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COMBINED PERTURBATION BOUNDS: I. EIGENSYSTEMS AND SINGULAR VALUE DECOMPOSITIONS*

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Abstract. In this paper we present some new combined perturbation bounds of eigenvalues and eigensubspaces for a Hermitian matrix H, particularly in an asymptotic sense, $\delta_{12}^2\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2+\sum_{i=1}^r(\lambda_i-\widetilde{\lambda}_i)^2\leq\|\Delta HU_1\|_F^2+O(\|\Delta HU_1\|_F^4)$, where λ_i denotes the eigenvalues of H and U_1 the eigensubspace corresponding to the eigenvalues λ_i, $i=1,2,\ldots,r$. The bound for each factor of eigensystems is optimal due to the sinΘ theorem and the Hoffman–Wielandt theorem. In addition, combined perturbation bounds for singular value decompositions and combined perturbation bounds in some, more general, measures are also obtained.

Key words. eigensystems, singular subspace, singular value, combined perturbation bound

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1. Introduction.

Let $C^{m\times n}$ denote the set of complex $m\times n$ matrices, A^* stand for the conjugate transpose of a matrix A, $\lambda(A)$ be the spectrum of A, and $\Re(A)$ be the column space of A. The Frobenius norm and spectral norm of a matrix A are denoted by $\|A\|_F$ and $\|A\|_2$, respectively.

Let H and $\widetilde{H}$ be two $n\times n$ Hermitian matrices with the following eigendecompositions:

$$H=\begin{pmatrix} U_1 & U_2\end{pmatrix}\begin{pmatrix}\Lambda_1 & 0\\ 0 & \Lambda_2\end{pmatrix}\begin{pmatrix}U_1^*\\ U_2^*\end{pmatrix},$$
$$\widetilde{H}=\begin{pmatrix} \widetilde{U}_1 & \widetilde{U}_2\end{pmatrix}\begin{pmatrix}\widetilde{\Lambda}_1 & 0\\ 0 & \widetilde{\Lambda}_2\end{pmatrix}\begin{pmatrix}\widetilde{U}_1^*\\ \widetilde{U}_2^*\end{pmatrix}, \tag{1.1}$$

where $U=\begin{pmatrix} U_1 & U_2\end{pmatrix}$, $\widetilde{U}=\begin{pmatrix} \widetilde{U}_1 & \widetilde{U}_2\end{pmatrix}$ are unitary, and

$$\Lambda_1=\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_r),\quad \Lambda_2=\operatorname{diag}(\lambda_{r+1},\lambda_{r+2},\ldots,\lambda_n),$$
$$\widetilde{\Lambda}_1=\operatorname{diag}(\widetilde{\lambda}_1,\widetilde{\lambda}_2,\ldots,\widetilde{\lambda}_r),\quad \widetilde{\Lambda}_2=\operatorname{diag}(\widetilde{\lambda}_{r+1},\widetilde{\lambda}_{r+2},\ldots,\widetilde{\lambda}_n). \tag{1.2}$$

Let

$$\delta_{ij}^{(k,l)}=\min_{\lambda\in\lambda(\Lambda_i),\widetilde{\lambda}\in\lambda(\widetilde{\Lambda}_j)}\frac{|\lambda-\widetilde{\lambda}|}{|\lambda|^k|\widetilde{\lambda}|^l},\quad i,j=1,2, \tag{1.3}$$

where k and l are nonnegative real numbers. For simplicity, we always use the notation $\delta_{ij}=\delta_{ij}^{(0,0)}$.

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The perturbation bounds for eigensystems, eigenspaces, and eigenvalues have been studied by many authors; e.g., see [1, 2, 3, 4, 6, 7, 8, 9, 10, 12]. The perturbation of the eigenspace $\Re(U_1)$ is measured by the canonical angle between the subspaces $\Re(U_1)$ and $\Re(\widetilde{U}_1)$ (e.g., see [8]), defined by

$$\Theta(U_1, \widetilde{U}_1) = arc\cos(U_1^* \widetilde{U}_1 \widetilde{U}_1^* U_1)^{1/2}.$$

The classical perturbation bound for the subspace was given as the $Sin\Theta$ theorem by Davis and Kahan [3].

THEOREM A (**sin**Θ theorem [3]). *Let H and $\widetilde{H} = H + \Delta H$ be two Hermitian matrices with the eigendecompositions* (1.1)–(1.2). *Then*

$$\delta_{12} \| \sin\Theta(U_1, \widetilde{U}_1)\|_F \le \|R\|_F, \tag{1.4}$$

where $R = \widetilde{H}U_1 - U_1\Lambda_1 = \Delta H U_1$.

The corresponding perturbation bound for eigenvalues is

$$\sum_{i=1}^{r} (\lambda_i - \widetilde{\lambda}_i)^2 \le \|R\|_F^2 \, . \tag{1.5}$$

When $r = n$, the bound (1.5) is the well-known Hoffman–Wielandt theorem [6].

The perturbation bounds in (1.4) and (1.5) are given in the absolute measure $\|R\|_F$. Recently, a relative-type perturbation bound was introduced. A general form of the relative bound is

$$\| \sin\Theta(U_1, \widetilde{U}_1)\|_F \le \alpha_{lk} \|H^{-l} \Delta H \widetilde{H}^{-k}\|_F \, , \tag{1.6}$$

where α_{lk} is a positive real number. Dopico, Moro, and Molera [4], Chen and Li [2], Li [8], and Londre and Rhee [9] studied the bound for $l = k = 1/2$, and Ipsen [7] studied it for the more general case.

In this paper we focus on perturbation bounds in a combined form of eigenspaces and eigenvalues. In particular, we shall show the new perturbation bound

$$\delta_{12}^2 \| \sin\Theta(U_1, \widetilde{U}_1)\|_F^2 + (1 - \| \sin\Theta(U_1, \widetilde{U}_1)\|_2^2) \sum_{i=1}^{r} (\lambda_i - \widetilde{\lambda}_i)^2 \le \|R\|_F^2 \, , \tag{1.7}$$

which, in an asymptotic sense, leads to

$$\delta_{12}^2 \| \sin\Theta(U_1, \widetilde{U}_1)\|_F^2 + \sum_{i=1}^{r} (\lambda_i - \widetilde{\lambda}_i)^2 \le \|R\|_F^2 + O(\|R\|_F^4), \tag{1.8}$$

where $\delta_{12} > 0$. The bounds (1.7) and (1.8) contain both the bound for eigenspaces and the bound for eigenvalues. In comparison with Davis and Kahan's theorem, (1.7) is sharper than the bound in Davis and Kahan's theorem and it also leads to the Hoffman–Wielandt theorem. On the other hand, the bound in (1.4) can be calculated when λ_i, $\widetilde{\lambda}_i$, and $\|R\|_F$ are known. In this case, a more precise bound for eigenspaces can be obtained from (1.7). In addition, we have obtained some new bounds in a relative sense and extensions to perturbation bounds for singular values and singular subspaces.

