Numerical PDEs (MA6612) Mid-Test, March 14, 2008

1. Consider the following boundary value problem

$$-u'' + 10u' + qu = f(x), \qquad x \in (0,1) \tag{1}$$
$$u(0) = 0, \quad u(1) = 1\,.$$

where q is a nonnegative constant.

(i) Describe the central finite difference scheme for the BVP (1)-(2) with a uniform mesh.

(ii) For $q \geq 0$, prove that the FD system has a unique solution under certain condition.

(iii) Show that if the solution $u \in C^4[0,1]$, the truncation error of method is $O(h^2)$.

(iv) For $q > 0$, show that if $u \in C^4[0,1]$ is a solution of the BVP (1)-(2), then

$$\max_j |u(x_j) - u_j| = O(h^2)$$

where $\{u_j\}$ denotes the finite difference solution defined in (i) and u is the exact solution of (1)-(2).

2. Present a second-order scheme for the following boundary value problem

$$-u'' + u = f(x), \qquad x \in (0,1) \tag{2}$$
$$u'(0) = u(1) = 0\,.$$

and truncation error of this scheme when $u \in C^4[0,1]$.

3. Present a variational model for the boundary value problem

$$-u'' + p(x)u' + q(x)u = f(x), \qquad x \in (0,1) \tag{3}$$
$$u(0) = u(1) = 0$$

and solve it with $p = 0$, $q = 2$ and $f = 1$ using a linear finite element method with three uniform elements.

4. Find $u \in V_2$ such that

$$\int_0^1 (u'v' - 2v)\,dx = 0 \qquad \forall v \in V_2$$

where

$$V_2 = \{u | \text{ polynomials of degree } \leq 3 \text{ and } u(0) = u(1) = 0\}\,.$$