Coefficient Identification of the Wave Equation Using the Alternating Directions Method

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In this study, we consider the problem of coefficient identification of the scalar wave equation. This problem is to determine the space-dependent unknown coefficient by means of the knowledge of simultaneous Dirichlet and Neumann data.

Let $\Omega \subset \mathbb{R}^n$ $(n = 2, 3)$ be a bounded domain with smooth boundary. A conventional problem is to find the function u such that

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} - \nabla \cdot (K \nabla u) = 0 & \text{in } \Omega \times (0, T]; \\ u = \dfrac{\partial u}{\partial t} = 0 & \text{on } \Omega \times \{0\}; \\ u = \overline{u} & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{1}$$

Here we assume that the coefficient function K belongs to $L^\infty(\Omega)$ and satisfies the condition $K(x) \geq C > 0$ for all $x \in \Omega$, where C is given positive constant. Our inverse problem is to determine the unknown coefficient function $K(x)$ with the knowledge of the Dirichlet data $\overline{u}$ and the Neumann data $\overline{q} := K \dfrac{\partial \tilde{u}}{\partial n}\Big|_{\partial\Omega \times (0,T)}$. The uniqueness and stability of this problem were guaranteed under the appropriate assumptions.

The purpose of this paper is to present an algorithm for the numerical resolution of our inverse problem. To determine the unknown coefficient function K, we adopt the direct variational method. The unknown coefficient function K is determined by minimizing the functional $F : L^\infty(\Omega) \to \mathbb{R}_+ := [0, +\infty)$, defined by

$$F(K) = \int_0^T \int_\Omega |K^{\frac{1}{2}} \nabla u - K^{-\frac{1}{2}} \sigma|^2 \, dx dt; \tag{2}$$

where $\sigma := K \nabla v$ with the solution v of the problem

$$\begin{cases} \dfrac{\partial^2 v}{\partial t^2} - \nabla \cdot (K \nabla v) = 0 & \text{in } \Omega \times (0, T]; \\ v = \dfrac{\partial v}{\partial t} = 0 & \text{on } \Omega \times \{0\}; \\ K \dfrac{\partial v}{\partial n} = \overline{q} & \text{on } \partial\Omega \times (0, T). \end{cases} \tag{3}$$

To find the minimum K, we make use of the alternating directions method presented by Kohn and Vogelius[1] for the impedance computed tomography. Their method consists of solving two boundary value problems and minimizing the functional alternately. For fixed u and σ, we notice that our functional (2) is minimized at $K = \sqrt{\int_0^T |\sigma(\cdot, t)|^2 dt / \int_0^T |\nabla u(\cdot, t)|^2 dt}$.

To find the unknown coefficient function, we summarize the following algorithm:

<u>Numerical algorithm</u>

1. Given an initial coefficient function K_0.
2. For $i = 0, 1, 2, \cdots\cdots$;
   (a) Solve the Dirichlet problem (1) with the coefficient K_i to find ∇u.
   (b) Solve the Neumann problem (3) with the coefficient K_i to find $\sigma = K_i \nabla v$
   (c) Update the coefficient function by $K_{i+1} = \sqrt{\dfrac{\int_0^T |\sigma(\cdot, t)|^2 dt}{\int_0^T |\nabla u(\cdot, t)|^2 dt}}$.

In the talk, we will show the efficacy of our algorithm by numerical experiments.

References

[1] R.Kohn and M.Vogelius, Relaxation of variational method for impedance computed tomography, Communications on Pure and Applied Mathematics 40, pp. 745–777, 1987.

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