
Volume 33, Issue 2, September 2012
ISSN 1063-5203
ELSEVIER
Applied and Computational Harmonic Analysis
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Appl. Comput. Harmon. Anal. 33 (2012) 204–225

Contents lists available at SciVerse ScienceDirect

Applied and Computational Harmonic Analysis

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A dual-chain approach for bottom–up construction of wavelet filters with any integer dilation

Charles K. Chui [a,1], Bin Han [b,2], Xiaosheng Zhuang [c,*,2]

[a] Department of Mathematics and Computer Sciences, University of Missouri–St. Louis, MO 63121, USA
[b] Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton T6G 2G1, Canada
[c] Institut für Mathematik, Technische Universität Berlin, 10623 Berlin, Germany

ARTICLE INFO

Article history:
Received 10 October 2010
Revised 28 September 2011
Accepted 14 November 2011
Available online 17 November 2011
Communicated by Ming-Jun Lai

Keywords:
Dual-chain method
Biorthogonal wavelets
Bottom–up wavelet filter construction
Lazy wavelets
Symmetry property
Integer dilation
Polyphase matrices
CBC algorithm

ABSTRACT

A dual-chain approach is introduced in this paper to construct dual wavelet filter systems with an arbitrary integer dilation $\mathsf{d} \geqslant 2$. Starting from a pair $(a, \tilde{a})$ of d-dual low-pass filters, with $(a_0, a_1) = (a, \tilde{a})$, a top–down chain of filters $a_0 \to a_1 \to \cdots \to a_r = \delta$ is constructed with consecutive d-dual pairs (a_j, a_{j+1}), $j = 1, \ldots, r-1$, and $\#(a_1) > \#(a_2) > \cdots > \#(a_r) = 1$, where $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$, and $\#(a_j)$ denotes the number of filter taps of a_j. This enables the formulation of the filter system $(a_r; b_{r,1}, \ldots, b_{r,\mathsf{d}-1}) =: (a_r; \vec{b}_r)$, with $\vec{b}_r = [\delta(\cdot - 1), \ldots, \delta(\cdot - \mathsf{d} + 1)]$, to be used as the second component of the initial filter system $((a_{r-1}; \vec{b}_{r-1}), (a_r; \vec{b}_r))$ of the bottom–up d-dual chain: $((a_{r-1}; \vec{b}_{r-1}), (a_r; \vec{b}_r)) \to ((a_{r-2}; \vec{b}_{r-2}), (a_{r-1}; \vec{b}^{\sharp}_{r-1})) \to \cdots \to ((a_0; \vec{b}_0), (a_1; \vec{b}^{\sharp}_1))$, constructed bottom–up iteratively, from $j = r$ to $j = 0$, by using both the d-duality property of (a_j, a_{j+1}), $j = 0, \ldots, r-1$ and the unimodular property of the polyphase Laurent polynomial matrix associated with the filter system $(a_j; \vec{b}_j)$. Then the desired dual wavelet filter systems, associated with a and $\tilde{a}$, are given by $(b_1, \ldots, b_{\mathsf{d}-1}) := (b_{0,1}, \ldots, b_{0,\mathsf{d}-1})$ and $(\tilde{b}_1, \ldots, \tilde{b}_{\mathsf{d}-1}) := (b^{\sharp}_{1,1}, \ldots, b^{\sharp}_{1,\mathsf{d}-1})$. More importantly, the constructive algorithm for this dual-chain approach can be appropriately modified to preserve the symmetry property of the initial d-dual pair $(a, \tilde{a})$. For any dilation factor d, the dual-chain algorithms developed in this paper provide two systematic methods for the construction of both biorthogonal wavelets and bottom–up wavelets with or without the symmetry property.

1. Introduction

The traditional application of wavelets to analyze signals and data is to decompose the given signal or data-set into multi-levels of frequency bands in order to facilitate effective analysis and processing. The given signal or data-set is represented in terms of some desirable scaling function, such as the mth order cardinal B-spline, and its companion "synthesis" wavelet. The (finite) sequence that governs the refinement relation of the dual scaling function, together with the (finite) sequence that defines the dual wavelet (in terms of the dual scaling function), is used as a filter pair for signal or data-set decomposition, where duality is required to assure perfect reconstruction. We may call the decomposition/reconstruction scheme for signal

* Corresponding author.
E-mail addresses: ckchui@stanford.edu (C.K. Chui), bhan@ualberta.ca (B. Han), xzhuang@math.tu-berlin.de (X. Zhuang).

[1] This author was also with the Department of Statistics, Stanford University, Stanford, CA 94305, USA. His research is supported by ARO Grants W911NF-07-1-0525 and W911NF-11-1-0426.
[2] Research was supported in part by NSERC Canada under Grant RGP 228051.