2. Combined bounds for eigensystems. In this section we study combined perturbation bounds for eigensystems.

LEMMA 2.1 (see [5]). *Let $T \in C^{n\times n}$ and $\Lambda_i = \mathrm{diag}(\lambda_1^{(i)}, \ldots, \lambda_n^{(i)}) \in C^{n\times n}$, $i = 1, 2, 3, 4$. Then there exists a permutation τ of $\langle n\rangle$ such that*

$$\sigma_n^2(T) \sum |\lambda_i^{(1)}\lambda_{\tau(i)}^{(2)} - \lambda_i^{(3)}\lambda_{\tau(i)}^{(4)}|^2 \le ||\Lambda_1 T\Lambda_2 - \Lambda_3 T\Lambda_4||_F^2, \tag{2.1}$$

where $\sigma_n(T)$ is the smallest singular value of T.

We have our main theorem below.

THEOREM 2.2. *Let H and $\widetilde{H} = H + \Delta H$ be two $n \times n$ nonsingular Hermitian matrices with the eigendecompositions* (1.1)–(1.2). *Then*

$$(\delta_{12}^2 - \delta_{11}^2)\|\sin\Theta(U_1, \widetilde{U}_1)\|_F^2 + r\delta_{11}^2 \le ||R||_F^2 \tag{2.2}$$

and

$$\delta_{12}^2\|\sin\Theta(U_1, \widetilde{U}_1)\|_F^2 + (1 - \|\sin\Theta(U_1, \widetilde{U}_1)\|_2^2)\sum_{i=1}^{r}(\lambda_i - \widetilde{\lambda}_i)^2 \le ||R||_F^2. \tag{2.3}$$

Proof. Left- and right-multiplying the equation $\widetilde{H} - H = \Delta H$ by $\widetilde{U}^*$ and U_1, respectively, leads to

$$\widetilde{\Lambda}\widetilde{U}^*U_1 - \widetilde{U}^*U_1\Lambda_1 = \widetilde{U}^*\Delta H U_1,$$

and in the block form,

$$\begin{pmatrix} \widetilde{\Lambda}_1\widetilde{U}_1^*U_1 - \widetilde{U}_1^*U_1\Lambda_1 \\ \widetilde{\Lambda}_2\widetilde{U}_2^*U_1 - \widetilde{U}_2^*U_1\Lambda_1 \end{pmatrix} = \widetilde{U}^*\Delta H U_1.$$

It follows that

$$\|\widetilde{\Lambda}_2\widetilde{U}_2^*U_1 - \widetilde{U}_2^*U_1\Lambda_1\|_F^2 + \|\widetilde{\Lambda}_1\widetilde{U}_1^*U_1 - \widetilde{U}_1^*U_1\Lambda_1\|_F^2 = ||\Delta H U_1||_F^2\,. \tag{2.4}$$

Since

$$\left|\left(\widetilde{\Lambda}_2\widetilde{U}_2^*U_1 - \widetilde{U}_2^*U_1\Lambda_1\right)_{ij}\right|^2 = (\widetilde{\lambda}_{i+r} - \lambda_j)^2|(\widetilde{U}_2^*U_1)_{ij}|^2 \ge \delta_{12}^2|(\widetilde{U}_2^*U_1)_{ij}|^2$$
$$i = 1, 2, \ldots, n-r;\ j = 1, 2, \ldots, r,$$

and

$$\left|\left(\widetilde{\Lambda}_1\widetilde{U}_1^*U_1 - \widetilde{U}_1^*U_1\Lambda_1\right)_{ij}\right|^2 = (\widetilde{\lambda}_i - \lambda_j)^2|(\widetilde{U}_1^*U_1)_{ij}|^2 \ge \delta_{11}^2|(\widetilde{U}_1^*U_1)_{ij}|^2,$$
$$i, j = 1, 2, \ldots, r,$$

we have

$$\delta_{12}^2\|\widetilde{U}_1^*U_2\|_F^2 + \delta_{11}^2\|\widetilde{U}_1^*U_1\|_F^2 \le \|\widetilde{\Lambda}_2\widetilde{U}_2^*U_1 - \widetilde{U}_2^*U_1\Lambda_1\|_F^2 + \|\widetilde{\Lambda}_1\widetilde{U}_1^*U_1 - \widetilde{U}_1^*U_1\Lambda_1\|_F^2\,. \tag{2.5}$$

Equation (2.2) is obtained by (2.4) and (2.5) and by noting the fact that

$$\|\widetilde{U}_1^*U_1\|_F^2 = r - \|\widetilde{U}_1^*U_2\|_F^2\,.$$

By Lemma 2.1,

$$\sigma_n^2(\widetilde{U}_1^* U_1) \sum_{i=1}^{r} |\widetilde{\lambda}_i - \lambda_i|^2 \leq \|\widetilde{\Lambda}_1 \widetilde{U}_1^* U_1 - \widetilde{U}_1^* U_1 \Lambda_1\|_F^2 \,. \tag{2.6}$$

By the C-S decomposition theorem (see, e.g., [11]), we have

$$\sigma_n^2(\widetilde{U}_1^* U_1) = 1 - \|\sin\Theta(U_1, \widetilde{U}_1)\|_2^2,$$

and therefore,

$$\delta_{12}^2 \|\sin\Theta(U_1, \widetilde{U}_1)\|_F^2 + (1 - \|\sin\Theta(U_1, \widetilde{U}_1)\|_2^2) \sum_{i=1}^{r} |\widetilde{\lambda}_i - \lambda_i| \leq ||R||_F^2,$$

which proves (2.3). ∎

Obviously the combined bounds in Theorem 2.2 contain perturbation bounds for both eigenspaces and eigenvalues. If we take $U_1 = U$ and $\widetilde{U}_1 = \widetilde{U}$, then $\|\sin\Theta(U_1, \widetilde{U}_1)\|_2 = \|\sin\Theta(U_1, \widetilde{U}_1)\|_F = 0$ and the bound (2.3) reduces to the Hoffman–Wielandt theorem. It is easy to obtain Davis and Kahan's sinΘ theorem from the bound (2.3) since $\|\sin\Theta(U_1, \widetilde{U}_1)\|_2 \leq 1$.

