Solution of Assignment 2

Q2. Solve the following minimization problem

$$\min_{u\in M_4^0} F(u) = \frac{1}{2}\int_0^1 u'^2 dx - \int_0^1 u dx,$$

where

$$M_4^0 = \{u|u \text{ polynomial of degree } \leq 4 \ \text{ and } \ u(0) = u(1) = 0\}.$$

Solution Since $u(0) = u(1) = 0$, for any $u \in M_4^0$, there exist constants, a, b, c,

$$u = x(x-1)(ax^2bxc)$$

Substituting $u = x(x-1)(ax^2 + bx + c)$ into the minimization function $F(u)$, we have

$$f(a,b,c) := F(u) = \frac{3}{70}a^2 + \frac{1}{10}ab + \frac{1}{10}ac + \frac{1}{20}a + \frac{1}{15}b^2 + \frac{1}{12}b + \frac{1}{6}bc + \frac{1}{6}c + \frac{1}{6}c^2.$$

The critical point of the function $f(a, b, c)$ satisfy the following equations

$$\begin{aligned}
\frac{\partial f}{\partial a} &= \frac{3}{25}a_0 + \frac{1}{10}b_0 + \frac{1}{10}c_0 + \frac{1}{20} = 0,\\
\frac{\partial f}{\partial b} &= \frac{1}{10}a_0 + \frac{2}{15}b_0 + \frac{1}{6}c_0 + \frac{1}{12} = 0,\\
\frac{\partial f}{\partial b} &= \frac{1}{10}a_0 + \frac{1}{6}b_0 + \frac{1}{3}c_0 + \frac{1}{6} = 0.
\end{aligned}$$

Solving this linear system gives

$$a_0 = 0, \quad b_0 = 0, \quad c_0 = \frac{1}{2}.$$

Therefore, the solution is $u = x(x-1)/2$. □

Q4. Show that the following problem

$$\begin{cases} -u'' = f(x), & x \in (0,1) \\ u(0) = u'(1) = 0 \end{cases}$$

is equivalent to the following variational formulation: to find $u \in V$ such that for any $v \in V$,

$$(u', v') = (f, v),$$

where $V = \{v \in H^1(0,1)|v(0) = 0\}$. (i). Formulate a finite element method for this problem by using piecewise linear functions. (ii). Determine the corresponding linear system of equations in the case of a uniform partition.

Solution (I) First, if u satisfies the differential problem, then by multiplying v in both sides and integrating it from 0 to 1, we have

$$\int_0^1 -u''v dx = \int_0^1 f v dx,$$

Integration by part yields that

$$\int_0^1 -u''v dx = -u'v\Big|_0^1 + \int_0^1 u'v' dx = \int_0^1 u'v' dx.$$

The last equation is obtained by using $v(0) = u'(1) = 0$ $(v \in V)$. Therefore u satisfies the variational problem.

(II) (More complicated part) If u satisfies the variational problem and $u \in H^2(0,1) \hookrightarrow C^1[0,1]$, then

$$\int_0^1 -u''v dx = -u'v\Big|_0^1 + \int_0^1 u'v' dx = -u'(1)v(1) + \int_0^1 fv dx,$$

which leads to

$$\int_0^1 (u'' + f)v dx = u'(1)v(1),$$

for all $v \in V$. Choose

$$v_n = \begin{cases} 0 & 0 \leq x \leq 1 - 1/n \\ nx - n + 1, & 1 - 1/n \leq x \leq 1 \end{cases} \in V.$$

We have

$$\int_0^1 (u'' + f)v_n dx = \int_{1-1/n}^1 (u'' + f)v_n dx \to 0$$

and $u'(1)v_n(1) = u'(1)$. Letting $n \to \infty$, we have $u'(1) = 0$ and

$$\int_0^1 (u'' + f)v dx = 0$$

Following the proof in classroom, we complete the proof. ∎

(i). Let $\Pi : 0 = x_0 < x_1 < \cdots < x_{n-1} < x_n = 1$ be a partition of the interval $[0,1]$, and $\phi_j(x)$ be the piecewise linear functions which are given by

$$\phi_j(x) = \begin{cases} (x - x_{j-1})/h, & x_{j-1} \leq x \leq x_j, \\ (x_j - x)/h, & x_j \leq x \leq x_{j+1}, \end{cases} \quad \text{for} \quad 1 \leq j < n,$$

$$\phi_n(x) = (x - x_{n-1})/h, \qquad x_{n-1} \leq x \leq x_n.$$

Let the coefficient matrix $A = (a_{ij})$, where $a_{ij} = (\phi_j', \phi_i')$ and $b_i = (f, \phi_i)$. Then by calculation

$$A = \frac{1}{h}\begin{bmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & & \\ & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 1 \end{bmatrix}_{n\times n}.$$

The finite element linear system can be written in matrix form: $A\,\alpha = b$. □

Q5. Consider the following problem

$$\begin{cases} -u''''(x) + q(x)u(x) = f(x), & x \in (0,1) \\ u(0) = u(1) = u'(0) = u'(1) = 0. \end{cases}$$

(i) Present an equivalent variational problem. (ii) Can we use linear element method t solve this problem? (iii) Use a quadratic FEM and write down the coefficient matrix of the linear system of equations when $q = 0$.