1063-5203/$ – see front matter
doi:10.1016/j.acha.2011.11.004

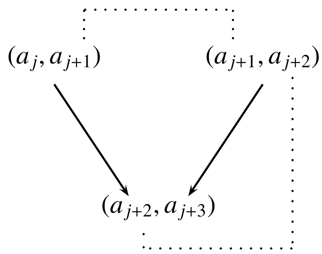

Fig. 1.1. Construction of dual pairs (a_{j+1}, a_{j+2}), (a_{j+2}, a_{j+3}) from previous pair (a_j, a_{j+1}), $j = 0, \ldots, r-3$.

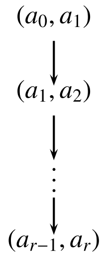

Fig. 1.2. $(a_0, a_1) := (a, \tilde{a})$ and $a_r = t_0\delta(\cdot - c_0)$, $t_0 \in \mathbb{C}$ and $c_0 \in \mathbb{Z}$.

processing and data analysis the "top–down" application of wavelets. Of course, when the dilation is extended from 2 to an arbitrary integer $\mathsf{d} > 2$, the number of wavelets and that of wavelet (band-pass) filters are both equal to $\mathsf{d} - 1$.

In the literature of filter banks for multirate systems (see, for example, [31]), scaling and wavelet functions are not considered, since only filter sequences are needed for digital signal processing and sub-band coding. In this regard, however, it is perhaps interesting to point out that there is indeed a one-to-one correspondence between filter banks and frequency-based wavelets in the distribution space, recently established by the second author in [19]. For other applications, such as curve subdivision and multi-level data interpolation, the top–down process of decomposing a finer data-set to coarser sets is reversed. As to wavelet curve subdivision, for instance, the wavelet subdivision scheme is applied to a coarse (ordered) set of control points to add new points and embed desirable curve features by using the filter pair of refinement sequence of the scaling function and the sequence that governs the synthesis wavelet, while the "decomposition" filter pair can be used for curve editing. In other words, the top–down "decomposition/reconstruction" is changed to bottom–up "subdivision/editing" (see [3,4]). For such bottom–up applications, since there is no data-set to be analyzed, the dual scaling function and dual wavelet are somewhat useless. Only the decomposition filter pair is used for editing (see [4]).

The objective of the present paper is to introduce an innovative approach, along with two effective algorithms, for the construction of the wavelet filter system and its dual, starting from a given dual pair of finite sequences $a = \{a(k)\}_{k\in\mathbb{Z}}$ and $\tilde{a} = \{\tilde{a}(k)\}_{k\in\mathbb{Z}}$, where duality will be defined in (1.4) relative to an arbitrary integer $\mathsf{d} \geqslant 2$. A typical example of a finite sequence $a = \{a(k)\}_{k\in\mathbb{Z}} : \mathbb{Z} \to \mathbb{C}$ is the refinement sequence of the mth order cardinal B-spline N_m for any $m \geqslant 2$, defined by

$$N_m(x) = \mathsf{d} \sum_{k\in\mathbb{Z}} a(k) N_m(\mathsf{d}x - k), \quad x \in \mathbb{R}, \tag{1.1}$$

or equivalently, $\widehat{N_m}(\mathsf{d}\xi) = \mathsf{a}(e^{-i\xi})\widehat{N_m}(\xi)$ in the frequency domain, where $\widehat{N_m}(\xi) := \int_{\mathbb{R}} N_m(x) e^{-ix\xi}\, dx$, and

$$\mathsf{a}(z) := \sum_{k\in\mathbb{Z}} a(k) z^k = \left(\frac{1 + z + \cdots + z^{\mathsf{d}-1}}{\mathsf{d}}\right)^m, \quad z \in \mathbb{C}\backslash\{0\}. \tag{1.2}$$

The innovation of our approach is the introduction of a dual-chain $a_0 \to a_1 \to \cdots \to a_{r-1} \to a_r$ with $(a_0, a_1) = (a, \tilde{a})$ such that the number of filter taps of a_j (more precisely, the length of the coefficient support interval of a_j, see definition in Section 2) is strictly decreasing as j runs from 1 to r and (a_{j-1}, a_j) is a dual pair for each $j = 1, \ldots, r$. Our first algorithm (see Theorem 1 and Algorithm 1 in Section 2) terminates at $j = r$, with a_r having only one tap, that is, $a_r = t_0\delta(\cdot - c_0)$ for some $t_0 \in \mathbb{C}\backslash\{0\}$ and $c_0 \in \mathbb{Z}$, where $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. This top–down dual-chain algorithm is illustrated in Fig. 1.1, where the dual pairs (a_{j+1}, a_{j+2}) and (a_{j+2}, a_{j+3}) are constructed from (a_j, a_{j+1}) and (a_{j+1}, a_{j+2}) respectively and iteratively, and the dual-chain so obtained is displayed in Fig. 1.2. This algorithm, however, does not necessarily capture any symmetry property of a given dual pair (a_0, a_1) in generating the wavelet filter system and its dual. By forcing the filter lengths of $a_1, \ldots, a_r$ to decrease by at least two taps at each iterative step in going down the dual-chain in Fig. 1.2, our second algorithm, to be stated as Algorithm 2 in Section 2, does assure preservation of symmetry or anti-symmetry of a_0 and a_1. This second algorithm, to be highlighted in Theorem 2, terminates at $j = r$, either with a_r that has a single filter tap as in the first algorithm, or else with the Laurent polynomial symbol $\mathsf{a}_r(z)$ of a_r given by the sum of only two of its "polyphase components" (see (1.6) for the definition); more precisely:

