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The superconvergence of Newton–Cotes rules for the Hadamard finite-part integral on an interval

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Abstract We study the general (composite) Newton–Cotes rules for the computation of Hadamard finite-part integral with the second-order singularity and focus on their *pointwise superconvergence phenomenon*, i.e., when the singular point coincides with some a priori known point, the convergence rate is higher than what is globally possible. We show that the superconvergence rate of the (composite) Newton–Cotes rules occurs at the zeros of a special function and prove the existence of the superconvergence points. Several numerical examples are provided to validate the theoretical analysis.

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1 Introduction

We consider the Hadamard finite-part integral of the form (see e.g., [13,16,25])

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$$\oint_a^b \frac{f(x)}{(x-s)^2}dx := \lim_{\varepsilon\to 0}\left\{\int_a^{s-\varepsilon}\frac{f(x)}{(x-s)^2}dx + \int_{s+\varepsilon}^{b}\frac{f(x)}{(x-s)^2}dx - \frac{2f(s)}{\varepsilon}\right\},\ s\in(a,b). \tag{1.1}$$

$f(x)$ is said to be finite-part integrable with respect to the weight $(x-s)^{-2}$ if the limit on the right hand side of the above equation exists. A sufficient condition for $f(x)$ to be finite-part integrable is that its first derivative $f'(x)$ is Hölder continuous. Throughout this paper, $\oint$ denotes an integral in the Hadamard finite-part sense and by contrast, $⨍$ a Cauchy principal value integral or a finite Hilbert transform. The Hadamard finite-part integral is related to the usual Cauchy principal value integral by

$$\oint_a^b \frac{f(x)}{(x-s)^2}dx = \frac{d}{ds}\left(⨍_a^b \frac{f(x)}{x-s}dx\right). \tag{1.2}$$

In many occasions, this identity has been used as an alternative definition of the Hadamard finite-part integral (cf. [5,9,7,11]). Integrals of the form (1.1) appear frequently in boundary element methods (BEMs) and other numerical computations [2,3,10,21]. The efficiency of BEMs often depends upon the efficiency of numerical evaluation of such Hadamard finite-part integrals. Numerous work has been devoted in developing efficient quadrature formulas, such as the Gaussian method [10,11,15,17,22], the (composite) Newton–Cotes method [6,13,19,23,25], the transformation method [5,7] and some other methods [9,12]. The Newton–Cotes rule is a commonly used one in many areas due to its ease of implementation and flexibility of mesh.

Newton–Cotes rules for Riemann integrals have been well studied. The accuracy of the (composite) Newton–Cotes rules for Riemann integrals is $O(h^{k+1})$ for odd k and $O(h^{k+2})$ for even k. However, the rules are less accurate for Hadamard finite-part integrals due to the hypersingularity of the integrand at the singular point s. The (composite) Newton–Cotes rules for Hadamard finite-part integrals were first studied in [13], where the error estimates obtained for the trapezoidal rule and Simpson's rule are much lower than their counterparts for Riemann integrals. Our numerical experiments show that, when the singular point s coincides with some a priori known point, Newton–Cotes rules can reach a higher-order convergence rate. We refer to this as *the pointwise superconvergence phenomenon* of the (composite) Newton–Cotes rules for Hadamard finite-part integrals.

The superconvergence of (composite) Newton–Cotes rules for Hadamard finite-part integrals was first studied in [23,25], where the superconvergence rate of the trapezoidal rule and Simpson's rule was presented, respectively. This paper focuses on the superconvergence of arbitrary degree Newton–Cotes rules for Hadamard finite-part integrals. We prove both theoretically and numerically that the (composite) Newton–Cotes rules reach the superconvergence rate $O(h^{k+1})$ when the local coordinate of the singular point s is the zero of the function

$$\mathcal{S}_k(\tau) := \psi_k'(\tau) + \sum_{i=1}^{\infty}[\psi_k'(2i+\tau) + \psi_k'(-2i+\tau)], \quad \tau \in (-1, 1), \tag{1.3}$$

where ψ_k is a function of second kind associated with a polynomial of equally-distributed zeros.

The rest of this paper is organized as follows. In Sect. 2, after introducing some basic formulas of the general (composite) Newton–Cotes rules and some notations, we present our main result of superconvergence. The complete proof is given in Sect. 3. In Sect. 4, we prove the existence of superconvergence points and present some properties of these superconvergence points. In Sect. 5, we present several numerical examples to validate our analysis. Finally, we give some concluding remarks in the last section.

2 The superconvergence of Newton–Cotes rules

Let $a = x_0 < x_1 < \cdots < x_n = b$ be a partition of interval $[a, b]$. To construct a piecewise Lagrange interpolation polynomial of degree k, we introduce a further partition at each subinterval,

$$x_i = x_{i0} < x_{i1} < \cdots < x_{ik} = x_{i+1}$$

and a linear transformation

$$x = \hat{x}_i(\tau) := (\tau + 1)(x_{i+1} - x_i)/2 + x_i, \quad \tau \in [-1, 1]$$

from the reference element $[-1, 1]$ to the subinterval $[x_i, x_{i+1}]$. Here we assume that both meshes are quasi-uniform. We define the piecewise Lagrange polynomial interpolation by

$$\mathcal{F}_{kn}(x) = \sum_{j=0}^{k} f(x_{ij}) \frac{l_{ki}(x)}{(x - x_{ij})l_{ki}'(x_{ij})}, \quad x \in [x_i, x_{i+1}], \tag{2.1}$$

where

$$l_{ki}(x) = \prod_{j=0}^{k}(x - x_{ij}).$$

Replacing $f(x)$ in (1.1) by $\mathcal{F}_{kn}(x)$ gives the general (composite) Newton–Cotes rule

$$\mathcal{Q}_{kn}(f) := \int_a^b \frac{\mathcal{F}_{kn}(x)}{(x-s)^2}dx = \sum_{i=0}^{n-1}\sum_{j=0}^{k} \omega_{ij}^{(k)} f(x_{ij}) = \int_a^b \frac{f(x)}{(x-s)^2}dx - \mathcal{E}_{kn}(f), \tag{2.2}$$

where $\mathcal{E}_{kn}(f)$ denotes the error functional and

$$\omega_{ij}^{(k)} = \frac{1}{l'_{ki}(x_{ij})} \int_{x_i}^{x_{i+1}} \frac{1}{(x-s)^2} \prod_{m=0, m\neq j}^{k} (x - x_{im}) dx. \tag{2.3}$$

The classical (composite) trapezoidal rule and Simpson's rule, two special cases of the quadrature formula (2.2), were studied by Linz [13] where explicit formulae of the Cotes coefficients $\omega_{ij}^{(k)} (k = 1, 2)$ were presented. The error estimate obtained in [13] is

$$|\mathcal{E}_{kn}(f)| \leq C\gamma^{-2}(h, s)h^k, \quad k = 1, 2, \tag{2.4}$$

where

$$\gamma(h, s) = \min_{0\leq i\leq n} \frac{|s - x_i|}{h}, \quad h = \max_{0\leq i\leq n-1} |x_{i+1} - x_i|. \tag{2.5}$$

The above estimate shows that the accuracy depends upon a factor $\gamma^{-2}(h, s)$, a quantity that tends to infinity when the singular point s approaches a mesh point. A more precise estimate was given in [19] where

$$|\mathcal{E}_{kn}(f)| \leq C \min\{\gamma^{-1}(h, s), |\ln \gamma(h, s)| + |\ln h|\}h^k, \quad k = 1, 2. \tag{2.6}$$

Here we present the error estimate for the (composite) Newton–Cotes rule with an arbitrary degree in the following theorem. The proof can be obtained along the line in [19,23].

