Direct Representation of Sources in the Poisson Equation in terms of the Surface Integral of the Boundary Data

Takaaki Nara and Shigeru Ando
The University of Tokyo

Let us consider the three-dimensional Poisson equation with point sources and a boundary condition on the unit sphere surface $\partial\Omega$:

$$\Delta V = -\sum_{i=1}^{N} q_i \delta(r - r_i, \theta - \theta_i, \phi - \phi_i), \qquad \frac{\partial V}{\partial r}\Big|_{r=1} = 0. \tag{1}$$

To estimate the source positions (r_i, θ_i, ϕ_i) and source strength q_i from boundary data of V on $\partial\Omega$ is our source inverse problem.

Via multipole expansion of V, the relation between the surface integral of the boundary data and the source parameters is obtained as follows:

$$c_n \equiv n \int_{\partial\Omega} V(\theta, \phi) \xi^n \cdot \sin\theta \, d\theta d\phi = \sum_{i=1}^{N} q_i \zeta_i^n, \tag{2}$$

where $\zeta_i \equiv r_i \cdot \sin\theta_i \cdot \mathrm{e}^{j\phi_i}$ which represent the projected position of the i-th source onto the $x-y$ plane, and $\xi \equiv \sin\theta \cdot \mathrm{e}^{j\phi}$. (2) is equivalent to the equation used in [1]. We can estimate $q_i, (r_i, \theta_i, \phi_i)$ $(i = 1, 2, ..., N)$ and N by the following algorithm:

Algorithm Assume that there are N different projected source positions $\zeta_1, ..., \zeta_N$ on the $x-y$ plane, $\zeta_1, ..., \zeta_N$ are the solutions of the following equation of N-th degree:

$$x^N - K_1 x^{N-1} + ... + (-1)^N K_N = 0, \tag{3}$$

$$\text{where} \quad \begin{pmatrix} K_N \\ \vdots \\ K_1 \end{pmatrix} = \begin{pmatrix} (-1)^{N+1} c_0 & \dots & -c_{N-2} & c_{N-1} \\ (-1)^{N+1} c_1 & \dots & -c_{N-1} & c_N \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^{N+1} c_{N-1} & \dots & -c_{2N-3} & c_{2N-2} \end{pmatrix}^{-1} \begin{pmatrix} c_N \\ \vdots \\ c_{2N-1} \end{pmatrix}. \tag{4}$$

Thus, the three-dimensional positions are obtained by projection onto two orthogonal planes. The strength of the sources are obtained from

$$\begin{pmatrix} q_1 \\ \vdots \\ q_N \end{pmatrix} = \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ \zeta_1^{N-1} & \dots & \zeta_N^{N-1} \end{pmatrix}^{-1} \begin{pmatrix} c_0 \\ \vdots \\ c_{N-1} \end{pmatrix}. \tag{5}$$

To examine that the number of the sources is exactly N, we should only check whether

$$c_{2N} = \sum_{i=1}^{N} q_i \zeta_i^{2N} \tag{6}$$

holds or not. If (6) holds, the number of the source is exactly N. If (6) does not hold, we reassume that the number of the source is $N+1$ and repeat (4), (5), and (6). By continuing this procedure, we can finally obtain the real source number and source parameters.

References

[1] A.El.Badia et.al., "An inverse source problem in potential analysis", Inverse Problems, Vol.16, No.3(2000), pp.651-653.