$$\mathrm{a}_r(z) = z^\beta \mathrm{a}_r^{[\beta]}\big(z^{\mathrm{d}}\big) + z^\gamma \mathrm{a}_r^{[\gamma]}\big(z^{\mathrm{d}}\big),$$

for some integers β, γ where $0 \leqslant \beta < \gamma \leqslant \mathrm{d}-1$.

To facilitate better understanding of the ideas and procedures presented in this paper, the refinement symbol $\mathrm{a}(z)$ in (1.2) for the spline setting will be used as a typical example, where the dual (or more precisely, d-dual) $\tilde{a}$ of a is given, in terms of its Laurent polynomial symbol $\tilde{\mathrm{a}}(z)^\star := \mathrm{a}_{2n}^I(z)/\mathrm{a}(z)$, where

$$\mathrm{a}_{2n}^I(z) = z^{(1-\mathrm{d})n}\left(\frac{1+z+\cdots+z^{\mathrm{d}-1}}{\mathrm{d}}\right)^{2n} \times \sum_{j=0}^{n-1}\left(\sum_{j_1+\cdots+j_{\mathrm{d}-1}=j}\prod_{k=1}^{\mathrm{d}-1}\binom{n+j_k-1}{j_k}\sin^{-2j_k}\left(\frac{k\pi}{\mathrm{d}}\right)\right)\left(\frac{1}{2}-\frac{z+z^{-1}}{4}\right)^j, \tag{1.3}$$

for any desirable integer $n \geqslant m/2$. Here, the standard notation $\tilde{\mathrm{a}}(z)^\star := \overline{\tilde{\mathrm{a}}(\bar{z}^{-1})}$, $z \in \mathbb{C}\backslash\{0\}$, is used and $\mathrm{a}_{2n}^I(z)$ is the unique Laurent polynomial symbol with minimal degree, whose coefficient sequence has the shortest support interval $[1-\mathrm{d}n, \mathrm{d}n-1]$, of the refinement sequence of the interpolating scaling function of order $2n$ and with dilation d (see [13,17,24, 29,32]). Moreover, a CBC (coset-by-coset) algorithm in [1,13,14] can be applied in general to construct the d-dual filter $\tilde{a}$ with an arbitrarily pre-assigned sum-rule order for a given filter a. One of the main objectives of this paper is to develop a general algorithm for deriving the corresponding band-pass filters from a pair of dual low-pass filters. Therefore, together with the CBC algorithms in [1,13,14], the algorithms developed in the present paper provide a complete computational scheme for the construction of univariate biorthogonal wavelets with arbitrary integer dilations, at least in the distribution sense as discussed in [19].

Recall that for any integer $\mathrm{d} \geqslant 2$, the d-duality of two finite sequences $u = \{u(k)\}_{k\in\mathbb{Z}}$ and $\tilde{u} = \{\tilde{u}(k)\}_{k\in\mathbb{Z}}$, relative to the dilation factor d, is defined by

$$\sum_{k\in\mathbb{Z}} \overline{u(k)}\tilde{u}(\mathrm{d}j+k) = \mathrm{d}^{-1}\delta(j), \quad j \in \mathbb{Z}, \tag{1.4}$$

and the symbol of u is defined by the Laurent polynomial

$$\mathrm{u}(z) := \sum_{k\in\mathbb{Z}} u(k)z^k, \quad z \in \mathbb{C}\backslash\{0\}. \tag{1.5}$$

Also recall that the polyphase components of the Laurent polynomial $\mathrm{u}(z)$ in (1.5) are defined by

$$\mathrm{u}^{[\gamma]}(z) := \sum_{k\in\mathbb{Z}} u(\mathrm{d}k+\gamma)z^k, \quad \gamma \in \mathbb{Z}. \tag{1.6}$$

Consequently, $\mathrm{u}(z)$ has the polyphase representation

$$\mathrm{u}(z) = \mathrm{u}^{[0]}\big(z^{\mathrm{d}}\big) + z\mathrm{u}^{[1]}\big(z^{\mathrm{d}}\big) + \cdots + z^{\mathrm{d}-1}\mathrm{u}^{[\mathrm{d}-1]}\big(z^{d}\big).$$

Throughout this paper, $\mathrm{u}^{[0]}(z), \ldots, \mathrm{u}^{[\mathrm{d}-1]}(z)$ will be called the *polyphase components* of $\mathrm{u}(z)$, and for convenience, they are also called the polyphase components of the sequence u itself. In other words, $\mathrm{u}^{[\gamma]}(z)$ is the symbol of the coset sequence $u^{[\gamma]} = \{u(\mathrm{d}k+\gamma)\}_{k\in\mathbb{Z}}$ of u.