Theorem 2.1 *Assume that $f(x) \in C^{k+\alpha}[a, b]$, $0 < \alpha \leq 1$, and $s \neq x_i$ for any $i = 0, 1, \ldots, n$. Then, for the general (composite) Newton–Cotes rule $\mathcal{Q}_{kn}(f)$ defined in (2.2), there exists a positive constant C, independent of h and s, such that*

$$|\mathcal{E}_{kn}(f)| \leq C|\ln \gamma(h, s)|h^{k+\alpha-1}, \tag{2.7}$$

where $\gamma(h, s)$ is defined in (2.5).

Compared with Riemann integrals, the global convergence rate of the (composite) Newton–Cotes rule for finite-part integrals is one order lower for odd k and two orders lower for even k. However, numerical results show that the error estimate in (2.7) is optimal.

The main issue of this paper is the superconvergence of the general (composite) Newton–Cotes rule. For simplicity, hereafter we always assume that both meshes $\{x_i\}$ and $\{x_{ij}\}$ are uniform. It is not difficult to extend our analysis to certain quasi-uniform meshes.

Let

$$\phi_k(\tau) = \prod_{j=0}^{k} (\tau - \tau_j) = \prod_{j=0}^{k} \left(\tau - \frac{2j-k}{k}\right) \tag{2.8}$$

and denote by ψ_k the function of second kind associated with ϕ_k, defined by

$$\psi_k(t) = \begin{cases} -\dfrac{1}{2} \displaystyle⨍_{-1}^{1} \dfrac{\phi_k(\tau)}{\tau - t} d\tau, & |t| < 1, \\ -\dfrac{1}{2} \displaystyle\int_{-1}^{1} \dfrac{\phi_k(\tau)}{\tau - t} d\tau, & |t| > 1. \end{cases} \tag{2.9}$$

It is known that if ϕ_k is the Legendre polynomial, ψ_k defines the Legendre function of the second kind (see e.g., [1]). By (1.2), we have

$$\psi'_k(t) = \begin{cases} -\dfrac{1}{2} \displaystyle⨎_{-1}^{1} \dfrac{\phi_k(\tau)}{(\tau - t)^2} d\tau, & |t| < 1, \\ -\dfrac{1}{2} \displaystyle\int_{-1}^{1} \dfrac{\phi_k(\tau)}{(\tau - t)^2} d\tau, & |t| > 1. \end{cases} \tag{2.10}$$

The superconvergence results of Newton–Cotes rules are given in the following theorem.

Theorem 2.2 *Assume $f(x) \in C^{k+1+\alpha}[a, b]$, $0 < \alpha \leq 1$ and $\tau^\star$ is a zero of $\mathcal{S}_k(\tau)$ defined by (1.3) and (2.10). Then, for the general (composite) Newton–Cotes rule $\mathcal{Q}_{kn}(f)$ defined in (2.2), there holds at $s = \hat{x}_i(\tau^\star)$*

$$|\mathcal{E}_{kn}(f)| \leq C[1 + \eta(s)h^{1-\alpha}]h^{k+\alpha}, \quad 0 < \alpha \leq 1 \tag{2.11}$$

for even k, and

$$|\mathcal{E}_{kn}(f)| \leq \begin{cases} C[1 + \eta(s)h^{1-\alpha}]h^{k+\alpha}, & 0 < \alpha < 1, \\ C[\eta(s) + |\ln h|]h^{k+1}, & \alpha = 1 \end{cases} \tag{2.12}$$

for odd k, where

$$\eta(s) = \max\{(b - s)^{-1}, (s - a)^{-1}\}. \tag{2.13}$$

By comparing Theorems 2.1 and 2.2, one can see that the superconvergence rate of the (composite) Newton–Cotes rules at certain points is one order higher than their global convergence rate. We list the superconvergence points, the zeros of $\mathcal{S}_k(\tau)$, with 16 digits in Table 1 for different k. The proof of Theorem 2.2 will be given in next section.

Table 1 Superconvergence points of Newton–Cotes rules

k	superconvergence points ($\tau_k^\star$)
$k=1$	± 0.6666666666666666
$k=2$	0
$k=3$	± 0.4176898586988372, ± 0.9323070644490695
$k=4$	0, ± 0.5543264529853550
$k=5$	± 0.1889629663325798, ± 0.6786253433205400, ± 0.9650849350320763

3 The proof of the main result

We begin the analysis by investigating the properties of ψ_k, defined in (2.9). In the following, C will denote a generic positive constant which is independent of h and s but which may depend on k, α and bounds of the derivatives of $f(x)$. Let P_l and Q_l denote the Legendre polynomial of degree l and the associated Legendre function of the second kind, respectively.

Lemma 3.1 *Let $\psi_k(t)$ be defined in (2.9). Then*

$$\psi_k(t) = \begin{cases} \sum_{i=1}^{k_1+1} \omega_{2i-1} Q_{2i-1}(t), & k = 2k_1, \\ \sum_{i=0}^{k_1} \omega_{2i} Q_{2i}(t), & k = 2k_1 - 1 \end{cases} \tag{3.1}$$

and

$$\psi_k'(t) = \begin{cases} \sum_{i=1}^{k_1} a_i Q_{2i}(t), & k = 2k_1, \\ \sum_{i=1}^{k_1} b_i Q_{2i-1}(t), & k = 2k_1 - 1, \end{cases} \tag{3.2}$$

where

$$\omega_i = \frac{2i+1}{2} \int_{-1}^{1} \phi_k(\tau) P_i(\tau) d\tau \tag{3.3}$$

and

$$\begin{aligned} a_i &= -(4i+1) \sum_{j=1}^{i} \omega_{2j-1}, \\ b_i &= -(4i-1) \sum_{j=1}^{i} \omega_{2j-2}. \end{aligned} \tag{3.4}$$

Proof For $k = 2k_1$,

$$\phi_k(\tau) = \prod_{j=0}^{2k_1}\left(\tau - \frac{j-k_1}{k_1}\right) = \tau\prod_{j=1}^{k_1}\left(\tau^2 - \frac{j^2}{k_1^2}\right)$$

and the polynomial $\phi_k(\tau)$ is an odd function. In terms of Legendre polynomials,

$$\phi_k(\tau) = \sum_{i=1}^{k_1+1}\omega_{2i-1}P_{2i-1}(\tau), \tag{3.5}$$

where ω_{2i-1} is defined in (3.3). The first part of (3.1) follows immediately from the definition of $\psi_k(t)$. Since

$$\sum_{i=1}^{k_1+1}\omega_{2i-1} = \sum_{i=1}^{k_1+1}\omega_{2i-1}P_{2i-1}(1) = \phi_k(1) = 0,$$

we can rewrite the first part of (3.1) by

$$\psi_k(t) = \sum_{i=1}^{k_1}\frac{a_i}{4i+1}[Q_{2i+1}(t) - Q_{2i-1}(t)]$$

with $a_i = -(4i+1)\sum_{j=1}^{i}\omega_{2j-1}$, which leads to the first part of (3.2) by using the recurrence relation (cf. [1])

$$Q'_{l+1}(\tau) - Q'_{l-1}(\tau) = (2l+1)Q_l(\tau), \quad l = 1, 2, 3, \ldots. \tag{3.6}$$

