Solutions to Assignment 1 of MA6612 (Numerical PDEs)

Q1. Define the finite difference equations

$$L_h u_j \equiv -\frac{\frac{u_{j+1}-u_j}{h_{j+1}} - \frac{u_j - u_{j-1}}{h_j}}{\frac{h_{j+1}+h_j}{2}} + p_j \frac{u_{j+1}-u_{j-1}}{h_{j+1}+h_j} + q_j u_j = f_j, \qquad j = 1, \cdots, N$$

$$u_0 = a, \qquad u_{N+1} = b,$$

where $p_j = p(x_j)$, $q_j = q(x_j)$, $f_j = f(x_j)$.

$$L_h u_j = -\frac{2 - h_{j+1}p_j}{h_{j+1}(h_j + h_{j+1})} u_{j+1} + \left(\frac{2}{h_j h_{j+1}} + q_j \right) u_j - \frac{2 + h_j p_j}{h_j (h_j + h_{j+1})} u_{j-1}.$$

Matrix-vector form

$$A\mathbf{u} = \mathbf{b},$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}, A = \begin{bmatrix} d_1 & e_1 & & & \\ c_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & e_{N-1} \\ & & & c_N & d_N \end{bmatrix}, \mathbf{b} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix} - \begin{bmatrix} c_1 a \\ 0 \\ \vdots \\ 0 \\ e_N b \end{bmatrix}$$

$$c_j = -\frac{2 + h_j p_j}{h_j(h_j + h_{j+1})}, \qquad d_j = \frac{2}{h_j h_{j+1}} + q_j, \qquad e_j = -\frac{2 - h_{j+1} p_j}{h_{j+1}(h_j + h_{j+1})}.$$

If $\max h_j = x_j - x_{j-1} < 2/p^*$, then the matrix A is strictly diagonally dominant. So the uniqueness is obtained.

Q2.

$$L_h u_j \equiv -\frac{u_{j-1} - 2u_j + u_{j+1}}{h^2} + q_j \frac{u_{j+1} - u_j}{h} = f_j, \qquad u_0 = g_0, \qquad u_{N+1} = g_1.$$

Matrix-vector form

$$A\mathbf{u} = \mathbf{b},$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}, \ A = \begin{bmatrix} a_1 & e_1 & & & \\ c_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & e_{N-1} \\ & & & c_N & a_N \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix} - \begin{bmatrix} c_1 g_0 \\ 0 \\ \vdots \\ 0 \\ e_N g_1 \end{bmatrix}$$

$$a_j = \frac{2}{h^2} - \frac{q_j}{h}, \qquad c_j = -\frac{1}{h^2}, \qquad e_j = -\frac{1}{h^2} + \frac{q_j}{h}.$$

Consider the following diagonally scale of A,

$$\tilde{A} = AD = \begin{bmatrix} a_1 d_1 & e_1 d_2 & & & \\ c_2 d_1 & a_2 d_2 & e_2 d_3 & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & e_{N-1} d_N \\ & & & c_N d_{N-1} & a_N d_N \end{bmatrix},$$

where

$$D = \text{diag}(d_1, d_2, \cdots, d_N).$$

We now require d_i to be such that $\tilde{A}$ is a strictly diagonally dominant matrix. For the new finite difference system

$$\tilde{A}\mathbf{v} = AD \cdot D^{-1}\mathbf{u} = \mathbf{b},$$

by the proof of Theorem 1.2, we have

$$\max_{1\le j\le N} |v_j| \le \tilde{K}\left\{\max_{1\le j\le N} |L_h u_j|\right\},$$

with

$$\tilde{K} = \frac{1}{\min\limits_{1\le j\le N}(|a_j d_j| - |c_j d_{j-1}| - |e_j d_{j+1}|)}.$$