By adopting the notation in (1.5)–(1.6), the duality relation of a and $\tilde{a}$, as defined by (1.4) for $u = a$ and $\tilde{u} = \tilde{a}$, is equivalent to the identity:

$$\sum_{\gamma=0}^{\mathrm{d}-1} \mathrm{a}^{[\gamma]}(z)\tilde{\mathrm{a}}^{[\gamma]}(z)^\star = \mathrm{d}^{-1}. \tag{1.7}$$

Returning to the first algorithm to be described in Section 2, though already illustrated in Fig. 1.1, if the sequence $a_r = \delta$ ($t_0 = 1$ and $c_0 = 0$ in Fig. 1.2) is the Kronecker delta sequence, then a_r is the refinement sequence of the delta “function” (or Dirac delta distribution), to be denoted by $\boldsymbol{\delta}(x)$, with the corresponding lazy wavelets

$$\eta^\gamma(x) := \mathrm{d}\boldsymbol{\delta}(\mathrm{d}x-\gamma), \quad \gamma = 1, \ldots, \mathrm{d}-1,$$

that satisfy

$$\eta^\gamma(x) = \mathrm{d}\sum_{k\in\mathbb{Z}} b_{r,\gamma}(k)\boldsymbol{\delta}(\mathrm{d}x-k),$$

where $b_{r,\gamma} := \{b_{r,\gamma}(k)\}_{k\in\mathbb{Z}}$ is obviously given by

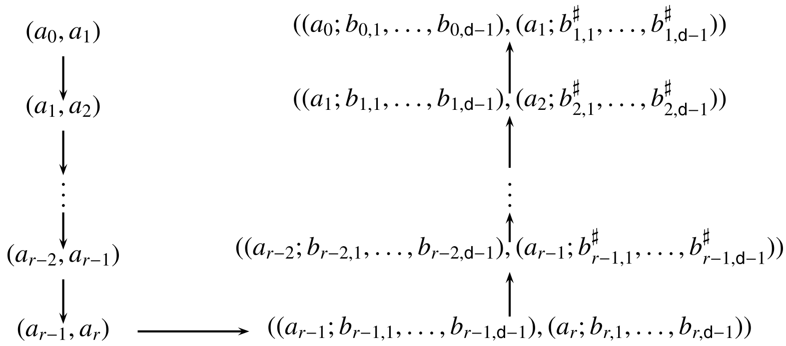

Fig. 1.3. Bottom–up construction of dual filter systems. Left: top–down chain. Right: bottom–up chain.

$$b_{r,\gamma}(k) = \delta(k-\gamma), \quad k \in \mathbb{Z};\ \gamma = 1, \dots, \mathrm{d}-1.$$

These wavelet filters $b_{r,\gamma} = \{\delta(k-\gamma)\}_{k\in\mathbb{Z}}$, $\gamma = 1, \dots, \mathrm{d}-1$, constitute the band-pass components of our initial filter system

$$(a_r; b_{r,1}, \dots, b_{r,\mathrm{d}-1}),$$

with low-pass component $a_r = \delta$, for constructing the filter systems

$$(a_j; b_{j,1}, \dots, b_{j,\mathrm{d}-1}) \tag{1.8}$$

iteratively, from $j = r-1$ to $j = r-2, \dots,$ and finally to $j = 0$, in going up the bottom–up dual-chain as illustrated in Fig. 1.3.

To describe this bottom–up procedure, let us consider the polyphase matrices

$$\mathbf{P}_j(z) := \begin{bmatrix} \mathrm{a}_j^{[0]}(z) & \mathrm{a}_j^{[1]}(z) & \cdots & \mathrm{a}_j^{[\mathrm{d}-1]}(z) \\ \mathrm{b}_{j,1}^{[0]}(z) & \mathrm{b}_{j,1}^{[1]}(z) & \cdots & \mathrm{b}_{j,1}^{[\mathrm{d}-1]}(z) \\ \vdots & \vdots & \ddots & \vdots \\ \mathrm{b}_{j,\mathrm{d}-1}^{[0]}(z) & \mathrm{b}_{j,\mathrm{d}-1}^{[1]}(z) & \cdots & \mathrm{b}_{j,\mathrm{d}-1}^{[\mathrm{d}-1]}(z) \end{bmatrix}, \quad j = 0, \dots, r, \tag{1.9}$$

associated with the filter bank systems in (1.8). (The interested reader is referred to [31] for further details on the application of polyphase decomposition to the construction of filter banks.)

