Solution of Assignment 3

Q1. Prove that there exists a constant $C > 0$ satisfying the following Pointcare's inequality,

$$\|u\|_{L^2} \leq C\|u'\|_{L^2}, \quad \forall\, u \in H_0^1(0,1).$$

Solution For $u \in H_0^1(0,1)$, there is $u = \int_0^x u'(t)dt$, hence we have following estimation

$$\begin{aligned}
\int_0^1 u^2 dx &\leq \int_0^1 \left(\int_0^x u'(t)dt \right)^2 dx \\
&\leq \int_0^1 \left(\int_0^x 1^2 dt \int_0^x |u'(t)|^2 dt \right) dx \\
&\leq \frac{1}{2} \int_0^1 |u'|^2 dx.
\end{aligned}$$

Therefore,

$$\|u\|_{L^2} \leq \frac{1}{\sqrt{2}} \|u'\|_{L^2}.$$

□

Q4. Solve the boundary value problem

$$-u'' + 2u = x$$

with the boundary conditions (i) $u(0) = 0,\ u(1) = 0$ and (ii) $u'(0) = 0,\ u(1) = 0$, using Legendre spectral collocation method in P_3.

Solution (i). Let $u = x(x-1)(ax+b)$, then

$$-u'' + 2u = (2x^3 - 2x^2 - 6x + 2)a + (2x^2 - 2x - 2)b.$$

Collocating the equation at the zeros of Legendre polynomial of degree 2 ($x^{\pm} = 1/2 \pm 1/2\sqrt{3}$) leads to

$$\begin{aligned}
\left(-\frac{7}{6} + \frac{19}{18}\sqrt{3} \right) a - \frac{7}{3} b = \frac{1}{2} - \frac{1}{6}\sqrt{3}, \\
\left(-\frac{7}{6} - \frac{19}{18}\sqrt{3} \right) a - \frac{7}{3} b = \frac{1}{2} + \frac{1}{6}\sqrt{3}.
\end{aligned}$$

The solution to above linear equations is

$$a = -\frac{3}{19}, \qquad b = -\frac{18}{133}.$$

(ii). The functions $u \in P_3$ which satisfies $u'(0) = u(1) = 0$ can be represented by $u = (ax^2 + bx + b)(x-1)$. For such $u \in P_3$,

$$-u'' + 2u = (2x^3 - 2x^2 - 6x + 2)a + (2x^2 - 4)b.$$

Collocating the equation at the zeros of Legendre polynomial of degree 2 leads to

$$\begin{aligned}
\left(-\frac{7}{6}+\frac{19}{18}\sqrt{3}\right)a-\left(\frac{10}{3}+\frac{\sqrt{3}}{3}\right)b=\frac{1}{2}-\frac{1}{6}\sqrt{3},\\
\left(-\frac{7}{6}-\frac{19}{18}\sqrt{3}\right)a-\left(\frac{10}{3}-\frac{\sqrt{3}}{3}\right)b=\frac{1}{2}+\frac{1}{6}\sqrt{3}.
\end{aligned}$$

The solution to above linear equations are

$$a=-\frac{39}{211},\qquad b=-\frac{18}{211}.$$

□

Q5. Solve the above problem by Hermite cubic spline collocation method with two uniform elements.

Solution For each small interval $[x_{j-1}, x_j]$, let $x = x_{j-1} + th$, $0 \le t \le 1$, then on this interval, the cubic spline interpolation function is

$$u_h(x)=u_h(x_{j-1}+th)=u_{j-1}(1+2t)(1-t)^2+u'_{j-1}ht(1-t)^2+u_j(3-2t)t^2+u'_jht^2(t-1).$$

Hence,

$$u''_h(x)=\frac{1}{h^2}\Big[(-6+12t)u_{j-1}+(-4+6t)hu'_{j-1}+(6-12t)u_j+(-2+6t)hu'_j\Big].$$

Collocation at $t=(1\pm1/\sqrt{3})/2$ (zeros of Legendre polynomial of degree 2) yields a linear system of equations, of $2N+2$ unkown variables (when including the two boundary conditions into the equations). For $N = 2$, i.e. a two-piece Hermite cubic spline collocation scheme, we have

