Conditional Stability and Regularization for Backward Heat Problem

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ABSTRACT

Consider the 1-D backward heat equation for the temperature field $u(x;t)$ with the Robin boundary:

$$\begin{cases} \frac{\partial u}{\partial t} = -\frac{\partial^2 u}{\partial x^2}, & (x,t) \in (0,1) \times (0,T) \\ -u_x(0,t) + hu(0,t) = 0, & t \in (0,T) \\ u_x(1,t) + Hu(1,t) = 0, & t \in (0,T) \\ u(x,0) = g_0(x), & x \in (0,1). \end{cases} \tag{0.1}$$

For some $g_0(x)$ given appropriately, there exists unique solution to (0.1). It is well-known that (0.1) is a ill-posed problem. That is, a little error in the initial data may lead to the nonexistence to (0.1). Also, the solution does not depend continuously on the initial data.

Assume the initial data is not given on the exact time $t = 0$, but given on some curve $t = \varepsilon f(x)$ with $|f(x)| < 1$ and $\varepsilon > 0$ small enough. Then we are led to the following problem:

$$\begin{cases} \frac{\partial v}{\partial t} = -\frac{\partial^2 v}{\partial x^2}, & (x,t) \in (0,1) \times (\varepsilon f(x),T) \\ -v_x(0,t) + hv(0,t) = 0, & t \in (\varepsilon f(0),T) \\ v_x(1,t) + Hv(1,t) = 0, & t \in (\varepsilon f(1),T) \\ v(x,\varepsilon f(x)) = \tilde{g}_0(x), & x \in (0,1). \end{cases} \tag{0.2}$$

The solution to (0.2) may not exist, provided $\tilde{g}_0(x) \neq u(x,\varepsilon f(x))$. The aim of this paper is to establish a stable method to approximate $u(x;t)$ in the domain $(x,t) \in (0,1) \times (\varepsilon,T)$ by $v(x;t)$, with the approximate initial data $\tilde{g}_0(x)$ given at $t = \varepsilon f(x)$.

Motivated by Ames and Payne's work in 1994, we first establish a conditional stability result for (0.2) with the existence assumption on $v(x;t)$, which means we can estimate $\|v(\cdot,t)\|_{L^2(0,1)}$ for $t \in (\varepsilon,T)$ by ε and $\tilde{g}_0(x)$. Based on this stability estimate, we propose a new strategy for the choice of regularizing parameter α to get the approximate regularized solution $\tilde{v}(x;t)$ to (0.2) for general $\tilde{g}_0(x)$. Then $u(x;t)$ is obtained from the measured data $\tilde{g}_0(x)$ approximately. The error analysis for $u(x;t) - \tilde{v}(x;t)$ is also given.

Our results generalize Ames and Payne's work and propose a new stable scheme to solve (0.1) with the noisy initial data. Also the method proposed in this paper can be applied to tackle other higher-dimensional backward heat problems.