For each $j = 1, \dots, r$, the duality relation of a_{j-1} and a_j, in terms of the polyphase formulation (1.7), is given by

$$\sum_{\gamma=0}^{\mathrm{d}-1} \mathrm{a}_{j-1}^{[\gamma]}(z)\mathrm{a}_j^{[\gamma]}(z)^\star = \mathrm{d}^{-1}.$$

Consequently, for each $j = 1, \dots, r$,

$$\big[\mathrm{a}_{j-1}^{[0]}(z), \dots, \mathrm{a}_{j-1}^{[\mathrm{d}-1]}(z)\big]\big(\mathbf{P}_j(z)\big)^\star = \big[\mathrm{d}^{-1}, q_{j,1}(z), \dots, q_{j,\mathrm{d}-1}(z)\big], \tag{1.10}$$

where $[\mathrm{a}_{j-1}^{[0]}(z), \dots, \mathrm{a}_{j-1}^{[\mathrm{d}-1]}(z)]$ is the first row of $\mathbf{P}_{j-1}(z)$ and $(\mathbf{P}_j(z))^\star$ denotes the matrix of Laurent polynomials such that $(\mathbf{P}_j(z))^\star = \overline{\mathbf{P}_j(\bar{z}^{-1})}^\top$ for $z \in \mathbb{C}\backslash\{0\}$, which is still a matrix of Laurent polynomials. Set

$$V_j(z) := \begin{bmatrix} 1 & -\mathrm{d}q_{j,1}(z) & -\mathrm{d}q_{j,2}(z) & \cdots & -\mathrm{d}q_{j,\mathrm{d}-1}(z) \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{1.11}$$

Then

$$\big[\mathrm{d}^{-1}, q_{j,1}(z), \dots, q_{j,\mathrm{d}-1}(z)\big]V_j(z) = \big[\mathrm{d}^{-1}, 0, \dots, 0\big]$$

and

$$\big[\mathrm{a}_{j-1}^{[0]}(z), \dots, \mathrm{a}_{j-1}^{[\mathrm{d}-1]}(z)\big] = [1, 0, \dots, 0]\big(\mathrm{d}\mathbf{P}_j(z)^\star V_j(z)\big)^{-1}.$$

That is, the first row of $\mathbf{P}_{j-1}(z)$ is precisely the same as the first row of $(\mathrm{d}\mathbf{P}_j(z)^\star V_j(z))^{-1}$.

Now, for the initial step $j = r$, since $\mathbf{P}_r(z) = I_{\mathsf{d}}$ is the $\mathsf{d} \times \mathsf{d}$ identity matrix, the functions $q_{r,1}(z), \ldots, q_{r,\mathsf{d}-1}(z)$ in (1.10) for $j = r$ are Laurent polynomials so that the matrix $(\mathsf{d}\mathbf{P}_r(z)^\star V_r(z))^{-1}$ with Laurent polynomial entries can be used to define $\mathbf{P}_{r-1}(z)$. Consequently, $q_{r-1,1}(z), \ldots, q_{r-1,\mathsf{d}-1}(z)$ in (1.10) for $j = r-1$ are also Laurent polynomials. Hence, $(\mathsf{d}\mathbf{P}_{r-1}(z)^\star V_{r-1}(z))^{-1}$ can be used to define $\mathbf{P}_{r-2}(z)$. In general, using the fact that the polynomial matrix $\mathbf{P}_j(z)^\star V_j(z) =: \mathbf{P}_j^\sharp(z)^\star$ is unimodular (i.e., $\det(\mathbf{P}_j(z)^\star V_j(z))$ is a nonzero monomial), we can go up the chain as shown in Fig. 1.3 to construct the filter system

$$\big((a_0; b_{0,1}, \ldots, b_{0,\mathsf{d}-1}), \big(a_1; b^\sharp_{1,1}, \ldots, b^\sharp_{1,\mathsf{d}-1}\big)\big).$$

This yields a pair of biorthogonal d-wavelet filter systems. But as already mentioned above, a pair of biorthogonal d-wavelet filters is always associated with an underlying frequency-based dual wavelet pair in the distribution space (see [19] for detail), we have also obtained a pair of biorthogonal d-wavelet families, at least in the distribution sense. For completeness of our presentation and to facilitate the discussion of our illustrative examples, we will include a brief discussion of biorthogonal wavelets in $L_2(\mathbb{R})$ in Section 2.3.