The proof for the second parts of (3.1) and (3.2) is similar. □

Lemma 3.2 *Let $\psi_k(t)$ be defined by (2.9). Then for $\tau \in (-1, 1)$ and $m \geq 1$, we have*

$$\sum_{i=m+1}^{\infty}\left[|\psi'_k(2i+\tau)| + |\psi'_k(-2i+\tau)|\right] \leq \frac{C}{m^{1+[1+(-1)^k]/2}} \tag{3.7}$$

and

$$\sum_{i=0}^{2m}|2(m-i)+\tau|^{\alpha}|\psi'_k(2(m-i)+\tau)| \leq \begin{cases} C, & 0 \leq \alpha < 1, \\ C(\ln m)^{[1-(-1)^k]/2}, & \alpha = 1. \end{cases} \tag{3.8}$$

Proof By the classical identity [1]

$$Q_l(t) = \frac{1}{2^{l+1}}\int_{-1}^{1}\frac{(1-\tau^2)^l}{(t-\tau)^{l+1}}d\tau, \quad |t| > 1, \quad l = 0, 1, 2, \ldots, \tag{3.9}$$

we get

$$|Q_l(t)| \le \frac{C}{(|t|-1)^{l+1}}, \quad |t| > 1$$

and by (3.2),

$$|\psi_k'(t)| \le \frac{C}{(|t|-1)^{2+[1+(-1)^k]/2}}, \quad |t| \ge 2, \tag{3.10}$$

which leads to (3.7) and (3.8). The proof is complete. □

Lemma 3.3 *Assume* $s \in (x_m, x_{m+1})$ *for some* m *and let* $c_i = 2(s - x_i)/h - 1$, $0 \le i \le n-1$. *Then, we have*

$$\psi_k'(c_i) = \begin{cases} -\dfrac{2^{k-1}}{h^k} \displaystyle⨍_{x_i}^{x_{i+1}} \frac{1}{(x-s)^2} \prod_{j=0}^{k}(x - x_{ij})dx, \ i = m, \\ -\dfrac{2^{k-1}}{h^k} \displaystyle\int_{x_i}^{x_{i+1}} \frac{1}{(x-s)^2} \prod_{j=0}^{k}(x - x_{ij})dx, \ i \ne m. \end{cases} \tag{3.11}$$

Proof By the definition (1.1), we have

$$\begin{aligned} & ⨍_{x_m}^{x_{m+1}} \frac{1}{(x-s)^2} \prod_{j=0}^{k}(x - x_{mj})dx \\ &= \lim_{\varepsilon \to 0} \left\{ \left(\int_{x_m}^{s-\varepsilon} + \int_{s+\varepsilon}^{x_{m+1}} \right) \frac{1}{(x-s)^2} \prod_{j=0}^{k}(x - x_{mj})dx - \frac{2}{\varepsilon} \prod_{j=0}^{k}(s - x_{mj}) \right\} \\ &= \left(\frac{h}{2}\right)^k \lim_{\varepsilon \to 0} \left\{ \left(\int_{-1}^{c_m - \frac{2\varepsilon}{h}} + \int_{c_m + \frac{2\varepsilon}{h}}^{1} \right) \frac{\phi_k(\tau)}{(\tau - c_m)^2} d\tau - \frac{h}{\varepsilon} \phi_k(c_m) \right\} \\ &= \left(\frac{h}{2}\right)^k ⨍_{-1}^{1} \frac{\phi_k(\tau)}{(\tau - c_m)^2} d\tau = -\frac{h^k}{2^{k-1}} \psi_k'(c_m), \end{aligned}$$

where the change of variable $x = \hat{x}_m(\tau)$ has been employed. The first identity in (3.11) is then verified. The second identity can be obtained by applying the approach to the correspondent Riemann integral. □

Lemma 3.4 *Assume* $f(x) \in C^{k+1+\alpha}[a, b]$, $0 < \alpha \le 1$, $n = 2m + 1$ *and* $s = \hat{x}_m(\tau_k^\star)$ *with* $\tau_k^\star \in (-1, 1)$ *being a zero of* $\mathcal{S}_k(\tau)$. *Then, for the general (composite) Newton–Cotes rule* $\mathcal{Q}_{kn}(f)$ *defined in (2.2), it holds that*

$$|\mathcal{E}_{kn}(f)| \le Ch^{k+\alpha} \tag{3.12}$$

for even k, and

$$|\mathcal{E}_{kn}(f)| \le \begin{cases} Ch^{k+\alpha}, & 0<\alpha<1, \\ C|\ln h|h^{k+1}, & \alpha = 1 \end{cases} \tag{3.13}$$

for odd k.

Proof Let $\hat{\mathcal{F}}_{k+1,n}(x)$ be a piecewise Lagrange interpolation polynomial of degree $k+1$ which interpolates $f(x)$ on the points $\{x_{i0}, x_{i1}, \ldots, x_{ik}, \widetilde{x}_{i,k+1}\}$ at each subinterval $[x_i, x_{i+1}]$, where $\widetilde{x}_{i,k+1}$ is an additional point in (x_i, x_{i+1}), such as $\widetilde{x}_{i,k+1} = (x_{i1} + x_{i0})/2$. Then the error functional can be split into two parts,

$$\mathcal{E}_{kn}(f) = \oint_a^b \frac{f(x) - \hat{\mathcal{F}}_{k+1,n}(x)}{(x-s)^2}dx + \oint_a^b \frac{\hat{\mathcal{F}}_{k+1,n}(x) - \mathcal{F}_{kn}(x)}{(x-s)^2}dx. \tag{3.14}$$

By Theorem 2.1, the first part can be bounded by $O(h^{k+\alpha})$ since $s = \hat{x}_m(\tau_k^\star)$ and $\gamma(h,s) = (1+\tau_k^\star)/2$ or $(1-\tau_k^\star)/2$, independent of h. Thus we only need to estimate the second part. Since both $\hat{\mathcal{F}}_{k+1,n}(x)$ and $\mathcal{F}_{kn}(x)$ are the interpolation to $f(x)$ on $\{x_{ij}\}$, we have

$$\hat{\mathcal{F}}_{k+1,n}(x) - \mathcal{F}_{kn}(x) = \beta_{ki}\prod_{j=0}^{k}(x - x_{ij}), \quad x \in [x_i, x_{i+1}], \tag{3.15}$$

where β_{ki} is the leading coefficient of $\hat{\mathcal{F}}_{k+1,n}(x)$. It follows from Lemma 3.3 that

$$\begin{aligned} \oint_a^b \frac{\hat{\mathcal{F}}_{k+1,n}(x) - \mathcal{F}_{kn}(x)}{(x-s)^2}dx &= -\frac{h^k}{2^{k-1}}\sum_{i=0}^{2m}\beta_{ki}\psi_k'(2(m-i)+\tau_k^\star) \\ &:= \mathcal{I}_1 + \mathcal{I}_2 + \mathcal{I}_3, \end{aligned} \tag{3.16}$$

where

$$\mathcal{I}_1 = -\frac{f^{(k+1)}(s)h^k}{2^{k-1}(k+1)!}\sum_{i=0}^{2m}\psi_k'(2(m-i)+\tau_k^\star),$$

$$\mathcal{I}_2 = -\frac{h^k}{2^{k-1}(k+1)!}\sum_{i=0}^{2m}\left[f^{(k+1)}(\hat{x}_i(0)) - f^{(k+1)}(s)\right]\psi_k'(2(m-i)+\tau_k^\star),$$

$$\mathcal{I}_3 = -\frac{h^k}{2^{k-1}}\sum_{i=0}^{2m}\left[\beta_{ki} - \frac{f^{(k+1)}(\hat{x}_i(0))}{(k+1)!}\right]\psi_k'(2(m-i)+\tau_k^\star).$$

