Inverse problems for nonstationary partial differential equations with a finite number of mesurements

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Abstract: We discuss inverse problems of determining coefficients in evolutionary partial differential equations by spatially restricted measurements. As evolutionary systems, we will consider

parabolic equation
hyperbolic equation
isotropic Lamé equation
Maxwell’s equations.

We will exclusively treat the inverse problems by a finite number of measurements. In order to state the inverse problem, we take a typical formulation for an inverse hyperbolic problem.

In a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$, with $\partial\Omega \in C^2$, we consider a hyperbolic equation with fixed initial and boundary conditions:

$$\partial_t^2 u(t,x) - \nabla(p(x)\nabla u(t,x)) = 0, \qquad 0 < t < T, x \in \Omega.$$

$$u(0,x) = a(x), \qquad \partial_t u(0,x) = 0, \qquad x \in \Omega$$

$$u(t,x) = b(t,x), \qquad 0 < t < T, x \in \partial\Omega.$$

Let $\omega \subset \Omega$ be a suitable subdomain of Ω, Γ a suitable subboundary of $\partial\Omega$, and $T > 0$ be given. Now we can formulate our inverse problem with finite measurements as follows: Determine $p = p(x)$ in (1) from measurements

$$u(t,x), \qquad 0 < t < T, \quad x \in \omega$$

or

$$\frac{\partial u}{\partial \nu}(t,x), \qquad 0 < t < T, x \in \partial\Omega.$$

Here $\nu = \nu(x)$ is the unit outward normal vector to $\partial\Omega$ at x, and $x \cdot x'$ denotes the scalar product in $\mathbb{R}^n$. We set $\frac{\partial u}{\partial \nu} = \nabla \cdot \nu$.

We are interested in two theoretical aspects, that is,

the uniqueness.
the stability.

For the inverse problem (1) - (4), we can more precisely state these two topics:

Is the map $p \mapsto u|_{(0,T)\times\gamma}$ injective? Which are conditions on $\gamma \subset \partial\Omega$ and $T > 0$ guaranteeing the injectiveness?

Let $u(p)$ be a solution in a suitable class to (1) - (3) associated with a coefficient p. Then can we estimate $p - q$ by $u(p) - u(q)|_{(0,T)\times\gamma}$? More precisely, which topologies should we choose for the stability? Moreover, which conditions are necessary for restoring stability if unconditional stability fails?

For this kind of inverse problems, Bukhgeim and Klibanov [2] proposed a methodology based on a Carleman estimate and has been studied by many authors. See also Bukhgeim [1], Isakov [5], Klibanov [6] and the references therein.

The main purpose of this talk is to show recent results on these uniqueness and the stability for the above evolutionary systems. However we do not intend to give a complete survey and centre on the results which have been obtained by the author and his colleagues.

We would like also to underline the motivation:

Why should we discuss the uniqueness and the stability?

Of course, we could insist that we would be interested in the uniqueness and the stability because of mathematical interests (In particular, the uniqueness problem is closely related with the classical unique continuation for a hyperbolic equation). Yet we herewith would like to emphasize that the numerical computations are grounded by such mathematical analysis for the inverse problem if the analysis is well executed. Although we do not here explain the practical importance of the inverse problems in detail, these come from the real worlds, and so numerical reconstruction of unknown coefficients should be eventually done.

We can mention a very common understanding: *If the original problem is "well-posed"* (*or has "stability"*), *then a "fine" numerical scheme can yield accurate numerical results*.

Development of "fine" numerical schemes is a serious issue, and one can concentrate on researches for numerical methods as long as the problem is "well-posed". The initial/boundary value problem for a hyperbolic equation falls within this happy case. In other words, in the well-posed problem, if numerical results are not good, then one can attribute the badness to the adopted numerical scheme (and possibly also to the choice of the model equation).

The situation is drastically different in the inverse problem: the inverse problem includes *intrinsic ill-posedness*. Such ill-posedness is caused inevitably by the fact that a usual observation is always restricted. For example, if in the inverse problem for (1) - (3), we took the "ideal" observation $u|_{\partial\Omega\times(0,T)}$, in place of (4), then no ill-posedness would happen. Thus, in order to execute reasonable numerical computations, we have to answer questions at least regarding the uniqueness and the stability. By the recent researches, we know that for example, in the inverse hyperbolic problems, ill-posedness is modest so that it may be avoidable according to conditions in the formulation. That is, such conditions can restore the well-posedness and the restoration may be extremely difficult without such conditions.

For numerical reconstruction of unknown functions, the Tikhonov regularization is widely used where a regularizing term stabilizes ill-posedness of the original problem. As a consequence of the introduction of regularizing term, optimal choice of regularizing parameters is an important problem. The stability can yield a practical strategy in choosing regularizing parameters (e.g., Cheng and Yamamoto [3]). Such support to the regularization theory is also a motivation for our current researches for the stability in inverse problems.

Now, in order to outline the main results on the uniqueness and stability for the above inverse problems, we will state only the uniqueness for the inverse problem (1) - (4). Let a subdomain $\gamma \subset \Omega$ satisfy

$$\partial\gamma \supset \{x \in \partial\Omega; (x - x_0)\cdot\nu(x) \ge 0\}$$

with some $x_0 \notin \overline{\Omega}$.

Then we can state the uniqueness: We assume that $p = q$ on $\partial\Omega$ and for constants $0 < \sigma_0 \le 1$, $\sigma_1 > 0$, and $M_0 \ge 0$,

$$p(x), q(x) \ge \sigma_1, \left|\frac{(\nabla p(x)\cdot(x - x_0))}{2p(x)}\right|, \left|\frac{(\nabla q(x)\cdot(x - x_0))}{2q(x)}\right| \le 1 - \sigma_0, |\nabla p(x)|, |\nabla q(x)| \le M_0, \quad x \in \overline{\Omega}.$$

Moreover let p, q, a, b be sufficiently smooth and satisfy compatible conditions so that

$$u(p), u(q) \in W^{4,\infty}((0,T)\times\Omega).$$

We choose $\kappa > 0$ such that

$$\kappa + \frac{M_0 \sup_{x\in\Omega}|x - x_0|}{\sqrt{\sigma_1}}\sqrt{\kappa} < \sigma_0\sigma_1.$$

We further assume that

$$(\nabla a(x)\cdot(x - x_0)) \neq 0, \qquad x \in \overline{\Omega}$$

and

$$T > \frac{1}{\sqrt{\kappa}}\sup_{x\in\overline{\Omega}}|x - x_0|.$$

Then $u(p)(t,x) = u(q)(t,x)$, $0 < t < T$, $x \in \gamma$, implies $p(x) = q(x)$, $x \in \Omega$.

Moreover, under suitable assmptions on a priori boundedness, we can prove the conditional stability: we can choose $\theta \in (0,1)$ such that

$$\|p - q\|_{L^2(\Omega)} \le C\sum_{j=2}^{3}\|\partial_t^j(u(p) - u(q))\|^{\theta}_{L^2((0,T)\times\gamma)}$$

provided that p, q are in a suitable bounded admissible set.