To appreciate and better understand the approach and motivation of this paper, let us make some remarks. Recall that a pair of biorthogonal d-wavelet filter systems $((a; b_1, \ldots, b_{\mathsf{d}-1}), (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_{\mathsf{d}-1}))$ satisfies the duality condition:

$$\mathbf{P}(z)\tilde{\mathbf{P}}(z)^\star = \mathsf{d}^{-1} I_{\mathsf{d}}, \tag{1.12}$$

where

$$\mathbf{P}(z) := \begin{bmatrix} \mathsf{a}^{[0]}(z) & \cdots & \mathsf{a}^{[\mathsf{d}-1]}(z) \\ \mathsf{b}_1^{[0]}(z) & \cdots & \mathsf{b}_1^{[\mathsf{d}-1]}(z) \\ \vdots & \ddots & \vdots \\ \mathsf{b}_{\mathsf{d}-1}^{[0]}(z) & \cdots & \mathsf{b}_{\mathsf{d}-1}^{[\mathsf{d}-1]}(z) \end{bmatrix}, \qquad \tilde{\mathbf{P}}(z) := \begin{bmatrix} \tilde{\mathsf{a}}^{[0]}(z) & \cdots & \tilde{\mathsf{a}}^{[\mathsf{d}-1]}(z) \\ \tilde{\mathsf{b}}_1^{[0]}(z) & \cdots & \tilde{\mathsf{b}}_1^{[\mathsf{d}-1]}(z) \\ \vdots & \ddots & \vdots \\ \tilde{\mathsf{b}}_{\mathsf{d}-1}^{[0]}(z) & \cdots & \tilde{\mathsf{b}}_{\mathsf{d}-1}^{[\mathsf{d}-1]}(z) \end{bmatrix}. \tag{1.13}$$

There are two major tasks in the construction of biorthogonal d-wavelet systems. The first task is to construct a pair $(a, \tilde{a})$ of d-dual filters such that a and $\tilde{a}$ have some desirable properties such as sum rules and short supports. The second major task is to derive all the band-pass filters $b_1, \ldots, b_{\mathsf{d}-1}, \tilde{b}_1, \ldots, \tilde{b}_{\mathsf{d}-1}$ from a given pair $(a, \tilde{a})$ of d-dual filters so that the pair $((a; b_1, \ldots, b_{\mathsf{d}-1}), (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_{\mathsf{d}-1}))$ satisfies the duality condition in (1.12).

The first task of constructing pairs of dual filters has been extensively studied in the literature, for example, see [1,5,7, 9,13,14,18,24,30] and the references therein. For a given primal filter, there are several methods available in the literature for constructing dual filters. Two particular methods are the lifting scheme in [30] and the CBC (coset-by-coset) algorithm in [1,13,14]. To explain these two methods a little bit in detail, we first look at the particular case $\mathsf{d} = 2$. When $\mathsf{d} = 2$, we have only one pair of band-pass filters b_1 and $\tilde{b}_1$, which, by a simple calculation, must take the following form:

$$\mathsf{b}_1(z) = cz^{2n-1}\tilde{\mathsf{a}}^\star(-z), \qquad \tilde{\mathsf{b}}_1(z) = \frac{1}{\bar{c}} z^{2n-1}\mathsf{a}^\star(-z), \tag{1.14}$$

where $c \in \mathbb{C}\backslash\{0\}$ and $n \in \mathbb{Z}$. For simplicity, one often takes $c = 1$ and $n = 1$ in (1.14); that is,

$$\mathsf{b}_1(z) = z\tilde{\mathsf{a}}^\star(-z), \qquad \tilde{\mathsf{b}}_1(z) = z\mathsf{a}^\star(-z). \tag{1.15}$$

Consequently, for the dilation factor $\mathsf{d} = 2$, a pair of biorthogonal 2-wavelet filter systems $((a; b_1), (\tilde{a}; \tilde{b}_1))$ is completely determined by a pair $(a, \tilde{a})$ of 2-dual filters, while their associated band-pass filters $b_1, \tilde{b}_1$ are almost uniquely determined by (1.15). Hence, for the case of the dilation factor 2, the major task is to construct a pair $(a, \tilde{a})$ of 2-dual filters with some desirable properties. Suppose that $((a; b), (\tilde{a}; \tilde{b}))$ is a given pair of biorthogonal 2-wavelet filter systems. By a simple argument from linear algebra, one can easily check that $(a, \tilde{a}^{new})$ is a pair of 2-dual filters if and only if $\tilde{\mathsf{a}}^{new}(z) = \tilde{\mathsf{a}}(z) + \mathbf{\Theta}(z^2)\tilde{\mathsf{b}}(z)$ for some Laurent polynomial $\mathbf{\Theta}$. In terms of pairs of biorthogonal 2-wavelet filter systems, the old system $((a; b), (\tilde{a}; \tilde{b}))$ is updated into a new system $((a; b^{new}), (\tilde{a}^{new}; \tilde{b}))$, where

$$\tilde{\mathsf{a}}^{new}(z) = \tilde{\mathsf{a}}(z) + \mathbf{\Theta}\big(z^2\big)\tilde{\mathsf{b}}(z), \qquad \mathsf{b}^{new}(z) = \mathsf{b}(z) - \mathbf{\Theta}\big(z^2\big)\mathsf{a}(z). \tag{1.16}$$