Example 2.1. Let

$$H = U\Lambda U^*, \qquad \widetilde{H} = (1+\epsilon)U\Lambda U^*,$$

where Λ is positive and diagonal and

$$\Lambda = \begin{pmatrix} I_r & 0 \\ 0 & 2I_{n-r} \end{pmatrix}, \qquad U = \begin{pmatrix} U_{11} & 0 \\ 0 & U_{22} \end{pmatrix}.$$

Then

$$\Delta H = \epsilon U \begin{pmatrix} I_r & 0 \\ 0 & 2I_{n-r} \end{pmatrix} U^*, \qquad \Delta H U_1 = \epsilon \begin{pmatrix} U_{11} \\ 0 \end{pmatrix}.$$

A simple calculation gives

$$\|R\|_F^2 = \|\Delta H U_1\|_F^2 = r\epsilon^2, \qquad \delta_{12} = 1 + 2\epsilon, \qquad \delta_{11} = \epsilon.$$

The bound (1.4) becomes

$$\|\sin\Theta(U_1\, \widetilde{U}_1)\|_F^2 \leq \frac{\|R\|_F^2}{\delta_{12}^2} = \frac{r\epsilon^2}{(1+2\epsilon)^2},$$

and from our bound (2.2),

$$\|\sin\Theta(U_1\, \widetilde{U}_1)\|_F^2 \leq \frac{\|R\|_F^2 - r\delta_{11}^2}{\delta_{12}^2 - \delta_{11}^2} = 0\,,$$

which leads to $\|\sin\Theta(U_1, \widetilde{U}_1)\|_F^2 = 0$.

When $\delta_{12} > 0$,

$$\|\sin\Theta(U_1, \widetilde{U}_1)\|_F \leq O(\|R\|_F), \qquad \sum_{i=1}^{r} (\lambda_i - \widetilde{\lambda}_i)^2 \leq \|R\|_F^2$$

and we obtain the asymptotic bound in (1.8). However, the following example shows the absolute bound

$$\delta_{12}^2\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2+\sum_{i=1}^{r}(\lambda_i-\widetilde{\lambda}_i)^2\le\|R\|_F^2$$

does not hold.

Example 2.2. Let

$$H=\begin{pmatrix}2&0&0\\0&1&0\\0&0&1\end{pmatrix}\quad\text{and}\quad\widetilde{H}=(1+\epsilon)\widetilde{U}^T H\widetilde{U},$$

where $\epsilon>0$ and

$$\widetilde{U}=\begin{pmatrix}\cos\theta&-\sin\theta&0\\\sin\theta&\cos\theta&0\\0&0&1\end{pmatrix}.$$

The eigenvalues of H and $\widetilde{H}$ are $2,1,1$ and $2(1+\epsilon),(1+\epsilon),(1+\epsilon)$, respectively. For $r=1$,

$$U_1=\begin{pmatrix}1\\0\\0\end{pmatrix}\quad\text{and}\quad\widetilde{U}_1=\begin{pmatrix}\cos\theta\\\sin\theta\\0\end{pmatrix}.$$

A simple calculation gives $\delta_{12}=1-\epsilon$ and

$$\begin{aligned}\delta_{12}^2\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2+\sum_{i=1}^{r}|\widetilde{\lambda}_i-\lambda_i|^2&=(1-\epsilon)^2\sin^2\theta+(2\epsilon)^2\\&>(1-\epsilon)^2\sin^2\theta+4\epsilon^2\cos^2\theta=||\Delta HU_1||_F^2=\|R\|_F^2.\end{aligned}$$

The perturbation bounds for eigenspaces in a relative measure have been studied by several authors; Dopico, Moro, and Molera [4] presented the relative perturbation bound

$$\delta_{12}^{(\frac{1}{2},\frac{1}{2})}\|\sin\Theta(U_1\,\widetilde{U}_1)\|_F\le\|H^{-1/2}\Delta H\widetilde{H}^{-1/2}\|_F$$

for nonsingular Hermitian matrices H and $\widetilde{H}$. A sharper bound obtained by Chen and Li [2] is

$$\frac{2\delta_{12}^{(\frac{1}{2},\frac{1}{2})}\delta_{21}^{(\frac{1}{2},\frac{1}{2})}}{\sqrt{\left(\delta_{12}^{(\frac{1}{2},\frac{1}{2})}\right)^2+\left(\delta_{21}^{(\frac{1}{2},\frac{1}{2})}\right)^2}}\|\sin\Theta(U_1\,\widetilde{U}_1)\|_F\le\|H^{-1/2}\Delta H\widetilde{H}^{-1/2}\|_F\,. \tag{2.7}$$

Li [8] and Londre and Rhee [9] studied perturbation bounds in a different relative measure. The perturbation bound in [8, 9] is given by

$$\delta_{12}^{(\frac{1}{2},\frac{1}{2})}\|\sin\Theta(U_1\,\widetilde{U}_1)\|_F\le\frac{\|H^{-1/2}\Delta HH^{-1/2}\|_F}{\sqrt{1-\mu_2}}, \tag{2.8}$$

where

$$\mu_2=\|H^{-1/2}\Delta HH^{-1/2}\|_2\,.$$

A modified bound in the relative measure given in [2] is

$$\min\left\{\delta_{12}^{(\frac{1}{2},\frac{1}{2})},\delta_{21}^{(\frac{1}{2},\frac{1}{2})}\right\}\|\sin\Theta(U_1,\widetilde{U}_1)\|_F \le \frac{\sqrt{2}}{2}\frac{\|H^{-1/2}\Delta H H^{-1/2}\|_F}{\sqrt{1-\mu_2}}. \tag{2.9}$$

The perturbation bound of eigenvalues in the more general relative measure $\|H^{-k}\Delta H\widetilde{H}^{-l}\|_F$ with any nonnegative numbers k and l was studied by Ipsen [7]. The perturbation bound given in [7] is

$$\sum_{i=1}^{n}|\lambda_i^{-k}\widetilde{\lambda}_{\tau(i)}^{1-l}-\lambda_i^{1-k}\widetilde{\lambda}_{\tau(i)}^{-l}|^2 \le \|H^{-k}\Delta H\widetilde{H}^{-l}\|_F^2. \tag{2.10}$$

No perturbation bound for eigenspaces has been obtained.