Then it is easy to show that

$$\max_{1\le j\le N} |u_j| \le K\left\{\max_{1\le j\le N} |L_h u_j|\right\},$$

with

$$K = \max_{1\le j\le N} |d_j|\tilde{K}.$$

It is easy to show using Taylor's theorem that $|Lu(x_j) - L_h u(x_j)| = O(h)$. By $u_0 - u(x_0) = 0$, $u_{N+1} - u(x_{N+1}) = 0$, we have

$$\max_{0\le j\le N+1} |u_j - u(x_j)| = O(h).$$

- If $q(x) > 0$ for $x \in (0,1)$, choose $d(x) = x(2-x)$ and $d_j = d(jh) = jh(2-jh)$. Then

$$\begin{aligned}|a_j d_j| - |c_j d_{j-1}| - |e_j d_{j+1}| &= \frac{1}{h^2}\{(2-hq_j)(2jh - j^2h^2) - 2(j-1)h + (j-1)^2h^2 \\ &\quad -(1-hq_j)(2(j+1)h - (j+1)^2h^2)\} \\ &= 2 + 2q_j - (2j+1)q_j h > 2\end{aligned}$$

  Thus we have

$$K < \frac{1}{2}.$$

- If $q(x) < 0$ for $x \in (0,1)$, choose $d(x) = 1 - x^2$ and $d_j = d(jh) = 1 - (jh)^2$. Then

$$\begin{aligned}|a_j d_j| - |c_j d_{j-1}| - |e_j d_{j+1}| &= \frac{1}{h^2}\{(2-hq_j)(1 - j^2h^2) - 1 + (j-1)^2h^2 \\ &\quad -(1-hq_j)(1 - (j+1)^2h^2)\} \\ &= 2 - (2j+1)q_j h > 2\end{aligned}$$

  Thus we have

$$K < \frac{1}{2}.$$

- For general case, let

$$d(x) = \int_0^x e^{\int_0^s (q(t)-1)dt} ds$$

  and choose

$$d_j = d(jh) = \int_0^{jh} e^{\int_0^s (q(t)-1)dt} ds.$$

Obviously,

$$d'(x) = e^{\int_0^x (q(t)-1)dt} > e^{-\alpha-1}, \qquad d''(x) = (q(x)-1)d'(x).$$

Then for sufficiently small h,

$$\begin{aligned}
|a_j d_j| - |c_j d_{j-1}| - |e_j d_{j+1}| &= \frac{1}{h^2}\{(2-hq_j)d_j - d_{j-1} - (1-hq_j)d_{j+1}\} \\
&= \frac{1}{h^2}\{2d_j - d_{j-1} - d_{j+1} + hq_j(d_{j+1} - d_j)\} \\
&= \frac{1}{h^2}\left\{-h^2 d_j'' - \frac{h^4}{12}d''''(\xi_j) + h^2 q_j d_j' + \frac{h^3}{2}q_j d''(\mu_j)\right\} \\
&= \frac{1}{h^2}\left\{-h^2((q_j-1)d_j') - \frac{h^4}{12}d''''(\xi_j) + h^2 q_j d_j' + \frac{h^3}{2}q_j d''(\mu_j)\right\} \\
&= \frac{1}{h^2}\left\{h^2 d_j' - \frac{h^4}{12}d''''(\xi_j) + \frac{h^3}{2}q_j d''(\mu_j)\right\} \\
&= d_j' + O(h) > \frac{1}{2}e^{-\alpha-1},
\end{aligned}$$

where $\xi_j \in ((j-1)h, (j+1)h)$ and $\mu_j \in (jh, (j+1)h)$. Thus we have

$$K < 2e^{\alpha+1} \max_{x\in(0,1)} |d(x)|.$$

Q3. Proof. Suppose A is singular. Then there exists a nonzero vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$. Without lost of generality, we assume $\|\mathbf{v}\|_\infty = 1$. Let $|v_i| = 1$, then $|v_j| \le |v_i| = 1,\ j = 1, 2, \cdots, n$. By $A\mathbf{v} = \mathbf{0}$, we have

$$|a_{ii}| \le \sum_{j=1, j\ne i}^{n} |a_{ij}||v_j| \le \sum_{j=1, j\ne i}^{n} |a_{ij}|,$$

which contradicts the strictly diagonally dominant condition.