Now we estimate these three terms one by one. First, by noting that $\mathcal{S}_k(\tau_k^\star) = 0$ and (1.3),

$$\mathcal{I}_1 = \frac{f^{(k+1)}(s)h^k}{2^{k-1}(k+1)!} \sum_{i=m+1}^{\infty} [\psi_k'(2i + \tau_k^\star) + \psi_k'(-2i + \tau_k^\star)],$$

which is bounded by $O(h^{k+1})$ for any positive integer k by (3.7) and by noting the fact $h = O(1/m)$. Secondly, since for $f(x) \in C^{k+1+\alpha}[a, b]$, $0 < \alpha \le 1$,

$$|f^{(k+1)}(\hat{x}_i(0)) - f^{(k+1)}(s)| \le C|2(m-i) + \tau_k^\star|^\alpha h^\alpha,$$

by Lemma 3.2, when k is odd, $\mathcal{I}_2$ is bounded by $O(h^{k+\alpha})$ for $0 < \alpha < 1$ and bounded by $O(|\ln h|h^{k+1})$ for $\alpha = 1$, and when k is even, $\mathcal{I}_2$ is always bounded by $O(h^{k+\alpha})$. To estimate $\mathcal{I}_3$, it suffices, by Lemma 3.2, to show that

$$\left|\beta_{ki} - \frac{f^{(k+1)}(\hat{x}_i(0))}{(k+1)!}\right| \le Ch^\alpha, \tag{3.17}$$

where β_{ki} is defined in (3.15). From the standard Lagrange interpolation formula,

$$\begin{aligned}\hat{\mathcal{F}}_{k+1,n}(x) = &\sum_{j=0}^{k} f(x_{ij}) \frac{(x - \widetilde{x}_{i,k+1})l_{ki}(x)}{(x - x_{ij})(x_{ij} - \widetilde{x}_{i,k+1})l_{ki}'(x_{ij})} \\ &+ \frac{f(\widetilde{x}_{i,k+1})l_{ki}(x)}{l_{ki}(\widetilde{x}_{i,k+1})}, \quad x \in [x_i, x_{i+1}],\end{aligned} \tag{3.18}$$

which implies

$$\beta_{ki} = \sum_{j=0}^{k} \frac{f(x_{ij})}{(x_{ij} - \widetilde{x}_{i,k+1})l_{ki}'(x_{ij})} + \frac{f(\widetilde{x}_{i,k+1})}{l_{ki}(\widetilde{x}_{i,k+1})}. \tag{3.19}$$

Taking $f(x) = \hat{\mathcal{F}}_{k+1,n}(x) = (x - \hat{x}_i(0))^l$ in (3.18) for $0 \le l \le k+1$, we have

$$(x - \hat{x}_i(0))^l = \sum_{j=0}^{k} \frac{(x_{ij} - \hat{x}_i(0))^l (x - \widetilde{x}_{i,k+1})l_{ki}(x)}{(x - x_{ij})(x_{ij} - \widetilde{x}_{i,k+1})l_{ki}'(x_{ij})} + \frac{(\widetilde{x}_{i,k+1} - \hat{x}_i(0))^l l_{ki}(x)}{l_{ki}(\widetilde{x}_{i,k+1})}.$$

By comparing the leading coefficients on both sides, we get

$$\delta_{l,k+1} = \sum_{j=0}^{k} \frac{(x_{ij} - \hat{x}_i(0))^l}{(x_{ij} - \widetilde{x}_{i,k+1})l_{ki}'(x_{ij})} + \frac{(\widetilde{x}_{i,k+1} - \hat{x}_i(0))^l}{l_{ki}(\widetilde{x}_{i,k+1})},$$

where $\delta_{l,k+1}$ is the Kronecker delta. Moreover, by Taylor's expansion,

$$\begin{aligned} f(x_{ij}) &= \sum_{l=0}^{k} \frac{f^{(l)}(\hat{x}_i(0))}{l!}(x_{ij}-\hat{x}_i(0))^l + \frac{f^{(k+1)}(\xi_{ij})}{(k+1)!}(x_{ij}-\hat{x}_i(0))^{k+1}, \\ f(\widetilde{x}_{i,k+1}) &= \sum_{l=0}^{k} \frac{f^{(l)}(\hat{x}_i(0))}{l!}(\widetilde{x}_{i,k+1}-\hat{x}_i(0))^l + \frac{f^{(k+1)}(\hat{\xi}_i)}{(k+1)!}(\widetilde{x}_{i,k+1}-\hat{x}_i(0))^{k+1}, \end{aligned} \tag{3.20}$$

where $\xi_{ij}, \hat{\xi}_i \in (x_i, x_{i+1})$. Substituting (3.20) into (3.19), we obtain

$$\begin{aligned} \beta_{ki} - \frac{f^{(k+1)}(\hat{x}_i(0))}{(k+1)!} &= \frac{1}{(k+1)!}\sum_{j=0}^{k} \frac{(x_{ij}-\hat{x}_i(0))^{k+1}(f^{(k+1)}(\xi_{ij})-f^{(k+1)}(\hat{x}_i(0))}{(x_{ij}-\widetilde{x}_{i,k+1})l'_{ki}(x_{ij})} \\ &\quad + \frac{(\widetilde{x}_{i,k+1}-\hat{x}_i(0))^{k+1}(f^{(k+1)}(\hat{\xi}_i)-f^{(k+1)}(\hat{x}_i(0)))}{(k+1)!\,l_{ki}(\widetilde{x}_{i,k+1})}. \end{aligned} \tag{3.21}$$

Thus (3.17) follows immediately by noting $f^{(k+1)}(x) \in C^{\alpha}[a,b]$ and the proof is complete. □

The Proof of Theorem 2.2 We assume $s = \hat{x}_m(\tau_k^\star)$ with its local coordinate $\tau_k^\star$ satisfying $\mathcal{S}_k(\tau_k^\star) = 0$. If $m = 0$ or $m = n-1$, the estimates in Theorem 2.2 can be directly obtained from Theorem 2.1 by noting $\eta(s) = O(h^{-1})$. Here we only consider the case $1 \le m < n/2$ since the proof for the case $n/2 \le m < n-1$ is similar. From (2.2),

$$\mathcal{E}_{kn}(f) = \,⨎_{a}^{x_{2m+1}} \frac{f(x)-\mathcal{F}_{kn}(x)}{(x-s)^2}dx + \int_{x_{2m+1}}^{b} \frac{f(x)-\mathcal{F}_{kn}(x)}{(x-s)^2}dx. \tag{3.22}$$

The first part can be estimated by Lemma 3.4. By the standard interpolation theory,

$$|f(x)-\mathcal{F}_{kn}(x)| \le Ch^{k+1}. \tag{3.23}$$

The second part of (3.22) is bounded by

$$\begin{aligned} \left|\int_{x_{2m+1}}^{b} \frac{f(x)-\mathcal{F}_{kn}(x)}{(x-s)^2}dx\right| &\le Ch^{k+1}\int_{x_{2m+1}}^{b} \frac{1}{(x-s)^2}dx \\ &= Ch^{k+1}\left(\frac{1}{x_{2m+1}-s} - \frac{1}{b-s}\right) \le C\eta(s)h^{k+1}. \end{aligned} \tag{3.24}$$

We obtain the desired estimates and the proof is complete. □

4 The existence of superconvergence points

In the above sections we have proved that the general (composite) Newton–Cotes rule achieves its superconvergence at zeros of the function $\mathcal{S}_k(\tau)$, which is related to the derivative of the function of second kind associated with $\phi_k(x)$, a polynomial of equally-distributed zeros. Those superconvergence points listed in Table 1 are obtained by solving the equation $\mathcal{S}_k(\tau) = 0$ and can be used for practical computation. Here we prove the existence of the zeros of $\mathcal{S}_k(\tau)$ for any positive integer k.

