h.

Q2. Solve the following minimization problem

$$\min_{u \in M_4^0} \frac{1}{2} \int_0^1 u'^2 \, dx - \int_0^1 u \, dx$$

where

$$M_4^0 = \{u | u \text{ polynomial of degree } \le 4 \text{ and } u(0) = u(1) = 0\}.$$

Q3. Solve the following two-point boundary value problem

$$-u'' + u = 10 \sin \pi x$$
$$u(0) = u(1) = 0$$

by using 8 linear elements with a uniform mesh.

Q4. Show that the problem

$$-u'' = f(x), \qquad on I = (0, 1)$$
$$u(0) = u'(1) = 0$$

is equivalent to the following variational formulation: Find $u \in V$ such that for any $v \in V$,

$$(u', v') = (f, v)$$

where $V = \{v \in H^1(I) : v(0) = 0\}$. (i) Formulate a finite element method for this problem using piecewise linear functions. (ii) Determine the corresponding linear system of equations in the case of a uniform partition.

Q5. Consider the following problem

$$u''''(x) + q(x)u(x) = f(x), \qquad 0 < x < 1$$
$$u(0) = u(1) = u'(0) = u'(1) = 0\,.$$

(i) present an equivalent variational problem. (ii) Can we use linear element method to solve this problem? (iii) Use a quadratic FEM and write down the coefficient matrix of the FEM system when $q = 0$.

Q6. Let P_3 a vector space which consists of all cubic polynomials. Find the basis functions $\{\phi_i\}_{i=1}^4$ such that for any $v(x) \in P_3$

$$v(x) = v(0)\phi_1(x) + v'(0)\phi_1(x) + v(1)\phi_3(x) + v'(1)\phi_4(x)\,.$$

Q7. Consider the two-point boundary value problem

$$-u'' + 9u = -9\cosh\left(\frac{3}{2}\right), \quad 0 < x < 1, \qquad u(0) = 0, \;\; u(1) = 0,$$

whose solution is

$$u(x) = \cosh\left(3x - \frac{3}{2}\right) - \cosh\left(\frac{3}{2}\right)$$

(a) Solve this problem using the finite element Galerkin method with piecewise quadratics on a uniform mesh.

(b) Examine the rate of convergence at the mesh points and at arbitrary points.

Q8. Find the approximation solution of Q7 in the space

$$V_N = \{u : u = \sum \alpha_i \phi_i(x)\} \qquad \phi_i = \sin i\pi x, \qquad i = 1, 2, 3\,.$$