Now we extend our analysis for combined perturbation bounds to these relative measures, instead of the measure $\|R\|_F$ used in Theorem 2.2. Since

$$H^{-k}\Delta H\widetilde{H}^{-l} = H^{-k}\widetilde{H}^{1-l}-H^{1-k}\widetilde{H}^{-l}\,,$$

multiplying on the left by U^* and the right by $\widetilde{U}$ gives

$$\Lambda^{-k}U^*\widetilde{U}\widetilde{\Lambda}^{1-l}-\Lambda^{1-k}U^*\widetilde{U}\widetilde{\Lambda}^{-l} = U^*H^{-k}\Delta H\widetilde{H}^{-l}\widetilde{U}\,,$$

which can be rewritten in the block form as

$$\begin{pmatrix} \Lambda_1^{-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{1-l}-\Lambda_1^{1-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{-l} & \Lambda_1^{-k}U_1^*\widetilde{U}_2\widetilde{\Lambda}_2^{1-l}-\Lambda_1^{1-k}U_1^*\widetilde{U}_2\widetilde{\Lambda}_2^{-l} \\ \Lambda_2^{-k}U_2^*\widetilde{U}_1\widetilde{\Lambda}_1^{1-l}-\Lambda_2^{1-k}U_2^*\widetilde{U}_1\widetilde{\Lambda}_1^{-l} & \Lambda_2^{-k}U_2^*\widetilde{U}_2\widehat{\Lambda}_2^{1-l}-\Lambda_2^{1-k}U_2^*\widetilde{U}_2\widetilde{\Lambda}_2^{-l} \end{pmatrix} = U^*H^{-k}\Delta H\widetilde{H}^{-l}\widetilde{U}\,.$$

It follows that

$$\begin{aligned} &\|\Lambda_1^{-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{1-l}-\Lambda_1^{1-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{-l}\|_F^2+\|\Lambda_1^{-k}U_1^*\widetilde{U}_2\widetilde{\Lambda}_2^{1-l}-\Lambda_1^{1-k}U_1^*\widetilde{U}_2\widetilde{\Lambda}_2^{-l}\|_F^2 \\ &+\|\Lambda_2^{-k}U_2^*\widetilde{U}_1\widetilde{\Lambda}_1^{1-l}-\Lambda_2^{1-k}U_2^*\widetilde{U}_1\widetilde{\Lambda}_1^{-l}\|_F^2+\|\Lambda_2^{-k}U_2^*\widetilde{U}_2\widetilde{\Lambda}_2^{1-l}-\Lambda_2^{1-k}U_2^*\widetilde{U}_2\widetilde{\Lambda}_2^{-l}\|_F^2 \\ &= \|U^*H^{-k}\Delta H\widetilde{H}^{-l}\widetilde{U}\|_F^2\,. \end{aligned} \tag{2.11}$$

We take the same approach as used for (2.5). Since

$$\begin{aligned} &|(\Lambda_1^{-k}U_1^*\widetilde{U}_2\widetilde{\Lambda}_2^{1-l}-\Lambda_1^{1-k}U_1^*\widetilde{U}_2\widetilde{\Lambda}_2^{-l})_{ij}|^2 = (\lambda_i^{-k}\widetilde{\lambda}_j^{1-l}-\lambda_i^{1-k}\widetilde{\lambda}_j^{-l})^2|(U_1^*\widetilde{U}_2)_{ij}|^2 \\ &\ge \left(\delta_{12}^{(k,l)}\right)^2|(U_1^*\widetilde{U}_2)_{ij}|^2 \end{aligned}$$

and

$$\begin{aligned} &|(\Lambda_2^{-k}U_2^*\widetilde{U}_1\widetilde{\Lambda}_1^{1-l}-\Lambda_2^{1-k}U_2^*\widetilde{U}_1\widetilde{\Lambda}_1^{-l})_{ij}|^2 = (\lambda_j^{-k}\widetilde{\lambda}_i^{1-l}-\lambda_j^{1-k}\widetilde{\lambda}_i^{-l})^2|(U_2^*\widetilde{U}_1)_{ij}|^2 \\ &\ge \left(\delta_{21}^{(k,l)}\right)^2|(U_2^*\widetilde{U}_1)_{ij}|^2, \end{aligned}$$

we obtain

$$\begin{aligned} &\left(\delta_{12}^{(k,l)}\right)^2\|U_1^*\widetilde{U}_2\|_F^2 \le \|\Lambda_1^{-k}U_1^*\widetilde{U}_2\widetilde{\Lambda}_2^{1-l}-\Lambda_1^{1-k}U_1^*\widetilde{U}_2\widetilde{\Lambda}_2^{-l}\|_F^2, \\ &\left(\delta_{21}^{(k,l)}\right)^2\|U_2^*\widetilde{U}_1\|_F^2 \le \|\Lambda_2^{-k}U_2^*\widetilde{U}_1\widetilde{\Lambda}_1^{1-l}-\Lambda_2^{1-k}U_2^*\widetilde{U}_1\widetilde{\Lambda}_1^{-l}\|_F^2\,. \end{aligned} \tag{2.12}$$

On the other hand, let

$$T = \begin{pmatrix} U_1^*\widetilde{U}_1 & \\ & U_2^*\widetilde{U}_2 \end{pmatrix}, \quad D_1 = \begin{pmatrix} \Lambda_1^{-k} & \\ & \Lambda_2^{-k} \end{pmatrix}, \quad D_3 = \begin{pmatrix} \Lambda_1^{1-k} & \\ & \Lambda_2^{1-k} \end{pmatrix}$$

and

$$D_2 = \begin{pmatrix} \widetilde{\Lambda}_1^{1-l} & \\ & \Lambda_2^{1-l} \end{pmatrix}, \quad D_4 = \begin{pmatrix} -\widetilde{\Lambda}_1^{-l} & \\ & -\widetilde{\Lambda}_2^{-l} \end{pmatrix}.$$

We have

$$\begin{aligned} &\|\Lambda_1^{-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{1-l} - \Lambda_1^{1-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{-l}\|_F^2 + \|\Lambda_2^{-k}U_2^*\widetilde{U}_2\widetilde{\Lambda}_2^{1-l} - \Lambda_2^{1-k}U_2^*\widetilde{U}_2\widetilde{\Lambda}_2^{-l}\|_F^2 \\ &= \|D_1TD_2 - D_3TD_4\|_F^2. \end{aligned} \tag{2.13}$$

By Lemma 2.1, there exists a permutation τ of $\langle n\rangle$ such that

$$\|D_1TD_2 - D_3TD_4\|_F^2 \geq \sigma_n^2(T)\sum_{i=1}^{n}|\lambda_i^{-k}\widetilde{\lambda}_{\tau(i)}^{1-l} - \lambda_i^{1-k}\widetilde{\lambda}_{\tau(i)}^{-l}|^2\,. \tag{2.14}$$