Q4. Let

$$L = \begin{bmatrix} 1 & & & & \\ -1 & 1 & & & \\ & \ddots & \ddots & & \\ & & -1 & 1 & \\ & & & -1 & 1 \end{bmatrix}.$$

Then

$$A = \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 0 & & & \\ 0 & 0 & 0 & & \\ & \ddots & \ddots & \ddots & \\ & & 0 & 0 & 0 \\ & & & 0 & 0 \end{bmatrix} = LL^T + D_0$$

$$L^{-T} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & 1 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad L^{-1}L^{-T} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 3 & \cdots & 3 \\ 1 & 2 & 3 & \cdots & 4 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 1 & 2 & 3 & \cdots & n \end{bmatrix}$$

By

$$\|L^{-T}\|_2^2 = \|L^{-1}L^{-T}\|_2 = \lambda_{\max}(L^{-1}L^{-T}) \le \operatorname{trace}(L^{-1}L^{-T}) = \frac{(n+1)n}{2} \le n^2,$$

$$x^T A x = x^T L L^T x + x^T D_0 x = (L^T x)^T (L^T x) + x_1^2 \ge \|L^T x\|_2^2$$

and

$$x^T x = \|x\|_2^2 = \|L^{-T} L^T x\|_2^2 \le \|L^{-T}\|_2^2 \|L^T x\|_2^2,$$

we have

$$x^T x \le n^2 x^T A x.$$

Q5.(a)

$$\frac{a_{j+1}+a_j}{2} = a_j + \frac{h}{2}a_j' + \frac{h^2}{4}a_j'' + \frac{h^3}{12}a'''(\xi_{1j}), \qquad \xi_{1j} \in (x_j, x_{j+1}),$$

$$\frac{a_j+a_{j-1}}{2} = a_j - \frac{h}{2}a_j' + \frac{h^2}{4}a_j'' - \frac{h^3}{12}a'''(\xi_{2j}), \qquad \xi_{2j} \in (x_{j-1}, x_j),$$

$$\frac{u(x_{j+1})-u(x_j)}{h} = u'(x_j) + \frac{h}{2}u''(x_j) + \frac{h^2}{6}u'''(x_j) + \frac{h^3}{24}u''''(\xi_{3j}), \qquad \xi_{3j} \in (x_j, x_{j+1}),$$

$$\frac{u(x_j)-u(x_{j-1})}{h} = u'(x_j) - \frac{h}{2}u''(x_j) + \frac{h^2}{6}u'''(x_j) - \frac{h^3}{24}u''''(\xi_{4j}), \qquad \xi_{4j} \in (x_{j-1}, x_j).$$

By

$$-\frac{1}{h}\left\{\frac{a_{j+1}+a_j}{2}\frac{u(x_{j+1})-u(x_j)}{h} - \frac{a_j+a_{j-1}}{2}\frac{u(x_j)-u(x_{j-1})}{h}\right\} + (au')'$$

$$= -\frac{h^2}{24}a_j\left(u''''(\xi_{3j}) + u''''(\xi_{4j})\right) - \frac{h^2}{6}a_j'u'''(x_j) - \frac{h^2}{4}a_j''u''(x_j) - \frac{h^2}{12}u'(x_j)\left(a'''(\xi_{1j}) + a'''(\xi_{2j})\right) + O(h^3),$$

thus

$$\max_{1\le j\le N}|L_h u(x_j) - f_j| = \max_{1\le j\le N}|L_h u(x_j) - Lu(x_j)| \le C_1 h^2.$$

(b)-(d)

$$L_h u_j = -\frac{a_{j-1}+a_j}{2h^2}u_{j-1} + \left(\frac{a_{j-1}+2a_j+a_{j+1}}{2h^2} + q_j\right)u_j - \frac{a_j+a_{j+1}}{2h^2}u_{j+1}.$$

Matrix-vector form

$$A\mathbf{u} = \mathbf{b},$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}, A = \begin{bmatrix} d_1 & e_1 & & & \\ c_2 & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & e_{N-1} \\ & & & c_N & d_N \end{bmatrix}, \mathbf{b} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix} - \begin{bmatrix} c_1 g_0 \\ 0 \\ \vdots \\ 0 \\ e_N g_1 \end{bmatrix}$$

$$c_j = -\frac{a_{j-1}+a_j}{2h^2}, \qquad d_j = \frac{a_{j-1}+2a_j+a_{j+1}}{2h^2} + q_j, \qquad e_j = -\frac{a_j+a_{j+1}}{2h^2}.$$

The matrix A is strictly diagonally dominant with no restriction on h. Then follow the lecture notes to show the stability. Obviously, $d_j > 0$ and $c_j = e_{j-1}$, A is symmetric.