Let $J := (-\infty, -1) \cup (-1, 1) \cup (1, +\infty)$ and the operator $\mathcal{W} : C(J) \to C(-1, 1)$ be defined by

$$\mathcal{W} f(\tau) = f(\tau) + \sum_{i=1}^{\infty} [f(2i + \tau) + f(-2i + \tau)], \quad \tau \in (-1, 1). \tag{4.1}$$

Obviously, $\mathcal{W}$ is a linear operator. By Lemma 3.1, ψ_k' is a linear combination of Q_l with $l \le k$ and therefore belongs to $C(J)$. By (1.3), we can write

$$\mathcal{S}_k(\tau) = \mathcal{W}\psi_k'(\tau). \tag{4.2}$$

Some properties of the operator $\mathcal{W}$ are summarized below.

Lemma 4.1 *Let the operator $\mathcal{W}$ be defined in (4.1) and $\tau \in (-1, 1)$. Then*

(i) $\mathcal{W} Q_0(\tau) = 0$;

(ii) *For $j > 0$ and $l \ge 0$, the differential operator $\mathcal{D}^j = d^j/d\tau^j$ and $\mathcal{W}$ are communicable with respect to function Q_l, i.e.,*

$$\mathcal{D}^j (\mathcal{W} Q_l)(\tau) = \mathcal{W}(Q_l^{(j)})(\tau); \tag{4.3}$$

(iii) *For $j > 0$,*

$$\mathcal{W}(P_1 Q_0^{(2j)})(\tau) > 0; \tag{4.4}$$

(iv) *For $j > 0$,*

$$\lim_{\tau \to 1^-} \mathcal{W} Q_{2j}(\tau) = \lim_{\tau \to -1^+} \mathcal{W} Q_{2j}(\tau) = 0. \tag{4.5}$$

Proof Since

$$Q_0(t) = \frac{1}{2} \ln \left| \frac{1+t}{1-t} \right|, \quad |t| \ne 1,$$

we have

$$\begin{aligned}\mathcal{W}Q_0(\tau) &= \frac{1}{2}\ln\frac{1+\tau}{1-\tau} + \frac{1}{2}\sum_{i=1}^{\infty}\left(\ln\frac{2i+1+\tau}{2i-1+\tau} + \ln\frac{2i-1-\tau}{2i+1-\tau}\right)\\ &= \lim_{i\to\infty}\frac{1}{2}\ln\frac{2i+1+\tau}{2i+1-\tau} = 0,\end{aligned}$$

which proves the part (i). By the classical identity [1]

$$Q_l(t) = \frac{1}{2^{l+1}}\int_{-1}^{1}\frac{(1-\tau^2)^l}{(t-\tau)^{l+1}}d\tau, \quad |x|>1, \quad l=0,1,2,\ldots,$$

we get

$$|Q_l^{(j)}(t)| \le \frac{C}{(|t|-1)^{l+1+j}}, \quad |t|>1, \quad j\ge 0$$

and the series in $\mathcal{W}Q_l(\tau)$ and $\mathcal{W}(Q_l^{(j)})(\tau)$ are convergent uniformly in any closed subset of $(-1,1)$, which implies the part (ii) with $l\ge 1$. By direct calculation,

$$\begin{aligned}\mathcal{W}(Q_0^{(j)})(\tau) &= \frac{(-1)^{j+1}(j-1)!}{2}\left\{\frac{1}{(\tau+1)^j} - \frac{1}{(\tau-1)^j} + \sum_{i=1}^{\infty}\left[\frac{1}{(2i+\tau+1)^j}\right.\right.\\ &\quad \left.\left. -\frac{1}{(2i+\tau-1)^j} + \frac{1}{(-2i+\tau+1)^j} - \frac{1}{(-2i+\tau-1)^j}\right]\right\}\\ &= \frac{(-1)^{j+1}(j-1)!}{2}\lim_{i\to\infty}\left[\frac{1}{(2i+1+\tau)^j} - \frac{1}{(-2i-1+\tau)^j}\right] = 0,\end{aligned}$$

which together with the part (i) proves the part (ii) with $l=0$. For the part (iii), since

$$\begin{aligned}P_1(t)Q_0^{(2j)}(t) &= \frac{(2j-1)!}{2}\left[\frac{1}{(t+1)^{2j}} + \frac{1}{(t-1)^{2j}}\right]\\ &\quad +(1-2j)Q_0^{(2j-1)}(t), \quad j=1,2,\ldots,\end{aligned}$$

applying the operator $\mathcal{W}$ to both sides of the above identity and using (i) and (ii), we find

$$\begin{aligned}\mathcal{W}(P_1Q_0^{(2j)})(\tau) &= \frac{(2j-1)!}{2}\left\{\frac{1}{(\tau+1)^{2j}} + \frac{1}{(\tau-1)^{2j}} + \sum_{i=1}^{\infty}\left[\frac{1}{(2i+\tau+1)^{2j}}\right.\right.\\ &\quad \left.\left. +\frac{1}{(2i+\tau-1)^{2j}} + \frac{1}{(-2i+\tau+1)^{2j}} + \frac{1}{(-2i+\tau-1)^{2j}}\right]\right\} > 0.\end{aligned}$$

Finally, we prove the part (iv). Since $P_l(t)$ and $Q_l(t)$ are the Legendre polynomial and associated function of second kind, we have the identity

$$Q_l(t) = P_l(t)Q_0(t) + f_{l-1}(t) = \frac{1}{2}\ln\left|\frac{1+t}{1-t}\right| P_l(t) + f_{l-1}(t), \quad |t| \neq 1,\ l \geq 1,$$

where $f_{l-1}(t)$ is a polynomial of degree not higher than $l-1$. Moreover,

$$\begin{aligned}
\lim_{\tau\to 1^-} \mathcal{W}Q_{2j}(\tau) &= \lim_{\tau\to 1^-}\left\{Q_{2j}(\tau) + \sum_{i=1}^{\infty}\left[Q_{2j}(2i+\tau) + Q_{2j}(-2i+\tau)\right]\right\} \\
&= \lim_{\tau\to 1^-}\left[Q_{2j}(\tau) - Q_{2j}(2-\tau)\right] \\
&= \lim_{\tau\to 1^-}\left[\frac{1}{2}\ln\frac{1+\tau}{1-\tau}P_{2j}(\tau) + f_{2j-1}(\tau) - \frac{1}{2}\right. \\
&\qquad \left.\ln\frac{3-\tau}{1-\tau}P_{2j}(2-\tau) - f_{2j-1}(2-\tau)\right] \\
&= \frac{1}{2}\lim_{\tau\to 1^-}\left[P_{2j}(2-\tau) - P_{2j}(\tau)\right]\ln(1-\tau) \\
&= \lim_{\tau\to 1^-} P_{2j}'(\xi_\tau)(1-\tau)\ln(1-\tau) = 0,
\end{aligned}$$

where $\xi_\tau \in (\tau, 2-\tau)$. By a similar argument, we reach

$$\lim_{\tau\to -1^+} \mathcal{W}Q_{2j}(\tau) = 0,$$

which concludes the proof. □

Lemma 4.2 *For* $j \geq i > 0$,

$$\mathcal{D}^{2j}(\mathcal{W}Q_{2i-1})(\tau) > 0 \tag{4.6}$$

and

$$\mathcal{D}^{2j+1}(\mathcal{W}Q_{2i})(\tau) > 0. \tag{4.7}$$

Proof Since

$$P_1(t) = t, \quad Q_1(t) = P_1(t)Q_0(t) - 1\,,$$

by Lemma 4.1 we have

$$\begin{aligned}
\mathcal{D}^{2j}(\mathcal{W}Q_1) &= \mathcal{W}(2jQ_0^{(2j-1)} + P_1Q_0^{(2j)}) = 2j\mathcal{D}^{2j-1}(\mathcal{W}Q_0) + \mathcal{W}(P_1Q_0^{(2j)}) \\
&= \mathcal{W}(P_1Q_0^{(2j)}) > 0
\end{aligned}$$