The above scheme in (1.16) is called the lifting scheme [9,30]. To apply (1.16), it requires that all the band-pass filters should be known in advance. Of course, this is trivial for $\mathsf{d} = 2$ as discussed above. However, for the dilation factor $\mathsf{d} > 2$, it is no longer trivial any more and the construction of band-pass filters critically lies on the second major task for deriving all band-pass filters from a given pair of dual filters. Moreover, the new filter $\tilde{a}^{new}$ constructed by the lifting scheme in (1.16) is generally not guaranteed to satisfy any order of sum rules. These shortcomings have been overcome by the CBC algorithm in [1,13,14]. The CBC algorithm can be applied starting with either a complete pair of biorthogonal wavelet filter systems or a pair of dual filters without knowing the band-pass filters in advance. To be more specific, let a be a finitely supported primal filter with at least one finitely supported d-dual filter. For any positive integer n, it is shown in [14, Theorem 3.4] that a finitely supported d-dual filter $\tilde{a}$ that satisfies the sum-rule condition of order at least n always exists, meaning that the symbol $\tilde{\mathsf{a}}(z)$ of $\tilde{a}$ is divisible by $(1 + z + \cdots + z^{\mathsf{d}-1})^n$. Furthermore, such dual filters $\tilde{a}$ having the shortest possible support, with or without any prescribed symmetry property, can be easily constructed by applying the CBC algorithm proposed in [1,13,14].

For a given primal filter a, to obtain a dual filter $\tilde{a}$ which has a preassigned order n of sum rules, it is easy to see that one only needs to solve a system of linear equations. The existence of a solution to such a system of linear equations is guaranteed by the CBC algorithm as long as the coefficient support of $\tilde{a}$ is long enough. Consequently, from the point of view of computation, the first major task for constructing pairs of dual filters can be reduced to a simple task of solving a system of linear equations. But, the more difficult problem is the second major task for deriving all the band-pass filters from a pair of dual filters. The main approach used in the literature is to study the matrix extension problem for matrices of Laurent polynomials. Currently, results on the matrix extension problem of Laurent polynomials have been used to derive band-pass filters from a pair of dual filters, for example, see [10,17,22,23,28,32,33] and references therein on the matrix extension problem with or without the symmetry constraint. However, the available algorithms in the literature to obtain the extension matrix are often quite complicated (for example, see [17,22,23,33]) and such algorithms only work for dimension one. If the symmetry property is required, then the corresponding algorithms for the matrix extension with symmetry generally are much more involved and there are many special cases to be considered. For the high-dimensional version of the matrix extension problem, there are barely any results even in dimension two.

The dual-chain approach proposed in this paper offers a quite simple way to address the two major tasks in the construction of biorthogonal wavelets. The top–down dual-chain in our algorithm can be built by simply solving a system of linear equations, while the existence of solutions to such a system of linear equations is guaranteed by Theorems 1 and 2. The bottom–up dual chain is even simpler since there is no system of linear equations involved. In particular, our dual chain approach keeps the same simplicity for the case of biorthogonal wavelet systems with the symmetry property. More importantly, the main ideas in our dual chain algorithms can be extended to high-dimensional biorthogonal wavelet systems. Though the existence of solutions to associated systems of linear equations has not been established so far in high dimensions, heuristically, a top–down dual chain in our algorithm, with or without the symmetry property, can often be built easily from many known pairs of high-dimensional dual filters by solving a system of linear equations. Now the same bottom–up dual-chain can be slightly modified to derive the associated high-dimensional band-pass filters, with or without the symmetry property. Our approach of using dual chains proposed in this paper provides a simple and computationally efficient way for constructing biorthogonal wavelets.

This paper is organized as follows. In Section 2, we introduce some necessary notations and describe our main results in terms of three theorems and two algorithms. Several examples are included in Section 3 to illustrate the effectiveness of the algorithms formulated in Section 2. Proofs of the results stated in Section 2 will be provided in Section 4, where two key lemmas are formulated and constructive proofs are given. Final remarks are stated in Section 5 for the interested reader for further study and investigation.

2. Main results

To facilitate the formulation of our main results in this paper, we first introduce the necessary notations and definitions.