Solution (i). Multiplying both sides by $v \in V = \{v \in H^2(0,1) | v(0) = v(1) = v'(0) = v'(1) = 0\}$ and integrating it from 0 to 1,

$$\int_0^1 u''''v dx = u'''v\Big|_0^1 dx - \int_0^1 u'''v'dx = -u''v'\Big|_0^1 + \int_0^1 u''v''dx = \int_0^1 u''v''dx.$$

The variational model is: to find $u \in V$ such that for any $v \in V$,

$$a(u,v) = (f,v)$$

where

$$a(u,v) = \int_0^1 u''v''dx + \int_0^1 quvdx\,.$$

(ii). The piecewise linear element method cannot be used for this problem, because the double derivatives in the variational model vanish for linear functions.

(iii). Let Π be a uniform partition $0 = x_0 < x_1 < \cdots < x_{2n-1} < x_{2n} = 1$, V_h^2 the space of continuous functions which are quadratic over each interval $[x_{2j-1}, x_{2j+1}]$. Let $h = 1/2n$, then the basis functions of V_h^2 are listed as follows,

$$\phi_{2j} = \begin{cases} -\dfrac{(x - x_{2j-1})(x - x_{2j+1})}{h^2}, & x_{2j-1} \le x \le x_{2j+1}, \\ 0, & \text{otherwise}, \\ & 1 \le j \le n-1, \end{cases}$$

$$\phi_{2j-1} = \begin{cases} \dfrac{(x - x_{2j-3})(x - x_{2j-2})}{2h^2}, & x_{2j-3} \le x \le x_{2j-1}, \\ \dfrac{(x - x_{2j})(x - x_{2j+1})}{2h^2}, & x_{2j-1} \le x \le x_{2j+1}, \\ 0, & \text{otherwise}, \\ & 2 \le j \le n-1, \end{cases}$$

$$\phi_1 = \begin{cases} \dfrac{x^2}{h^2}, & 0 \le x \le x_1, \\ \dfrac{(x - x_2)(x - x_3)}{2h^2}, & x_1 \le x \le x_3, \\ 0, & \text{otherwise}, \end{cases}$$

$$\phi_{2n} = \begin{cases} \dfrac{(x - x_{2n-3})(x - x_{2n-2})}{2h^2}, & x_{2n-3} \le x \le x_{2n-1}, \\ \dfrac{(1-x)^2}{h^2}, & x_{2n-1} \le x \le x_{2n}, \\ 0, & \text{otherwise}, \end{cases}$$

When $q = 0$, by direct calculation we obtain the coefficient matrix A, defined by $A_{ij} = (\phi_i'', \phi_j'')$,

$$A_{2i,2j} = \begin{cases} \dfrac{8}{h^3}, & j = i, \\ 0, & j \neq i, \end{cases} \qquad A_{2i,2j-1} = \begin{cases} -\dfrac{4}{h^3}, & j - i = 0, 1, \\ 0, & j - i \neq 0, 1, \end{cases}$$

$$A_{2i-1,2j-1} = \begin{cases} \dfrac{4 + 2\delta_{i1} + 2\delta_{in}}{h^3}, & j = i, \\ \dfrac{2}{h^3}, & |j - i| = 1, \\ 0, & \text{otherwise.} \end{cases}$$

In matrix form,

$$A = \frac{1}{h^3} \begin{bmatrix} 6 & -4 & 2 & & & & & & & & \\ -4 & 8 & -4 & 0 & & & & & & & \\ 2 & -4 & 4 & -4 & 2 & & & & & & \\ & 0 & -4 & 8 & -4 & 0 & & & & & \\ & & 2 & -4 & 4 & -4 & 2 & & & & \\ & & & \ddots & \ddots & \ddots & \ddots & \ddots & & & \\ & & & & 0 & -4 & 8 & -4 & 0 & & \\ & & & & & 2 & -4 & 4 & -4 & 2 \\ & & & & & & 0 & -4 & 8 & -4 \\ & & & & & & & 2 & -4 & 6 \end{bmatrix}_{(2n-1)\times(2n-1)} .$$