and

$$\mathcal{D}^{2j+1}(\mathcal{W}Q_2) = \mathcal{W}(Q_2^{(2j+1)} - Q_0^{(2j+1)} + Q_0^{(2j+1)}) = \mathcal{W}(3Q_1^{(2j)})$$
$$= 3\mathcal{D}^{2j}(\mathcal{W}Q_1) > 0,$$

which verifies (4.6) and (4.7) with $i = 1$. In the general case, we have

$$Q_{2i-1}^{(2j)}(t) = \sum_{k=1}^{i-1}[Q_{2k+1}^{(2j)}(t) - Q_{2k-1}^{(2j)}(t)] + Q_1^{(2j)}(t)$$
$$= \sum_{k=1}^{i-1}(4k+1)Q_{2k}^{(2j-1)}(t) + Q_1^{(2j)}(t),$$
$$Q_{2i}^{(2j+1)}(t) = \sum_{k=1}^{i}[Q_{2k}^{(2j+1)}(t) - Q_{2k-2}^{(2j+1)}(t)] + Q_0^{(2j+1)}(t)$$
$$= \sum_{k=1}^{i}(4k-1)Q_{2k-1}^{(2j)}(t) + Q_0^{(2j+1)}(t)$$

and therefore,

$$\mathcal{D}^{2j}(\mathcal{W}Q_{2i-1}) = \sum_{k=1}^{i-1}(4k+1)\mathcal{D}^{2j-1}(\mathcal{W}Q_{2k}) + \mathcal{D}^{2j}(\mathcal{W}Q_1),$$
$$\mathcal{D}^{2j+1}(\mathcal{W}Q_{2i}) = \sum_{k=1}^{i}(4k-1)\mathcal{D}^{2j}(\mathcal{W}Q_{2k-1}).$$

By the mathematical induction (4.6) and (4.7) hold for all positive integers j, i with $j \geq i$. □

Now we show the existence of superconvergence points in the following theorem.

Theorem 4.3 *For any positive integer k, the function $\mathcal{S}_k(\tau)$, defined in (1.3), has at least one zero in* $(-1, 1)$.

Proof Clearly we see from the classical orthogonal function theory that

$$Q_l(-t) = (-1)^{l+1}Q_l(t), \quad |t| \neq 1, \quad l = 0, 1, 2, \ldots$$

and moreover, by Lemma 3.1

$$\psi_k'(-t) = (-1)^{k+1}\psi_k'(t).$$

It follows from (1.3) that

$$\mathcal{S}_k(-\tau) = (-1)^{k+1}\mathcal{S}_k(\tau), \quad \tau \in (-1, 1). \tag{4.8}$$

When k is even, $\tau^\star = 0$ is a zero of the function $\mathcal{S}_k(\tau)$. Now we need to prove the case that k is odd. Let $k = 2k_1 - 1$ and $\mathcal{C}_k(\tau)$ be the function of τ, defined by

$$\mathcal{C}_k(\tau) = \mathcal{W}\psi_k(\tau). \tag{4.9}$$

On the one hand, by an argument similar to that of (4.8), we have

$$\mathcal{C}_k(-\tau) = (-1)^k\mathcal{C}_k(\tau), \tag{4.10}$$

which implies that $\mathcal{C}_k(\tau)$ vanishes at $\tau = 0$ when k is odd. By the second formula in (3.1) and the part (i) of Lemma 4.1, we obtain

$$\mathcal{C}_k(\tau) = \sum_{i=0}^{k_1} \omega_{2i}\mathcal{W}Q_{2i}(\tau) = \sum_{i=1}^{k_1} \omega_{2i}\mathcal{W}Q_{2i}(\tau), \tag{4.11}$$

which together with the part (iv) of Lemma 4.1 yields

$$\lim_{\tau\to 1^-} \mathcal{C}_k(\tau) = 0. \tag{4.12}$$

By Rolle's theorem, the first derivative of $\mathcal{C}_k(\tau)$ has at least one zero in (0, 1). On the other hand, by the part (ii) of Lemma 4.1 and by (4.2),

$$\mathcal{C}_k'(\tau) = \mathcal{S}_k(\tau). \tag{4.13}$$

As a result, $\mathcal{S}_k(\tau)$ has at least one zero in (0, 1) when k is odd. The proof is complete. □

Theorem 4.4 *Let* $\{a_i\}$ *and* $\{b_i\}$ *be defined in* (3.4)*. If* $a_i, b_i > 0$*, then* $\mathcal{S}_k(\tau)$ *has at most* $k - (-1)^k$ *distinct zeros in* $(-1, 1)$*.*

Proof For $k = 2k_1$, by Lemma 3.1, we have

$$\mathcal{S}_k(\tau) = \mathcal{W}\psi_k'(\tau) = \sum_{i=1}^{k_1} a_i\mathcal{W}Q_{2i}(\tau). \tag{4.14}$$

It follows from Lemma 4.1, Lemma 4.2 and the assumption $a_i > 0$ that

$$\mathcal{D}^{k+1}\mathcal{S}_k(\tau) = \sum_{i=1}^{k_1} a_i\mathcal{D}^{2k_1+1}(\mathcal{W}Q_{2i})(\tau) > 0. \tag{4.15}$$

Similarly, for $k = 2k_1 - 1$, we have

$$\mathcal{D}^{k+1}\mathcal{S}_k(\tau) = \sum_{i=1}^{k_1} b_i \mathcal{D}^{2k_1}(\mathcal{W}Q_{2i-1})(\tau) > 0. \tag{4.16}$$

Hence, $\mathcal{D}^{k+1}\mathcal{S}_k(\tau) > 0$ for any positive integer k, which implies that $\mathcal{S}_k(\tau)$ has at most $k+1$ distinct zeros in $[-1, 1]$. Otherwise, if $\mathcal{S}_k(\tau)$ has $k+2$ or more distinct zeros in $[-1, 1]$, by Rolle's Theorem, $\mathcal{D}^{k+1}\mathcal{S}_k(\tau)$ has at least one zero in $(-1, 1)$, which contradicts with $\mathcal{D}^{k+1}\mathcal{S}_k(\tau) > 0$.

In the case of k being even, by (4.14) and the part (iv) of Lemma 4.1, we see that

$$\lim_{\tau\to 1^-} \mathcal{S}_k(\tau) = \lim_{\tau\to -1^+} \mathcal{S}_k(\tau) = 0,$$

which shows that $\mathcal{S}_k(\tau)$ has two zeros at $\tau = \pm 1$. Thus, in this case, $\mathcal{S}_k(\tau)$ has at most $k - 1 = k - (-1)^k$ zeros in $(-1, 1)$. The proof is then complete. □

In Theorem 4.4, we have presented an upper bound for the number of the zeros of $\mathcal{S}_k(\tau)$ when $\psi_k'(t)$ is a positive linear combination of $Q_i(t)(1 \le i \le k)$. Our numerical test shows that the condition

$$a_i, b_i > 0$$

always holds for any positive k, although we cannot provide a theoretical analysis. We list in Table 2 the numbers of zeros of $\mathcal{S}_k(\tau)$, denoted by N_k, until $k = 15$. We see that for $k \le 15$, the upper bound given by Theorem 4.4 is reached except for the three cases where $k = 11, 13, 15$. As an example, we present the graph of the function $\mathcal{S}_{11}(\tau)$ in Fig. 1 where $\mathcal{S}_{11}(\tau)$ has been truncated when the absolute value is larger than $5.0E-3$. One can see from Fig. 1 that $\mathcal{S}_{11}(\tau)$ has only eight distinct zeros in $(-1, 1)$. The graph of the function $\mathcal{S}_{15}(\tau)$ is shown in Fig. 2 where we can see that $\mathcal{S}_{15}(\tau)$ has only four zeros.