$$\begin{aligned}
\left(\frac{76}{9}\sqrt{3}+1\right)u_0+\left(\frac{75+73\sqrt{3}}{36}\right)u'_0+\left(-\frac{76}{9}\sqrt{3}+1\right)u_{\frac{1}{2}}+\left(\frac{-75+73\sqrt{3}}{36}\right)u'_{\frac{1}{2}}&=\frac{1}{4}\left(1-\frac{1}{\sqrt{3}}\right),\\
\left(-\frac{76}{9}\sqrt{3}+1\right)u_0+\left(\frac{75-73\sqrt{3}}{36}\right)u'_0+\left(\frac{76}{9}\sqrt{3}+1\right)u_{\frac{1}{2}}+\left(\frac{-75-73\sqrt{3}}{36}\right)u'_{\frac{1}{2}}&=\frac{1}{4}\left(1+\frac{1}{\sqrt{3}}\right)\\
\left(\frac{76}{9}\sqrt{3}+1\right)u_{\frac{1}{2}}+\left(\frac{75+73\sqrt{3}}{36}\right)u'_{\frac{1}{2}}+\left(-\frac{76}{9}\sqrt{3}+1\right)u_1+\left(\frac{-75+73\sqrt{3}}{36}\right)u'_1&=\frac{1}{4}\left(3-\frac{1}{\sqrt{3}}\right)\\
\left(-\frac{76}{9}\sqrt{3}+1\right)u_0+\left(\frac{75-73\sqrt{3}}{36}\right)u'_0+\left(\frac{76}{9}\sqrt{3}+1\right)u_{\frac{1}{2}}+\left(\frac{-75-73\sqrt{3}}{36}\right)u'_{\frac{1}{2}}&=\frac{1}{4}\left(3+\frac{1}{\sqrt{3}}\right)\\
u_0=0,\quad u_1=0,\qquad \big(u'_0=0,\quad u_1=0\quad &\text{for boundary condition (ii)}\big).
\end{aligned}$$

By solving above linear system of equations, we obtain

$$u_0=0,\ \ u'_0=\frac{520707}{3867175},\ \ u_{\frac{1}{2}}=\frac{219}{4238},\ \ u'_{\frac{1}{2}}=\frac{72}{1825},\ \ u_1=0,\ \ u'_1=-\frac{1143693}{3867175}.$$

$$\left(\begin{array}{l}\text{For boundary condition (ii), we can solve}\\ u_0=\dfrac{520707}{6154561},\ \ u'_0=0,\ \ u_{\frac{1}{2}}=\dfrac{524577}{6154561},\ \ u'_{\frac{1}{2}}=-\dfrac{236811}{6154561},\ \ u_1=0,\ \ u'_1=-\dfrac{2200656}{6154561}.\end{array}\right)$$

□

Q6. Consider the equation $u'' + 2u' - u = x$ with the boundary conditions (i) $u(0) = 1$, $u(3) = 4$, or (ii) $u'(0) = 2$, $u'(1) + 3u(1) = 5$. Derive a variational formulation of the problem for each case.

Solution ((i). Let $v \in H_0^1(0,3)$. Multiplying v on both sides of the equation and integrating it over $[0,3]$ give

$$\int_0^3 (u'' + 2u' - u)v dx = \int_0^1 xv dx$$

Using integration by part,

$$-\int_0^3 (u'v' + 2u'v - uv)dx = \int_0^1 xv dx\,.$$

The variation model is to find $u \in V = \{u \in H^1(0,3),\, u(0) = 1, u(3) = 4\}$ such that for any $v \in H_0^1(0,3)$,

$$a(u,v) = (x,v)$$

where

$$a(u,v) = -\int_0^3 (u'v' + 2u'v - uv)dx\,.$$

(ii) Similarly, for $v \in H^1(0,1)$,

$$u'(1)v(1) - u'(0)v(0) - \int_0^1 (u'v' + 2u'v - uv)dx = \int_0^1 xv dx$$

and moreover,

$$(5 - 3u(1))v(1) - 2v(0) - \int_0^1 (u'v' + 2u'v - uv)dx = \int_0^1 xv dx$$

The variation model is to find u such that such that for any $v \in H^1(0,1)$,

$$a(u,v) = (x,v)$$

where

$$a(u,v) = (5 - 3u(1))v(1) - 2v(0) - \int_0^1 (u'v' + 2u'v - uv)dx\,.$$

Q7. Use the Legendre spectral collocation method with a polynomial of degree 4 to solve the equation

$$u'' + 4u' + 3u = 10e^{-2x},$$

with the boundary conditions $u(0) = 0$ and $u(1) = 1$. Compare your numerical results with the analytic solution.

Solution Let $u = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, then $u' = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3$, $u'' = 2a_2 + 6a_3x + 12a_4x^2$. The zeros of Legendre polynomial of degree 3 are

$$x_1 = \frac{1}{2}\left(1 - \sqrt{\frac{3}{5}}\right), \quad x_2 = \frac{1}{2}, \quad x_3 = \frac{1}{2}\left(1 + \sqrt{\frac{3}{5}}\right).$$

Collocating the equation at these points yields the following linear system of equations (together with boundary conditions),

$$3a_0+(4+3x_j)a_1+(2+8x_j+3x_j^2)a_2+(6x_j+12x_j^2+3x_j^3)a_3+(12x_j^2+16x_j^3+3x_j^4)a_4 = 10e^{-2x_j}, \quad j = 1,2,3,$$

and

$$a_0 = 0 \quad a_0 + a_1 + a_2 + a_3 + a_4 = 1.$$

By solving above linear system of equations, we obtain

$$a_0 = 0, \;\; a_1 = 1.6976, \;\; a_2 = 0.9838, \;\; a_3 = -3.0117, \;\; a_4 = 1.3303. \tag{0.1}$$

Comparing the approximate solution $u_{app} = \sum_{j=0}^{4} a_j x^j$ with the analytic solution of the differential equation $u_{anl} = 5.8332e^{-x} + 4.1668e^{-3x} - 10e^{-2x}$, the maximal error between $[0, 1]$ is 0.0168.