Formulas for Reconstructing Conductivity and its Normal Derivative at the Boundary from the Localized Dirichlet to Neumann Map

Kazumi TANUMA[*][†]

Let $\Omega \in \mathbb{R}^n$ $(n \geq 2)$ be a bounded domain with Lipschitz boundary $\partial\Omega$. Physically Ω is considered as an isotropic, static and conductive medium with conductivity $\gamma \in L^\infty(\Omega)$. When an electric potential $f \in H^{1/2}(\partial\Omega)$ is applied to the boundary $\partial\Omega$, the potential u solves the Dirichlet problem

$$\nabla \cdot (\gamma \nabla u) = 0 \quad \text{in} \quad \Omega; \qquad u|_{\partial\Omega} = f. \tag{1}$$

Assume that there is a constant $\delta > 0$ such that $\gamma(x) \geq \delta$ (a.e. $x \in \Omega$). Then, there exists a unique weak solution $u \in H^1(\Omega)$ to (1). Define the Dirichlet to Neumann map $\Lambda_\gamma : H^{1/2}(\partial\Omega) \longrightarrow H^{-1/2}(\partial\Omega)$ by

$$< \Lambda_\gamma f, g > = \int_\Omega \gamma \nabla u \cdot \nabla v \, dx \quad (g \in H^{1/2}(\partial\Omega)),$$

where u is the solution to (1), v is any $v \in H^1(\Omega)$ satisfying $v|_{\partial\Omega} = g$ and $< \, ; \, >$ is the bilinear pairing between $H^{1/2}(\partial\Omega)$ and $H^{-1/2}(\partial\Omega)$. Note that $\Lambda_\gamma f = \gamma \nabla u \cdot n$ when $f \in H^{3/2}(\partial\Omega)$, $\gamma \in C^1(\overline{\Omega})$ and $\partial\Omega$ is C^2, where n is the unit outer normal to $\partial\Omega$. Hence $\Lambda_\gamma f$ is the current flux across $\partial\Omega$ produced by the potential f on $\partial\Omega$.

The problem of determining conductivity of the medium from the measurements of the electric potential on the boundary and the corresponding current flux across the boundary is expressed as Inverse Problem "Determine $\gamma(x)$ from Λ_γ". Since this problem was posed by A.P.Calderon, many results on uniqueness, stability, reconstruction have been proved. Some of the previous works on reconstrusction can be seen in [B],[N],[NT1],[SU].

In this talk, we give three kinds of formulas for reconstructing conductivity and its normal derivative from the localized Dirichlet to Neumann map. These results have been obtained in the joint work with Gen Nakamura.

Assume that $\partial\Omega$ is flat at around $x = 0 \in \partial\Omega$ and that $\Omega, \partial\Omega$ are given by

$$\Omega = \{x_n > 0\}; \quad \partial\Omega = \{x_n = 0\}$$

---

[*] Division of Mathematical Sciences, Osaka Kyoiku University, Japan.

[†] Department of Mathematics, Faculty of Engineering, Gunma University, Japan (from January, 2002).

locally around $x = 0$, where $x = (x', x_n) = (x_1, \cdots, x_{n-1}, x_n)$.

Let $t = (t', 0) = (t_1, \cdots, t_{n-1}, 0)$ be any unit tangent to $\partial\Omega$ at $x = 0$.

Theorem 1 (Pointwise Reconstruction ([NT2])). Soppose that $D^{\alpha'}_{x'} D^{\alpha_n}_{x_n} \gamma$ is continuous around $x = 0$ for any multi-index (α', α_n) such that $|\alpha'| + 2\alpha_n \le 2$. Letting $\eta(x') \in C^4_0(\mathbb{R}^{n-1})$ satisfy

$$0 \le \eta \le 1, \quad \int_{\mathbb{R}^{n-1}} \eta^2 \, dx' = 1, \quad \operatorname{supp} \eta \subset \{|x'| < 1\},$$

we take

$$\varphi_N(x') = e^{\sqrt{-1} N x' \cdot t'} \eta(\sqrt{N} x'), \qquad \psi_N(x') = e^{\sqrt{-1} \frac{N}{2} x' \cdot t'} \eta(\sqrt{N} x')$$

for any positive integer N. Then,

$$\begin{aligned} &\lim_{N \to \infty} N^{\frac{n-1}{2}} \left[4 < \Lambda_\gamma \psi_N, \overline{\psi_N} > - 2 < \Lambda_\gamma \varphi_N, \overline{\varphi_N} > \right] \\ &= \frac{\partial}{\partial x_n} \gamma(0) + 3\gamma(0) \int_{\mathbb{R}^{n-1}} \left[|\nabla_{x'} \eta|^2 - (t' \cdot \nabla_{x'} \eta)^2 \right] dx'. \end{aligned} \tag{2}$$