It has been proved theoretically in [23,25] that the numbers of the superconvergence points in the trapezoidal rule ($k = 1$) and Simpson's rule ($k = 2$) reach the upper bounds. Theoretical analysis for the familiar Simpson's 3/8 rule ($k = 3$) is given below.

Theorem 4.5 $\mathcal{S}_3(\tau)$ *has only four distinct zeros in* $(-1, 1)$.

Proof A straightforward calculation gives

$$\psi_4'(t) = \frac{8}{5}Q_3(t) + \frac{8}{45}Q_1(t)\,.$$

By Theorem 4.4, $\mathcal{S}_3(\tau)$ has at most four distinct zeros in $(-1, 1)$. Note that

$$|\phi_3(\tau)| = |(\tau^2 - 1/9)(\tau^2 - 1)| \le \frac{16}{81}, \quad \tau \in (-1, 1).$$

Table 2 The number of zeros of $\mathcal{S}_k(\tau)$

k	N_k	$k-(-1)^k$	Upper bound reached or not
1	2	2	Y
2	1	1	Y
3	4	4	Y
4	3	3	Y
5	6	6	Y
6	5	5	Y
7	8	8	Y
8	7	7	Y
9	10	10	Y
10	9	9	Y
11	8	12	N
12	11	11	Y
13	8	14	N
14	13	13	Y
15	4	16	N

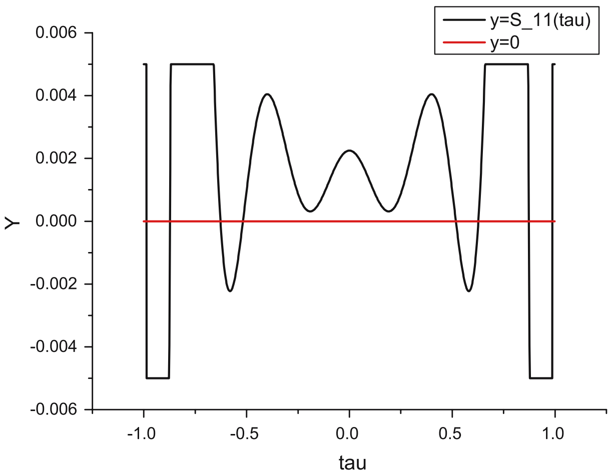

Fig. 1 The function $\mathcal{S}_{11}(\tau)$ in $(-1, 1)$

By an argument similar to that for (3.7), we obtain

$$\sum_{i=1}^{\infty}\left[|\psi_3'(2i+\tau)|+|\psi_3'(-2i+\tau)|\right] \leq \frac{16}{81(1-\tau^2)}, \quad \tau \in (-1, 1)$$

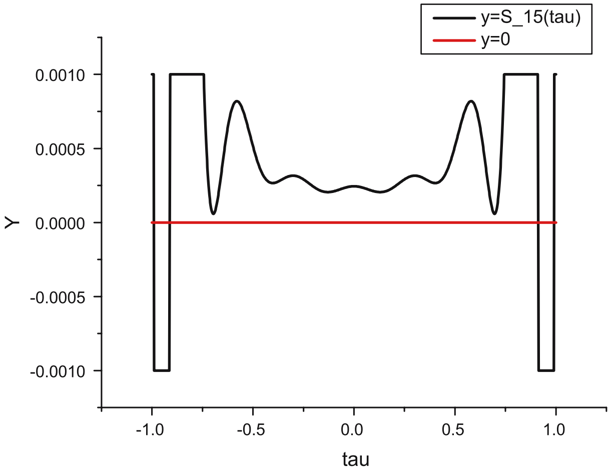

Fig. 2 The function $\mathcal{S}_{15}(\tau)$ in $(-1, 1)$

and therefore,

$$\mathcal{S}_3(0) = \psi_3'(0) + \sum_{i=1}^{\infty}[\psi_3'(2i) + \psi_3'(-2i)] \geq \psi_3'(0) - \frac{16}{81} = \frac{56}{81} > 0$$

and

$$\mathcal{S}_3\left(\frac{1}{2}\right) = \psi_3'\left(\frac{1}{2}\right) + \sum_{i=1}^{\infty}[\psi_3'(2i + \frac{1}{2}) + \psi_3'(-2i + \frac{1}{2})] \leq \psi_3'\left(\frac{1}{2}\right) + \frac{64}{243} < 0.$$

Also note that $\lim_{\tau \to 1^-} \mathcal{S}_3(\tau) = +\infty$. Thus, $\mathcal{S}_3(\tau)$ has two distinct zeros in $(0, 1)$ and by (4.8), $\mathcal{S}_3(\tau)$ has another two zeros in $(-1, 0)$, which completes the proof. □

5 Numerical examples

In this section, we present some numerical examples to confirm our theoretical analysis given in the above sections.

Example 5.1 First we consider the finite-part integral

$$\int_0^1 \frac{x^6}{(x-s)^2} dx, \quad s \in (0, \ 1) . \tag{5.1}$$

By (1.1), the exact solution is

$$\frac{6}{5} + \frac{3}{2}s + 2s^2 + 3s^3 + 6s^4 + \frac{1}{s-1} + 6s^5 \ln \frac{1-s}{s}.$$

Table 3 The error of $\mathcal{Q}_{3n}(f)$ and $\mathcal{Q}_{4n}(f)$ for evaluating (5.1) at $s = x_{[n/2]} + (\tau + 1)h/2$

	$\mathcal{Q}_{3n}(f)$			$\mathcal{Q}_{4n}(f)$		
n	$\tau = 0$	$\tau = \tau_{31}^{\star}$	$\tau = \tau_{32}^{\star}$	$\tau = 1/3$	$\tau = \tau_{41}^{\star}$	$\tau = \tau_{42}^{\star}$
4	2.17425E-02	7.02034E-03	1.29230E-03	8.38864E-04	1.87756E-05	1.33747E-05
8	2.21968E-03	4.36954E-04	7.01566E-05	4.70630E-05	6.64231E-07	5.25892E-07
16	2.47923E-04	2.72741E-05	4.10229E-06	2.78098E-06	2.18362E-08	1.76760E-08
32	2.92062E-05	1.70385E-06	2.48203E-07	1.68865E-07	6.98379E-10	5.69424E-10
64	3.54134E-06	1.06472E-07	1.52658E-08	1.04003E-08	2.20682E-11	1.80458E-11
h^{α}	3.044	4.000	4.023	4.021	4.984	4.980

Table 4 The error of $\mathcal{Q}_{3n}(f)$ and $\mathcal{Q}_{4n}(f)$ for evaluating (5.1) at $s = x_{n-1} + (\tau + 1)h/2$