□

Q6. Let P_3 be a vector space which consists of all cubic polynomials. Find the basis functions $\{\phi_j\}_{j=1}^4$ such that for any $v(x) \in P_3$,

$$v(x) = v(0)\phi_1(x) + v'(0)\phi_2(x) + v(1)\phi_3(x) + v'(1)\phi_4(x).$$

Solution Let

$$\phi_1(x) = (2x + 1)(x - 1)^2, \quad \phi_2(x) = x(x - 1)^2, \quad \phi_3(x) = x^2(3 - 2x), \quad \phi_4(x) = x^2(x - 1).$$

It is easy to verify the following identities,

$$\begin{aligned} \phi_1(0) = 1, &\quad \phi_1'(0) = \phi_1(1) = \phi_1'(1) = 0, \\ \phi_2'(0) = 1, &\quad \phi_2(0) = \phi_2(1) = \phi_2'(1) = 0, \\ \phi_3(1) = 1, &\quad \phi_3(0) = \phi_3'(0) = \phi_3'(1) = 0, \\ \phi_4'(1) = 1, &\quad \phi_4(0) = \phi_4'(0) = \phi_4(1) = 0. \end{aligned}$$

Therefore, every $v \in P_3$ can be represented by

$$v(x) = v(0)\phi_1(x) + v'(0)\phi_2(x) + v(1)\phi_3(x) + v'(1)\phi_4(x).$$

□

Q8. Find the approximate solution of the following problem

$$\begin{cases} -u'' + 9u = -9\cosh\left(\dfrac{3}{2}\right), & x \in (0,1) \\ u(0) = u(1) = 0, \end{cases}$$

in the finite dimensional space

$$V_N = \{u \,|\, u = \sum_{j=1}^{N} \alpha_j \sin j\pi x, \quad \alpha_j \in \mathbb{R}\}.$$

Solution Let $\phi_j(x) = \sin j\pi x, \quad j = 1, \cdots, n$. Easy to formulate the variational problem

$$\int_0^1 u'v'dx + 9\int_0^1 uvdx = -9\int_0^1 \cosh\left(\frac{3}{2}\right)vdx, \qquad \forall v \in H_0^1(0,1).$$

i.e.,

$$a(u,v) = (-9\cosh\left(\frac{3}{2}\right), v), \qquad \forall v \in H_0^1(0,1).$$

Denote the left hand side of above equation of $a(u,v)$, insert $v = \phi_i(x)$ into the equation, then

$$\sum_{j=1}^{N} a(\phi_j', \phi_i')\alpha_j = b_i, \qquad i = 1, \cdots, n.$$

where by calculation, the coefficients take the value

$$\begin{aligned} a(\phi_j', \phi_i') = &\pi\, ij\left(\frac{\pi\delta_{ij}}{2} + \frac{1}{i+j}\sin\frac{i+j}{2}\pi + \frac{1}{\delta_{ij} + (i-j)}\sin\frac{i-j}{2}\pi\right) \\ &+ \frac{9}{\pi}\left(\frac{\pi\delta_{ij}}{2} - \frac{1}{i+j}\sin\frac{i+j}{2}\pi + \frac{1}{\delta_{ij} + (i-j)}\sin\frac{i-j}{2}\pi\right) \end{aligned}$$

and

$$b_i = -9\int_0^1 \cosh\left(\frac{3}{2}\right)\sin i\pi x dx = \frac{9}{i\pi}\left[(-1)^i - 1\right]\cosh\left(\frac{3}{2}\right).$$

For $N = 3$, as an example, the linear system of equations can be written in matrix form,

$$A\alpha = b, \tag{0.1}$$

where

$$A = \begin{bmatrix} \dfrac{\pi^2+9}{2} & \dfrac{4\pi^2+36}{3\pi} & 0 \\ \dfrac{4\pi^2+36}{3\pi} & \dfrac{4\pi^2+9}{2} & \dfrac{36(\pi^2+1)}{5\pi} \\ 0 & \dfrac{36(\pi^2+1)}{5\pi} & \dfrac{9(\pi^2+1)}{2} \end{bmatrix},$$

$$b = \begin{bmatrix} -\dfrac{18}{\pi}\cosh\left(\dfrac{3}{2}\right) \\ 0 \\ -\dfrac{6}{\pi}\cosh\left(\dfrac{3}{2}\right) \end{bmatrix},$$

by calculation,

$$\alpha = \cosh\left(\frac{3}{2}\right)\begin{bmatrix} -\dfrac{36(900\pi^4 - 2839\pi^2 - 2304)}{\pi(\pi^2+9)(900\pi^4 - 4759\pi^2 - 19584)} \\ \dfrac{25920}{900\pi^4 - 4759\pi^2 - 19584} \\ -\dfrac{20}{3}\dfrac{180\pi^4 + 5269\pi^2 + 2304}{\pi(\pi^2+1)(900\pi^4 - 4759\pi^2 - 19584)} \end{bmatrix}.$$