By the C-S decomposition theorem [11], it is easy to see that

$$\sigma_n^2(T) \geq \min\{\sigma_n^2(U_1^*\widetilde{U}_1),\ \sigma_n^2(U_2^*\widetilde{U}_2)\} = 1 - \|\sin\Theta(U_1,\widetilde{U}_1)\|_2^2\,.$$

It follows that

$$\begin{aligned} &(1 - \|\sin\Theta(U_1,\widetilde{U}_1)\|_2^2)\sum_{i=1}^{r}|\lambda_i^{-k}\widetilde{\lambda}_{\tau(i)}^{1-l} - \lambda_i^{1-k}\widetilde{\lambda}_{\tau(i)}^{-l}|^2 \\ &\leq \|\Lambda_1^{-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{1-l} - \Lambda_1^{1-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{-l}\|_F^2 + \|\Lambda_2^{-k}U_2^*\widetilde{U}_2\widetilde{\Lambda}_2^{1-l} - \Lambda_2^{1-k}U_2^*\widetilde{U}_2\widetilde{\Lambda}_2^{-l}\|_F^2. \end{aligned} \tag{2.15}$$

Substituting (2.12) and (2.15) into (2.11) leads to a combined bound in the following theorem.

THEOREM 2.3. *Let H and $\widetilde{H} = H + \Delta H$ be two $n\times n$ nonsingular Hermitian matrices with the eigendecompositions* (1.1)–(1.2). *Then*

$$\begin{aligned} &\left((\delta_{12}^{(k,l)})^2 + (\delta_{21}^{(k,l)})^2\right)\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2 \\ &\qquad + (1 - \|\sin\Theta(U_1,\widetilde{U}_1)\|_2^2)\sum_{i=1}^{n}(\lambda_i^{-k}\widetilde{\lambda}_{\tau(i)}^{1-l} - \lambda_i^{1-k}\widetilde{\lambda}_{\tau(i)}^{-l})^2 \\ &\qquad \leq ||H^{-k}\Delta H\widetilde{H}^{-l}||_F^2. \end{aligned} \tag{2.16}$$

By an analogous approach, we can obtain

$$\begin{aligned} &\left(\delta_{22}^{(k,l)}\right)^2\|U_2^*\widetilde{U}_2\|_F^2 \leq \|\Lambda_2^{-k}U_2^*\widetilde{U}_2\widetilde{\Lambda}_2^{1-l} - \Lambda_2^{1-k}U_2^*\widetilde{U}_2\widetilde{\Lambda}_2^{-l}\|_F^2, \\ &\left(\delta_{11}^{(k,l)}\right)^2\|U_1^*\widetilde{U}_1\|_F^2 \leq \|\Lambda_1^{-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{1-l} - \Lambda_1^{1-k}U_1^*\widetilde{U}_1\widetilde{\Lambda}_1^{-l}\|_F^2\,. \end{aligned} \tag{2.17}$$

It is easy to see that

$$\|\sin\Theta(U_1,\widetilde{U}_1)\|_F = \|U_1^*\widetilde{U}_2\|_F = \|U_2^*\widetilde{U}_1\|_F\,.$$

By the definition of $\cos\Theta$,

$$\|\cos\Theta(U_1,\widetilde{U}_1)\|_F = \|U_1^*\widetilde{U}_1\|_F\,, \qquad \|\cos\Theta(U_2,\widetilde{U}_2)\|_F = \|U_2^*\widetilde{U}_2\|_F,$$

and again by the C-S decomposition theorem [11], we have

$$\|\cos\Theta(U_2,\widetilde{U}_2)\|_F^2 = \|\cos\Theta(U_1,\widetilde{U}_1)\|_F^2 + n - 2r$$

and

$$\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2 = r - \|\cos\Theta(U_1,\widetilde{U}_1)\|_F^2\,.$$

A new bound is given in the following theorem. The proof can be obtained by following the proof of Theorem 2.3 and replacing (2.15) by (2.17).

THEOREM 2.4. *Let H and $\widetilde{H} = H + \Delta H$ be two $n \times n$ nonsingular Hermitian matrices with the eigendecompositions* (1.1)–(1.2). *Then*

$$\begin{aligned}&\Big((\delta_{12}^{(k,l)})^2 + (\delta_{21}^{(k,l)})^2 - (\delta_{11}^{(k,l)})^2 - (\delta_{22}^{(k,l)})^2\Big)\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2\\ &\qquad + r(\delta_{11}^{(k,l)})^2 + (n-r)(\delta_{22}^{(k,l)})^2\\ &\qquad \le \|H^{-k}\Delta H\widetilde{H}^{-l}\|_F^2.\end{aligned} \tag{2.18}$$

Remark 2.1. Let H and $\widetilde{H}$ be Hermitian and the eigenvalues of H and $\widetilde{H}$ be enumerated by

$$\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n \quad \text{and} \quad \widetilde{\lambda}_1 \le \widetilde{\lambda}_2 \le \cdots \le \widetilde{\lambda}_n,$$

and assume that

$$\lambda_r < \lambda_{r+1}, \qquad \widetilde{\lambda}_r < \widetilde{\lambda}_{r+1}.$$

When the perturbation is small enough, we always have

$$\sum_{i=1}^n |\lambda_i^{-k}\widetilde{\lambda}_{\tau(i)}^{1-l} - \lambda_i^{1-k}\widetilde{\lambda}_{\tau(i)}^{-l}|^2 \le \sqrt{(\delta_{12}^{(k,l)})^2 + (\delta_{21}^{(k,l)})^2}$$

for some permutation τ of $\langle n\rangle$ and, therefore,

$$\begin{aligned}&\Big((\delta_{12}^{(k,l)})^2 + (\delta_{21}^{(k,l)})^2\Big)\,\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2 + \|\cos\Theta(U_1,\widetilde{U}_1)\|_2^2\sum_{i=1}^n |\lambda_i^{-k}\widetilde{\lambda}_{\tau(i)}^{1-l} - \lambda_i^{1-k}\widetilde{\lambda}_{\tau(i)}^{-l}|^2\\ &\ge \Big(\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2 + 1 - \|\sin\Theta(U_1,\widetilde{U}_1)\|_2^2\Big)\sum_{i=1}^n |\lambda_i^{-k}\widetilde{\lambda}_{\tau(i)}^{1-l} - \lambda_i^{1-k}\widetilde{\lambda}_{\tau(i)}^{-l}|^2\\ &\ge \sum_{i=1}^n |\lambda_i^{-k}\widetilde{\lambda}_{\tau(i)}^{1-l} - \lambda_i^{1-k}\widetilde{\lambda}_{\tau(i)}^{-l}|^2,\end{aligned}$$

which implies that the bound (2.16) is strictly sharper than the bound in (2.10) for the eigenvalue perturbation. Since $\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2 = \min\{r\,, n-r\}$, we have the following corollary.