	$\mathcal{Q}_{3n}(f)$			$\mathcal{Q}_{4n}(f)$		
n	$\tau = 0$	$\tau = \tau_{31}^{\star}$	$\tau = \tau_{32}^{\star}$	$\tau = 1/3$	$\tau = \tau_{41}^{\star}$	$\tau = \tau_{42}^{\star}$
4	4.05885E-02	1.14342E-02	3.19438E-03	1.23661E-03	6.74612E-06	1.60185E-05
8	5.86964E-03	9.07630E-04	3.20220E-04	8.24835E-05	1.95370E-07	1.42385E-06
16	7.87763E-04	6.91079E-05	3.25084E-05	5.31798E-06	3.19862E-08	1.02629E-07
32	1.02024E-04	5.45507E-06	3.47064E-06	3.37465E-07	2.62336E-09	6.84634E-09
64	1.29830E-05	4.67528E-07	3.90471E-07	2.12508E-08	1.83556E-10	4.41483E-10
h^{α}	2.974	3.544	3.152	3.989	3.837	3.955

We use the quadrature rules $\mathcal{Q}_{3n}(f)$ and $\mathcal{Q}_{4n}(f)$ defined by (2.2) to compute the approximate value of (5.1), respectively. The error $|\mathcal{E}_{3n}(f)|$ at $s = x_{[n/2]} + (\tau+1)h/2$ with $\tau = 0, \tau_{31}^{\star}, \tau_{32}^{\star}$ is presented in the left half of Table 3. The error $|\mathcal{E}_{4n}(f)|$ at $s = x_{[n/2]} + (\tau + 1)h/2$ with $\tau = 1/3, \tau_{41}^{\star}, \tau_{42}^{\star}$ is presented in the right half of Table 3. Here $\tau = 0$ is not a superconvergence point for $|\mathcal{E}_{3n}(f)|$ and $\tau = 1/3$ is not a superconvergence point for $|\mathcal{E}_{4n}(f)|$, while $\tau = \tau_{31}^{\star}, \tau_{32}^{\star}$ and $\tau = \tau_{41}^{\star}, \tau_{42}^{\star}$ are superconvergence points in $(-1, 0]$ for the quadrature rules $|\mathcal{E}_{3n}(f)|$ and $|\mathcal{E}_{4n}(f)|$, respectively, as given in Table 1. Numerical estimates of the convergence order are given in the last row, which are calculated from the last two meshes. Clearly the convergence orders at superconvergence points are $O(h^4)$ and $O(h^5)$, respectively, one order higher than those at non-superconvergence points, which confirms our theoretical analysis in Theorem 2.2. Numerical results at $s = x_{n-1} + (\tau + 1)h/2$ are given in Table 4. One can see that at all three points, the convergence order of $|\mathcal{E}_{3n}(f)|$ is about $O(h^3)$ and the convergence order of $|\mathcal{E}_{4n}(f)|$ is $O(h^4)$, which coincides with our theoretical analysis since $\eta(s) = O(h^{-1})$ in this case.

Example 5.2 Secondly we consider an example with less regularity. Let $f(x) = x^4 + |x|^{4+\alpha}$, $0 < \alpha \leq 1$, $a = -1$, $b = 1$ and $s = 0$. In this case, $f(x) \in C^{4+\alpha}[-1, 1]$ and the exact value of the finite-part integral (1.1) is $(12 + 2\alpha)/(9 + 3\alpha)$. We still

Table 5 The error of $\mathcal{Q}_{3n}(f)$ for approximating $\ne\!\!\!\!\int_{-1}^{1}(x^4+|x|^{4+\alpha})x^{-2}dx$

n	Mesh I $\alpha=1/3$	Mesh I $\alpha=1/2$	Mesh II $\alpha=1/3$	Mesh II $\alpha=1/2$
5	2.52030E-02	2.34679E-02	2.45329E-03	3.23223E-03
11	2.17520E-03	1.97976E-03	1.04782E-04	1.36483E-04
23	2.21515E-04	1.99574E-04	9.07829E-06	1.05391E-05
47	2.44334E-05	2.19794E-05	8.75597E-07	9.06468E-07
95	2.81169E-06	2.54003E-06	8.66786E-08	8.03395E-08
h^α	3.119	3.113	3.337	3.496

Table 6 The error of $\mathcal{Q}_{3n}(f)$ for approximating $\ne\!\!\!\!\int_{-1}^{1}(x^4+|x|^{3+\alpha})x^{-2}dx$

n	Mesh I $\alpha=1/3$	Mesh I $\alpha=1/2$	Mesh II $\alpha=1/3$	Mesh II $\alpha=1/2$
5	5.11454E-02	4.42643E-02	1.71760E-03	1.47647E-03
11	7.15172E-03	5.49088E-03	3.49950E-04	2.82703E-04
23	1.18016E-03	7.99203E-04	6.24167E-05	4.46119E-05
47	2.11485E-04	1.25997E-04	1.17377E-05	7.43705E-06
95	3.96034E-05	2.07507E-05	2.26922E-06	1.27763E-06
h^α	2.417	2.602	2.371	2.541

use quadrature rule $\mathcal{Q}_{3n}(f)$. Here two meshes strategies, denotes by Mesh I and Mesh II, respectively, are adopted. In the first, s is always placed at the midpoint of some subinterval, non-superconvergence point, and in the second, s is placed at the superconvergence point $\tau_{31}^{\star}$ same as used in Example 5.1. Both meshes are uniform except two subintervals near the ending points. Numerical results are presented in Table 5. One can see that the convergence orders in Mesh I and II are $O(h^3)$ and $O(h^{3+\alpha})$, respectively, which is in good agreement with our theoretical analysis.

Example 5.3 Finally, we consider an example in which $f(x)=x^4+|x|^{3+\alpha}(0<\alpha\le 1)$, $a=-1$ and $b=1$. In this case, $f(x)\in C^{3+\alpha}[-1,\ 1]$ and the exact value of the finite-part integral (1.1) is $(10+2\alpha)/(6+3\alpha)$. Here we use the same meshes and singular point s as in Example 5.2. Numerical results are given in Table 6. We find that the superconvergence phenomenon disappears since the convergence rates are about $O(h^{2+\alpha})$ in all four cases, which implies that the assumption on the regularity of $f(x)$ in Theorem 2.2 cannot be weakened.

6 Concluding remarks

We have shown both theoretically and numerically the superconvergence of the general (composite) Newton–Cotes rules for the evaluation of Hadamard finite-part integrals. The convergence rate at the superconvergence points is one order higher than the global convergence rate. In this paper the (composite) Newton–Cotes rules are obtained by replacing the integrand function $f(x)$ with its piecewise Lagrange interpolation. According to (1.2), these Newton–Cotes rules can also be obtained by differentiating with respect to s the corresponding (composite) Newton–Cotes rules for Cauchy principle value integrals. Moreover, it is possible to extend the approach in this paper to the Cauchy principal value integral to obtain certain superconvergence result.

The superconvergence phenomenon has been extensively studied for solving partial differential equations and singular integral equations by finite element method and collocation method, see e.g., [4, 8, 14, 18]. The former gives a solution of a higher-order accuracy at certain superconvergence points and the latter produces a solution with a higher-order accuracy when some special points are used as collocation points. A popular approach is the spectral method with Gaussian type collocation points, which has been used for both partial different equations and singular integral equations. The results in this paper show a possible way to improve the accuracy of the collocation method for Hadamard finite-part integral equations by choosing the superconvergence points to be the collocation points. A collocation method based on the Simpson's rule and its superconvergence points was used in [24] to solve an integral equation of Hadamard kernel. Numerical results show that the method is of higher-order accuracy. However, no theoretical analysis has been done.

In some practical applications, the integrand function in (1.1) is given by $f(x) = w(x)g(x)$ where $w(x)$ is a weight function which may have certain kind of singularities at the endpoints a and b. In this case, Gaussian quadrature rules may have advantages due to the nature of the weight function of orthogonal polynomials.

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