Hereafter we assume that $D^{\alpha'}_{x'} D^{\alpha_n}_{x_n} \gamma$ is continuous around $x = 0$ for any multi-index (α', α_n) such that $|\alpha'| + \alpha_n \le 3, \alpha_n \le 1$. Also we take $\eta(x')$ to be any function in $C^3_0(\mathbb{R}^{n-1})$ compactly supported in a neighborhood of $x' = 0$ and put

$$\varphi_N(x') = e^{\sqrt{-1} N x' \cdot t'} \eta(x'), \qquad \psi_N(x') = e^{\sqrt{-1} \frac{N}{2} x' \cdot t'} \eta(x').$$

Theorem 2 (Reconstruction in a Weak Form).

$$\lim_{N \to \infty} \left[4 < \Lambda_\gamma \psi_N, \overline{\psi_N} > - 2 < \Lambda_\gamma \varphi_N, \overline{\varphi_N} > \right] = \int_{\mathbb{R}^{n-1}} \frac{\partial \gamma}{\partial x_n}(x', 0) \, \eta^2(x') \, dx'. \tag{3}$$

Theorem 3 (Reconstruction of Fourier transform of conductivity). Let $\omega' \in \mathbb{R}^{n-1}$. Then

$$\begin{aligned} &\lim_{N \to \infty} \left[4 < \Lambda_\gamma (\psi_N e^{\sqrt{-1} x' \cdot \omega'}), \overline{\psi_N} > - 2 < \Lambda_\gamma (\varphi_N e^{\sqrt{-1} x' \cdot \omega'}), \overline{\varphi_N} > \right] \\ &= \int_{\mathbb{R}^{n-1}} \eta^2(x') \frac{\partial \gamma}{\partial x_n}(x', 0) \, e^{\sqrt{-1} x' \cdot \omega'} dx' \\ &\quad + (t' \cdot \omega') \int_{\mathbb{R}^{n-1}} \eta^2(x') \gamma(x', 0) \, e^{\sqrt{-1} x' \cdot \omega'} dx'. \end{aligned} \tag{4}$$

Our standpoint is to reconstruct normal derivative of the conductivity directly from the localized Dirichlet to Neumann map. In fact, the supports of φ_N and ψ_N in Theorem 1 and those in Theorem 2,3 are in the neighborhoods of $x = 0$. Hence all the Dirichlet to Neumann maps are localized around $x = 0$. Moreover, in the formulas (2) and (4), the factors $\int_{\mathbb{R}^{n-1}} \left[|\nabla \eta|^2 - (t' \cdot \nabla \eta)^2 \right] dx'$ and $(t' \cdot \omega')$ are controllable (except the case $n = 2$), that is, these factors are determined explicitly from the Dirichlet data. Then from (2) we obtain a 2×2

system of equations which can be solved for $\gamma(0)$ and $\frac{\partial\gamma}{\partial x_n}(0)$ simultaneously. In (4), for given $\omega' \in \mathbb{R}^{n-1}$ we may take $t' \in \mathbb{R}^{n-1}$ so that $t' \cdot \omega' = 0$ $(n > 2)$. Moreover in [B], [N] and [NT1], to recover the normal derivative of γ at $x = 0$, one needs to know not only the value $\gamma(0)$ but also all the values of γ in a neighborhood of $x = 0$ on $\partial\Omega$. Our formulas here need not any information of γ but only some regularity assumption on γ around $x = 0$. Recently [KY] has overcome this point in their inductive reconstruction. There are results on reconstruction of elastic tensor for the isotropic and anisotropic elasticity from the localized Dirichlet to Neumann map ([R],[NT3]).

References

[B] R. M. Brown, Recovering the conductivity at the boundary from the Dirichlet to Neumann map: a pointwise result (preprint)

[KY] H. Kang and K. Yun, Boundary determination of conductivities and Riemannian metrics via local Dirichlet-to-Neumann operator (preprint)

[N] A. I. Nachman, Global uniqueness for a two dimensional inverse boundary value problem, Ann. of Math., 142(1995)71-96.

[NT1] G. Nakamura and K. Tanuma, Local determination of conductivity at the boundary from Dirichlet to Neumann map, Inverse Problems 17 (2001)405-419

[NT2] G. Nakamura and K. Tanuma, Direct determination of the derivatives of conductivity at the boundary from the localized Dirichlet to Neumann map, Comm. Korean Math. Soc.16 (2001)415-425

[NT3] G. Nakamura and K. Tanuma, Reconstruction of elastic tensor of anisotropic elasticity at the boundary from the localized Dirichlet to Neumann map (preprint)

[R] R. L. Robertson, Boundary identifiability of residual stress via the Dirichlet to Neumann map, Inverse Problems 13 (1997)1107-1119.

[SU] J. Sylvester and G. Uhlmann, Inverse boundary value problem at the boundary-continuous dependence, Comm. Pure Appl. Math., 61 (1988)197-219.