COROLLARY 2.5. *Under the same assumption as in Theorem* 2.4*,*

$$({\delta_{12}^{(k,l)}}^2 + {\delta_{21}^{(k,l)}}^2)\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2 \le \begin{cases} \|H^{-k}\Delta H\widetilde{H}^{-l}\|_F^2 - (n-2r){\delta_{22}^{(k,l)}}^2, & 2r \le n,\\ \|H^{-k}\Delta H\widetilde{H}^{-l}\|_F^2 - (2r-n){\delta_{11}^{(k,l)}}^2, & 2r > n.\end{cases} \tag{2.19}$$

In comparison with the perturbation bound in (2.7), our bound (2.19) is sharper. The following corollary gives two combined perturbation bounds, in terms of the relative measure $\|H^{-k}\Delta H H^{-l}\|_F^2/(1-\mu_2)^{2l}$, which are sharper than the corresponding bounds obtained in [2, 8, 9].

COROLLARY 2.6. *If* $\mu_2 = \left\|H^{-1/2}\Delta H H^{-1/2}\right\|_2 < 1$, *then*

$$\begin{aligned}\Big((\delta_{12}^{(k,l)})^2 + (\delta_{21}^{(k,l)})^2 - (\delta_{11}^{(k,l)})^2 - (\delta_{22}^{(k,l)})^2\Big)\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2 \\ + r(\delta_{11}^{(k,l)})^2 + (n-r)(\delta_{22}^{(k,l)})^2 \\ \le \frac{\left\|H^{-k}\Delta H H^{-l}\right\|_2^2}{(1-\mu_2)^{2l}}\end{aligned} \tag{2.20}$$

and

$$\begin{aligned}\Big((\delta_{12}^{(k,l)})^2 + (\delta_{21}^{(k,l)})^2\Big)\|\sin\Theta(U_1,\widetilde{U}_1)\|_F^2 \\ + (1-\|\sin\Theta(U_1,\widetilde{U}_1)\|_2^2)\sum_{i=1}^{n}|\lambda_i^{-k}\widetilde{\lambda}_{\tau(i)}^{1-l} - \lambda_i^{1-k}\widetilde{\lambda}_{\tau(i)}^{-l}|^2 \\ \le \frac{\left\|H^{-k}\Delta H H^{-l}\right\|_2^2}{(1-\mu_2)^{2l}}.\end{aligned} \tag{2.21}$$

Proof. Taking the same approach as in [2], one may deduce that

$$||H^{-k}\Delta H\widetilde{H}^{-l}||_F \le \frac{\left\|H^{-k}\Delta H H^{-l}\right\|_2}{(1-\mu_2)^{l}}, \tag{2.22}$$

which together with Theorems 2.2 and 2.3 gives the desired bound. □

3. Combined bounds for singular value decompositions. Let A, $\widetilde{A} \in C^{m\times n}$ have the singular value decompositions (SVDs)

$$A = U\Sigma V^* = \begin{pmatrix} U_1 & U_2 \end{pmatrix}\begin{pmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{pmatrix}\begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix} \tag{3.1}$$

and

$$\widetilde{A} = \widetilde{U}\widetilde{\Sigma}\widetilde{V}^* = \begin{pmatrix} \widetilde{U}_1 & \widetilde{U}_2 \end{pmatrix}\begin{pmatrix} \widetilde{\Sigma}_1 & 0 \\ 0 & \widetilde{\Sigma}_2 \end{pmatrix}\begin{pmatrix} \widetilde{V}_1^* \\ \widetilde{V}_2^* \end{pmatrix}, \tag{3.2}$$

where $U = (\widetilde{U}_1\ \widetilde{U}_2)$ and $\widetilde{U} = (\widetilde{U}_1\ \widetilde{U}_2)$ are $m\times m$ unitary, $V = (\widetilde{V}_1\ \widetilde{V}_2)$ and $\widetilde{V} = (\widetilde{V}_1\ \widetilde{V}_2)$ are $n\times n$ unitary, and

$$\Sigma_1 = \operatorname{diag}(\sigma_1,\sigma_2,\ldots,\sigma_r), \quad \Sigma_2 = \operatorname{diag}(\sigma_{r+1},\sigma_{r+2},\ldots,\sigma_n), \tag{3.3}$$

$$\widetilde{\Sigma}_1 = \operatorname{diag}(\widetilde{\sigma}_1,\widetilde{\sigma}_2,\ldots,\widetilde{\sigma}_r), \quad \widetilde{\Sigma}_2 = \operatorname{diag}(\widetilde{\sigma}_{r+1},\widetilde{\sigma}_{r+2},\ldots,\widetilde{\sigma}_n). \tag{3.4}$$

Let $\sigma(A) = \lambda(\sqrt{A^*A})$ be the set of singular values of A and

$$\sigma_{ext}(\Sigma_2) = \begin{cases} \sigma(\Sigma_2)\cup\{0\} & \text{if } m>n, \\ \sigma(\Sigma_2) & \text{if } m=n. \end{cases}$$

The perturbation of singular subspaces is usually measured by the angle between the subspaces $\Re(U_1)$ and $\Re(\widetilde{U}_1)$ and the angle between the subspaces $\Re(V_1)$ and $\Re(\widetilde{V}_1)$, denoted by

$$\|\sin\Theta\|_F = \|U_1^*\widetilde{U}_2\|_F, \qquad \|\sin\Phi\|_F = \|V_1^*\widetilde{V}_2\|_F,$$

respectively. Let

$$\epsilon_{ij}^{(k,l)} = \min_{\lambda\in\sigma_{ext}(\Sigma_i),\widetilde{\mu}\in\sigma(\widetilde{\Sigma}_j)} \frac{|\mu-\widetilde{\mu}|}{|\mu|^k|\widetilde{\mu}|^l}, \quad i,j=1,2.$$

Similarly we use the notation $\epsilon_{ij} = \epsilon_{ij}^{(0,0)}$.

The perturbation bound of singular subspace was given by Wedin [13] and the perturbation bound for singular values can be found in the literature. We summarize the results in the following theorem.

