Solution to Midtest of MA6612 (Numerical PDEs)

Q1.(i)

$$L_h u_j \equiv -\frac{u_{j-1}-2u_j+u_{j+1}}{h^2} + 10\frac{u_{j+1}-u_{j-1}}{2h} + qu_j = f_j, \qquad u_0 = 0, \qquad u_{N+1} = 1.$$

(ii) Proof of the uniqueness: Matrix-vector form

$$A\mathbf{u} = \mathbf{b},$$

where

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_{N-1} \\ u_N \end{bmatrix}, \; A = \begin{bmatrix} a & e & & & \\ c & \ddots & \ddots & & \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & e \\ & & & c & a \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_{N-1} \\ f_N \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ e \end{bmatrix}$$

$$a = \frac{2}{h^2} + q, \qquad c = -\frac{1+5h}{h^2}, \qquad e = -\frac{1-5h}{h^2}.$$

Suppose $\mathbf{b} = 0$, we only need to show that $\mathbf{u} = 0$. Let

$$|u_{j_*}| = \max_{1 \le j \le N} |u_j|,$$

then

$$|au_{j_*}| \le |cu_{j_*-1}| + |eu_{j_*+1}| \le |c+e||u_{j_*}|,$$

i.e.,

$$q|u_{j_*}| = 0.$$

If $q = 0$, it is easy to show that

$$|u_{j_*}| = |u_{j_*-1}| = |u_{j_*+1}|.$$

Thus we have

$$|u_1| = |u_2| = \cdots = |u_N|.$$

By $au_1 + eu_2 = 0$ and $|a| \neq |e|$, we know $u_1 = u_2 = 0$. Thus $\mathbf{u} = 0$. If $q > 0$, it is obviously A is strictly diagonally dominant.

(iii) $Lu(x_j) - L_h u(x_j) = \frac{1}{12}u''''(\xi_j)h^2 - \frac{5}{3}u'''(\eta_j)h^2 = O(h^2), \qquad \xi_j, \eta_j \in (x_{j-1}, x_{j+1}).$

(iv) Following the lecture notes (theorem 1.2), the linear FD operator L_h is stable and therefore, $\max |u(x_j) - u_j| \le O(h^2)$.

Q2.

$$L_h u_j \equiv -\frac{u_{j-1}-2u_j+u_{j+1}}{h^2} + u_j = f_j,$$

$$u_{N+1} = 0.$$

$$\frac{u_1 - u_0}{h} - \frac{h}{2}(u_0 - f(0)) = 0\,.$$

The truncation error is

$$|L_h u(x_j) - Lu(x_j)| = \frac{1}{12}|u''''(\xi_j)|h^2 = O(h^2), \qquad 1 \le j \le N.$$

$$\frac{u(h)-u(0)}{h} - \frac{h}{2}(u(0) - f(0)) - u'(0) = \frac{1}{6}u'''(\xi)h^2 = O(h^2).$$

Q3. Variational model.

$$\int_0^1 (-u'' + pu' + qu)v dx = \int_0^1 fv dx, \qquad \forall v \in H_0^1(I), \qquad I = (0,1).$$

$$\Rightarrow \int_0^1 (u'v' + pu'v + quv) dx = \int_0^1 fv dx.$$

Find $u \in H_0^1(I)$ s.t.,

$$a(u,v) = (f,v), \qquad \forall v \in H_0^1(I),$$

where

$$a(u,v) = \int_0^1 (u'v' + pu'v + quv) dx, \qquad (f,v) = \int_0^1 fv dx.$$

By a linear finite element method with three uniform elements, we have the FEM system

$$\left(3\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} \frac{2}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{3} \end{bmatrix}\right)\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{1}{3} \end{pmatrix}.$$

$$\phi_i = \begin{cases} \frac{x - x_{i-1}}{h}, \ x_{i-1} \le x \le x_i \\ \frac{x_{i+1} - x}{h}, \ x_i \le x \le x_{i+1} \\ 0, \ \text{otherwise} \end{cases}$$

Solving the system gives $u = 0.0938(\phi_1 + \phi_2)$.

Q4.

$$V_2 = \{u | \text{ polynomials of degree} \le 3, u(0) = u(1) = 0\}$$

Thus,

$$V_2 = \text{span}\{\phi_1 = x(x-1), \ \phi_2 = x^2(x-1)\}.$$

Choose $v = \phi_1, \phi_2$, respectively. We have

$$\int_0^1 (\alpha_1\phi_1 + \alpha_2\phi_2)'\phi_i' dxi = \int_0^1 2\phi_i dx$$

or $Av = b$, where

$$a_{11} = \int_0^1 (x - 1 + x)^2 dx = \frac{1}{3}, \quad a_{12} = a_{21} = \int_0^1 (2x-1)(3x^2 - 2x) dx = \frac{1}{6}$$

$$a_{22} = \int_0^1 (3x^2 - 2x)^2 dx = \frac{2}{15}, \quad b_1 = -1/3, \quad b_2 = -1/6\,.$$

The system becomes

$$\begin{bmatrix} \frac{1}{3} & \frac{1}{6} \\ \frac{1}{6} & \frac{2}{15} \end{bmatrix}\begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{6} \end{bmatrix} \Rightarrow \alpha_1 = -1, \ \alpha_2 = 0.$$

Therefore

$$u = x(1-x).$$