MA6612 (Numerical Analysis PDEs) Assignment 3

Q1. Prove the pointcare's inequality

$$\|u\|_{L^2} \leq c\|u'\|_{L^2} \qquad u \in H_0^1(I)\,.$$

Q2*. Consider the 2-point BVP

$$\begin{aligned} -(p(x)u')' + q(x)u = f(x), &\quad x \in (0,1), \\ \lambda_0 u(0) - p(0)u'(0) = \nu_0, &\quad \lambda_1 u(1) + p(1)u'(1) = \nu_1, \end{aligned} \tag{1}$$

where

$$0 < p_* \leq p(x) \leq p^*, \quad 0 < q_* \leq q(x) \leq q^*, \quad x \in [0,1], \qquad \lambda_0 \geq 0, \lambda_1 \geq 0.$$

Let

$$a(u,v) = l(v), \quad v \in H^1(0,1),$$

be the weak form of (1) with

$$a(\phi,\psi) = (p\phi',\psi') + (q\phi,\psi) + \lambda_0\phi(0)\psi(0) + \lambda_1\phi(1)\psi(1),$$

and

$$l(v) = (f,v) + \lambda_0\nu_0 v(0) + \lambda_1\nu_1 v(1).$$

a) Show that there is a positive constant α, that depends only on p_* and q_*, such that

$$a(\phi,\phi) \geq \alpha\|\phi\|^2_{H^1(0,1)}, \quad \phi \in H^1(0,1), \tag{2}$$

where

$$\|\phi\|_{H^1(0,1)} = \left(\|\phi\|^2_{L^2(0,1)} + \|\phi'\|^2_{L^2(0,1)}\right)^{1/2}.$$

Show that there is a positive constant β, that depends only on p^*, q^*, λ_0, and λ_1, such that

$$a(\phi,\psi) \leq \beta\|\phi\|_{H^1(0,1)}\|\psi\|_{H^1(0,1)}, \quad \phi,\psi \in H^1(0,1).$$

Hint: To bound $|\phi(\gamma)|$ and $|\psi(\gamma)|$, $\gamma = 0,1$, use the following Sobolev inequality

$$\max_{x\in[0,1]} |v(x)| \leq \sqrt{3}\|v\|_{H^1(0,1)}, \quad v \in H^1(0,1).$$

b) Use (2) to show that the matrix A in the system of the Galerkin equations is positive definite. (This implies existence and uniqueness of the Galerkin solution.)

c) Show that if $U \in S_h \subset H^1(0,1)$ is the Galerkin approximation to the solution u of (1), then

$$\|u - U\|_{H^1(0,1)} \leq C \inf_{v\in S_h} \|u - v\|_{H^1(0,1)},$$

where C is a positive constant that depends only on p_*, p^*, q_*, q^*, λ_0, and λ_1.

d) Use c) to show that if S_h is the space of piecewise linear polynomials, that is,

$$S_h = \{v \in C[0,1] : v|_{[x_{i-1},x_i]} \in P_1, i = 1, \ldots, N+1\},$$

and the exact solution $u \in C^2[0,1]$, then

$$\|u - U\|_{H^1(0,1)} \leq Ch \max_{x\in[0,1]} |u''(x)|.$$

Hint: Use the following approximation result: If $u \in C^2[0,1]$ and $\tilde{u}$ is its piecewise linear interpolant, that is, $\tilde{u} \in S_h$ and $\tilde{u}(x_i) = u(x_i)$, $i = 0, \ldots, N+1$, then

$$\|u - \tilde{u}\|_{H^1(0,1)} \leq \sqrt{2}\, h \max_{x\in[0,1]} |u''(x)|.$$

Q3. Solve the boundary value problem in Q7 of Assignment 2 by linear FEM method with uniform meshies and $N = 10, 20, 40$, respectively. Present the accuracy (convergence rate) of the FEM solution in L_2-norm, H^1-norm and L^∞-norm.

Q4. Solve the boundary value problem

$$-u'' + 2u = x$$

with the boundary conditions (i) $u(0) = 0$, $u(1) = 0$ and $u'(0) = 0$, $u(1) = 0$, using Legendre spectral collocation method in P_3.

Q5. Solve the above problem by Hermite cubic spline collocation method with two uniform elements.

Q6. Consider the equation

$$u'' + 2u' - u = x$$

with the boundary conditions (i) $u(0) = 1$, $u(3) = 4$, OR (ii) $u'(0) = 2$, $u'(1)+3u(1) = 5$. Derive a variational formulation of the problem for each case.

Q7. Use the Legendre spectral collocation method with a polynomial of degree 4 for solving the equation

$$u'' + 4u' + 3u = 10e^{-2x}$$

with the boundary conditions $u(0) = 0$ and $u(1) = 1$. Compare your numerical results with the analytic solution.