Q6.(a) For $1 \le j \le N$,

$$L_h u(x_j) - f_j = L_h u(x_j) + u''(x_j) - qu(x_j) = -\frac{h^2}{12}u''''(\xi_j), \qquad \xi_j \in (x_{j-1}, x_{j+1}).$$

For $j = 0$,

$$\begin{aligned} L_h u(x_j) - f_j &= (1+\frac{hq}{2})u(0) - \frac{u(h)-u(0)}{h} - \alpha - \frac{h}{2}f(0) \\ &= (1+\frac{hq}{2})u(0) - u'(0) - \frac{h}{2}u''(0) - \frac{h^2}{6}u'''(\xi_0) - \alpha - \frac{h}{2}f(0) \\ &= (1+\frac{hq}{2})u(0) - (-\alpha + u(0)) - \frac{h}{2}(qu(0) - f(0)) - \frac{h^2}{6}u'''(\xi_0) - \alpha - \frac{h}{2}f(0) \\ &= -\frac{h^2}{6}u'''(\xi_0), \qquad \xi_0 \in (0, h). \end{aligned}$$

For $j = N+1$,

$$\begin{aligned} L_h u(x_j) - f_j &= (1+\frac{hq}{2})u(1) + \frac{u(1)-u(1-h)}{h} - \beta - \frac{h}{2}f(1) \\ &= (1+\frac{hq}{2})u(1) + u'(1) - \frac{h}{2}u''(1) + \frac{h^2}{6}u'''(\xi_{N+1}) - \beta - \frac{h}{2}f(1) \\ &= (1+\frac{hq}{2})u(1) + (\beta - u(1)) - \frac{h}{2}(qu(1) - f(1)) + \frac{h^2}{6}u'''(\xi_{N+1}) - \beta - \frac{h}{2}f(1) \\ &= \frac{h^2}{6}u'''(\xi_{N+1}), \qquad \xi_{N+1} \in (1-h, 1). \end{aligned}$$

Thus,

$$\max_{0 \le j \le N+1} |L_h u(x_j) - f_j| = O(h^2).$$

(b)-(d)
Matrix-vector form

$$A\mathbf{u} = \mathbf{b},$$

where

$$\mathbf{u} = \begin{bmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \\ u_{N+1} \end{bmatrix}, A = \begin{bmatrix} d_0 & c_0 & & & & \\ c & d & c & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & d & c \\ & & & & c_{N+1} & d_{N+1} \end{bmatrix}, \mathbf{b} = \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \\ f_{N+1} \end{bmatrix},$$

$$d_0 = 1 + \frac{hq}{2} + \frac{1}{h},\ c_0 = -\frac{1}{h},\ c = -\frac{1}{h^2},\ d = \frac{2}{h^2} + q,\ c_{N+1} = -\frac{1}{h},\ d_{N+1} = 1 + \frac{hq}{2} + \frac{1}{h}.$$

The matrix A is strictly diagonally dominant with no restriction on h. Then follow the lecture notes to show the stability. Let

$$\tilde{A} = \begin{bmatrix} \frac{1}{h}d_0 & \frac{1}{h}c_0 & & & & \\ c & d & c & & & \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ & & & c & d & c \\ & & & & \frac{1}{h}c_{N+1} & \frac{1}{h}d_{N+1} \end{bmatrix},$$

then $\tilde{A}$ is symmetric.

Q8. Let

$$B = I_n - \frac{1}{2}A = \begin{bmatrix} 0 & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & \frac{1}{2} & & \\ & \ddots & \ddots & \ddots & \\ & & \frac{1}{2} & 0 & \frac{1}{2} \\ & & & \frac{1}{2} & 0 \end{bmatrix}.$$

By Gershgorin circle theorem, all the eigenvalues of B,

$$|\lambda| \leq 1.$$

Since the matrix B is symmetric, then all the eigenvalues of B are real. Obviously $\det(I_n - B) \neq 0$ (it suffices to prove that the system $(I_n - B)v = 0$ has only the zero solution) and $\det(-I_n - B) \neq 0$, i.e., ± 1 are not eigenvalues. The spectral radius of B is less than 1, $\rho(B) < 1$, so we have

$$(I_n - B)^{-1} = \sum_{i=0}^{\infty} B^i.$$

Because $B^i, i = 0, 1, \cdots, \infty$ are nonnegative, therefore

$$A^{-1} = \frac{1}{2}(I_n - B)^{-1} = \frac{1}{2}\sum_{i=0}^{\infty} B^i$$

is nonnegative.