THEOREM B. *Let A, $\widetilde{A} \in C^{m\times n}$ have the SVDs* (3.1)–(3.4). *Then*

$$\epsilon_{12}^2(\|\sin\Theta\|_F^2 + \|\sin\Phi\|_F^2) \le \|R\|_F^2 + \|S\|_F^2 \tag{3.5}$$

and

$$2\sum_{i=1}^{r}(\sigma_i - \widetilde{\sigma}_{\tau(i)})^2 \le \|R\|_F^2 + \|S\|_F^2, \tag{3.6}$$

where

$$R = A\widetilde{V}_1 - \widetilde{U}_1\widetilde{\Sigma}_1 = -E\widetilde{V}_1, \qquad S = A^*\widetilde{U}_1 - \widetilde{V}_1\widetilde{\Sigma}_1 = -E^*\widetilde{U}_1.$$

To obtain a combined perturbation bound for SVDs, we consider the Jordan–Wielandt matrices

$$\mathbb{H} = \begin{pmatrix} 0 & A^* \\ A & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{\mathbb{H}} = \begin{pmatrix} 0 & \widetilde{A}^* \\ \widetilde{A} & 0 \end{pmatrix}. \tag{3.7}$$

Let

$$\mathbb{U} = \frac{1}{\sqrt{2}}\begin{pmatrix} V_1 & V_1 & V_2 & V_2 \\ U_1 & -U_1 & U_2 & -U_2 \end{pmatrix} = \begin{pmatrix} \mathbb{U}_1 & \mathbb{U}_2 \end{pmatrix}$$

and

$$\widetilde{\mathbb{U}} = \frac{1}{\sqrt{2}}\begin{pmatrix} \widetilde{V}_1 & \widetilde{V}_1 & \widetilde{V}_2 & \widetilde{V}_2 \\ \widetilde{U}_1 & -\widetilde{U}_1 & \widetilde{U}_2 & -\widetilde{U}_2 \end{pmatrix} = \begin{pmatrix} \widetilde{\mathbb{U}}_1 & \widetilde{\mathbb{U}}_2 \end{pmatrix},$$

where

$$\mathbb{U}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} V_1 & V_1 \\ U_1 & -U_1 \end{pmatrix} \quad \text{and} \quad \widetilde{\mathbb{U}}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} \widetilde{V}_1 & \widetilde{V}_1 \\ \widetilde{U}_1 & -\widetilde{U}_1 \end{pmatrix}.$$

Then the eigendecomposition of H and $\widetilde{H}$ can be rewritten as

$$\mathbb{H} = \mathbb{U}\begin{pmatrix} \Sigma_1 & & & \\ & -\Sigma_1 & & \\ & & \Sigma_2 & \\ & & & -\Sigma_2 \end{pmatrix}\mathbb{U}^* \quad \text{and} \quad \widetilde{\mathbb{H}} = \widetilde{\mathbb{U}}\begin{pmatrix} \widetilde{\Sigma}_1 & & & \\ & -\widetilde{\Sigma}_1 & & \\ & & \widetilde{\Sigma}_2 & \\ & & & -\widetilde{\Sigma}_2 \end{pmatrix}\widetilde{U}^*,$$

respectively. Applying Theorems 2.2 and 2.3 to the matrices $\mathbb{H}$ and $\widetilde{\mathbb{H}}$, we have the following combined perturbation bounds for singular values and singular subspaces.

THEOREM 3.1. *Let A and $\widetilde{A} = A + E$ be two $n \times n$ nonsingular matrices with the SVDs* (3.1)–(3.4). *Then there is a permutation τ in $\langle n \rangle$ such that*

$$\epsilon_{12}^2 \left(\| \sin\Phi(U_1, \widetilde{U}_1)\|_F^2 + \| \sin\Theta(V_1, \widetilde{V}_1) \|_F^2 \right) + \left(2 - \| \sin\Phi(U_1, \widetilde{U}_1)\|_2^2 - \| \sin\Theta(V_1, \widetilde{V}_1)\|_2^2 \right) \sum_{i=1}^{r} (\sigma_i - \widetilde{\sigma}_{\tau(i)})^2 \le \|R\|_F^2 + \|S\|_F^2 \tag{3.8}$$

and

$$(\epsilon_{12}^2 - \epsilon_{11}^2)(\| \sin\Phi(U_1, \widetilde{U}_1)\|_F^2 + \| \sin\Theta(V_1, \widetilde{V}_1)\|_F^2) + 2r\epsilon_{11}^2 \le \|R\|_F^2 + \|S\|_F^2. \tag{3.9}$$

In an asymptotic sense, (3.9) becomes

$$\epsilon_{12}^2 \left(\| \sin\Phi(U_1, \widetilde{U}_1)\|_F^2 + \| \sin\Theta(V_1, \widetilde{V}_1)\|_F^2 \right) + 2 \sum_{i=1}^{n} |\sigma_i - \widetilde{\sigma}_i|^2 \le (\|R\|_F^2 + \|S\|_F^2) + O((\|R\|_F^2 + \|S\|_F^2)^2). \tag{3.10}$$

From the SVDs (3.1) and (3.2), we obtain the left polar decomposition of the matrices A and $\widetilde{A}$, defined by

$$A = QP_l \quad \text{and} \quad \widetilde{A} = \widetilde{Q}\widetilde{P}_l, \tag{3.11}$$

and, similarly, the right polar decomposition

$$A = P_r Q \quad \text{and} \quad \widetilde{A} = \widetilde{P}_r \widetilde{Q}, \tag{3.12}$$

where Q is called the unitary polar factor of A and P_l and P_r are called the left and right Hermitian factor, respectively. It is noted that Wedin's $\sin\theta$ theorem is given in an absolute measure $\|R\|_F^2 + \|S\|_F^2$. The perturbation bounds for singular values and singular subspaces in some relative measures was studied in [4], where the relative measures $\|P_l^{-k} E \widetilde{P}_r^{-l}\|_F$ and $\|\widetilde{P}_l^{-l} E P_r^{-k}\|_F$ are used. The extension to combined bounds is given in the following theorem and the proof is similar to the proofs for Theorems 2.3 and 2.4.

THEOREM 3.2. *Let A and $\widetilde{A} = A + E$ be two $n \times n$ nonsingular matrices with the SVDs* (3.1)–(3.4). *Then*

$$\left(\left(\epsilon_{21}^{(k,l)}\right)^2 + \left(\epsilon_{12}^{(k,l)}\right)^2 \right) \left(\| \sin\Phi(U_1, \widetilde{U}_1)\|_F^2 + \| \sin\Theta(V_1, \widetilde{V}_1)\|_F^2 \right) + \left(2 - \| \sin\Phi(U_1, \widetilde{U}_1)\|_2^2 - \| \sin\Theta(V_1, \widetilde{V}_1)\|_2^2 \right) \sum_{i=1}^{n} (\sigma_i - \widetilde{\sigma}_{\tau(i)})^2 \le \|P_l^{-k} E \widetilde{P}_r^{-l}\|_F^2 + \|\widetilde{P}_l^{-l} E P_r^{-k}\|_F^2 \tag{3.13}$$

and

$$\left(\left(\epsilon_{21}^{(k,l)}\right)^2 + \left(\epsilon_{12}^{(k,l)}\right)^2 - \left(\epsilon_{11}^{(k,l)}\right)^2 - \left(\epsilon_{22}^{(k,l)}\right)^2 \right) \left(\| \sin\Phi(U_1, \widetilde{U}_1)\|_F^2 + \| \sin\Theta(V_1, \widetilde{V}_1)\|_F^2 \right) + 2r \left(\epsilon_{11}^{(k,l)}\right)^2 + 2(n-r) \left(\epsilon_{22}^{(k,l)}\right)^2 \le \|P_l^{-k} E \widetilde{P}_r^{-l}\|_F^2 + \|\widetilde{P}_l^{-l} E P_r^{-k}\|_F^2, \tag{3.14}$$

where P_l*,* P_r*,* $\widetilde{P}_l$*, and* $\widetilde{P}_r$ *are defined in* (3.11) *and* (3.12)*.*

Remark 3.1. A perturbation bound in Theorem 3.4 of [4] is as follows:

$$\left(\epsilon_{21}^{(\frac{1}{2},\frac{1}{2})}\right)^2 \left(\|\sin\Phi(U_1,\widetilde{U}_1)\|_F^2 + \|\sin\Theta(V_1,\ \widetilde{V}_1)\|_F^2\right) \\ \leq \|P_l^{-\frac{1}{2}} E \widetilde{P}_r^{-\frac{1}{2}}\|_F^2 + \|\widetilde{P}_l^{-\frac{1}{2}} E P_r^{-\frac{1}{2}}\|_F^2,$$

which can also be obtained from Theorem 3.2.

Finally, we consider the bounds for the right singular subspaces as in [9]. Let

$$\delta \equiv ||EA^\dagger||_2, \quad \delta_F \equiv ||EA^\dagger||_F.$$

Let A and $\widetilde{A}$ have the SVDs (3.1)–(3.4) and let

$$\varsigma_{ij} = \min_{\mu\in\sigma(\Sigma_i),\widetilde{\mu}\in\sigma(\widetilde{\Sigma}_j)} \frac{|\widetilde{\mu}^2-\mu^2|}{\widetilde{\mu}\mu}.$$

Let $H = A^*A$ and $\widetilde{H} = \widetilde{A}^*\widetilde{A}$. Then H and $\widetilde{H}$ have the eigendecompositions

$$H = (V_1, V_2)\begin{pmatrix} \Sigma_1^2 & 0 \\ 0 & \Sigma_2^2 \end{pmatrix}\begin{pmatrix} V_1^* \\ V_2^* \end{pmatrix}, \qquad \widetilde{H} = \begin{pmatrix} \widetilde{V}_1 & \widetilde{V}_2 \end{pmatrix}\begin{pmatrix} \widetilde{\Sigma}_1^2 & 0 \\ 0 & \widetilde{\Sigma}_2^2 \end{pmatrix}\begin{pmatrix} \widetilde{V_1}^* \\ \widetilde{V_2}^* \end{pmatrix}.$$

Applying Corollary 2.6 to H and $\widetilde{H}$ with $l = k = 1/2$ by the same argument as in Theorem 2.1 of [9], we obtain the following estimate:

$$(\varsigma_{21}^2 + \varsigma_{12}^2 - \varsigma_{11}^2 - \varsigma_{22}^2)||\sin\Theta(V_1,\widetilde{V}_1)||_F^2 + r\varsigma_{11}^2 + (n-r)\varsigma_{22}^2 \leq \frac{(2\delta_F+\delta_F^2)^2}{1-3\delta} \tag{3.15}$$

for $\delta < 1/3$. A simpler form of (3.15) is

$$||\sin\Theta(V_1,\widetilde{V}_1)||_F \leq \frac{2\delta_F+\delta_F^2}{\sqrt{(1-3\delta)(\varsigma_{21}^2+\varsigma_{12}^2)}}\,. \tag{3.16}$$

It is easy to see that the bound in (3.16) is always sharper than those in Theorem 2.1 of [9].

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REFERENCES

[1] J. L. Barlow and I. Slapnicar, *Optimal perturbation bounds for the Hermitian eigenvalue problem*, Linear Algebra Appl., 309 (2000), pp. 19–43.
[2] X. Chen and W. Li, *A note on the perturbation bounds of eigenspaces for Hermitian matrices*, J. Comput. Appl. Math., 196 (2006), pp. 338–346.
[3] C. Davis and W. M. Kahan, *The rotation of eigenvectors by a perturbation.* III, SIAM J. Numer. Anal., 7 (1970), pp. 1–46.
[4] F. M. Dopico, J. Moro, and J. M. Molera, *Weyl-type relative perturbation bounds for eigensystems of Hermitian matrices*, Linear Algebra Appl., 309 (2000), pp. 3–18.
[5] L. Elsner and S. Friedland, *Singular values, doubly stochastic matrices, and applications*, Linear Algebra Appl., 220 (1995), pp. 161–169.
[6] A. J. Hoffman and H. W. Wielandt, *The variation of spectrum of a normal matrix*, Duke Math. J., 20 (1953), pp. 37–39.

[7] I. C. F. Ipsen, *A note on unifying absolute and relative perturbation bounds*, Linear Algebra Appl., 358 (2003), pp. 239–253.

[8] R.-C. Li, *Relative perturbation theory.* II. *Eigenspace and singular subspace variations*, SIAM J. Matrix Anal. Appl., 20 (1999), pp. 471–492.

[9] T. Londre and N. H. Rhee, *A note on relative perturbation bounds*, SIAM J. Matrix Anal. Appl., 21 (1999), pp. 357–361.

[10] R. Mathias and K. Veselic, *A relative perturbation bound for positive definite matrices*, Linear Algebra Appl., 270 (1998), pp. 315–321.

[11] G. Stewart and J. Sun, *Matrix Perturbation Theory*, Academic Press, Boston, 1990.

[12] N. Truhar and I. Slapnicar, *Relative perturbation bound for invariant subspaces of graded indefinite Hermitian matrices*, Linear Algebra Appl., 301 (1999), pp. 171–185.

[13] P. A. Wedin, *Perturbation bounds in connection with the singular value decomposition*, BIT, 12 (1972), pp. 99–111.