In this paper, the dilation factor is an arbitrary integer $\mathsf{d} \geqslant 2$ which will be fixed throughout our presentation. We will always consider the scalar fields $\mathbb{F} = \mathbb{R}$ and $\mathbb{F} = \mathbb{C}$ of real and complex numbers, respectively, although the same results are valid for other scalar fields, such as $\mathbb{F} = \mathbb{Q}$, the field of rational numbers. All sequences u in this paper are assumed to be finite sequences; that is, $u : \mathbb{Z} \to \mathbb{F}$ is finitely supported on $\mathbb{Z}$. For such sequences, we will use the notation $u(k)$ for the kth term (or component) of the sequence u and write $u = \{u(k)\}_{k\in\mathbb{Z}}$. The symbol of this sequence u is defined in (1.5). If $u(m)u(n) \neq 0$ and $u(k) = 0$ for all $k < m$ or $k > n$, then $\mathsf{u}(z)$ is a Laurent polynomial of (Laurent polynomial) order $(n-m+1)$ or (Laurent polynomial) degree $(n-m)$, and we will also call the interval $[m,n]$ the *coefficient support interval* of $\mathsf{u}(z)$ (or of the sequence u itself), and write $\mathrm{suppintv}(u) := [m,n]$ to denote the support interval of the coefficient sequence u. The length of $\mathrm{suppintv}(u)$ will be defined by $|\mathrm{suppintv}(u)| := n - m$. Hence, the support interval of a sequence with a single nonzero term has length zero. On the other hand, when u is considered as a (digital) filter (and the terms of the sequence are called filter taps), the number of filter taps of the filter u, denoted by $\#(u)$, is given by $|\mathrm{suppintv}(u)| + 1$.

2.1. Dual-chain without symmetry constraint

Let us first discuss the construction of filter systems which do not necessarily have the symmetry property.

A family $\{a_j \mid j = 0, \dots, r\}$ of finite sequences is said to constitute a *chain of consecutive* d*-dual filters* (or *dual-chain*, for short), denoted by: $a_0 \to a_1 \to \cdots \to a_{r-1} \to a_r$, if the chain satisfies the following conditions:

(1) each (a_j, a_{j+1}) is a pair of d-dual filters, i.e., (a_j, a_{j+1}) satisfies (1.4) with $u = a_j$ and $\tilde{u} = a_{j+1}$, for $j = 0, \dots, r-1$;
(2) $\mathrm{suppintv}(a_{j+1}) \subsetneq \mathrm{suppintv}(a_j)$ for $j = 1, \dots, r-1$;
(3) $\mathsf{a}_r(z) = t_0 z^{c_0}$ for some nonzero constant $t_0 \in \mathbb{F}$ and some $c_0 \in \mathbb{Z}$, where $\mathsf{a}_r(z)$ is the symbol of a_r.

Item (1) describes the duality property of the top–down dual chain, while Item (2) ensures that the length of the coefficient support interval of a_j is strictly decreasing as the integer index j increases, so that the chain terminates with a one-tap filter, as precisely stated in Item (3).

Given a pair $(a, \tilde{a})$ of d-dual filters (i.e., $a, \tilde{a}$ satisfy (1.4) with $u = a$ and $\tilde{u} = \tilde{a}$), the following theorem assures the existence of a chain of consecutive d-dual filters.

Theorem 1. *Let $(a, \tilde{a})$ be a pair of d-dual filters. Then there exists a sequence of finitely supported filters $a_0, a_1, \dots, a_r$ such that $a_0 \to a_1 \to \cdots \to a_{r-1} \to a_r$ is a chain of consecutive d-dual filters with $(a_0, a_1) = (a, \tilde{a})$.*

When the construction of the chain of consecutive d-dual filters has been completed, we now begin to construct the corresponding chain of d-dual wavelet filter systems, by starting from the bottom, and using the last filter sequences $(a_r; b_{r,1}, \dots, b_{r,\mathsf{d}-1})$, where, as described in Section 1, the new wavelet filter components $b_{r,1}, \dots, b_{r,\mathsf{d}-1}$ are simply the governing sequences of the $d-1$ lazy wavelets corresponding to the scaling function $\boldsymbol{\delta}(x)$ (if $a_r = \delta = \{\delta(k)\}_{k\in\mathbb{Z}}$). An overview of the method of construction of the other new wavelet filter components $b_{r-1,1}, \dots, b_{r-1,\mathsf{d}-1}$ is also given in Section 1, see (1.16)–(1.19), with $j = r$. From this initial filter system, we may now build the bottom–up chain, as illustrated in Fig. 1.3, the construction of which is also highlighted in Section 1 in (1.16)–(1.19), but now iteratively from $j = r-1$, to $j = r-2$, and finally to $j = 1$, to complete the construction of the desired d-dual wavelet filter systems. The description in Section 1 is formulated as Algorithm 1 to be stated below.

The approach introduced in this paper is the construction of two chains: the first being the top–down chain for constructing the dual filter pair (a_{r-1}, a_r), while the second being the bottom–up chain for constructing the dual wavelet filter systems $(b_1, \dots, b_{\mathsf{d}-1})$ and $(\tilde{b}_1, \dots, \tilde{b}_{\mathsf{d}-1})$, associated with a given dual filter pair $(a, \tilde{a})$. We may also call these two chains a dual pair of chains (or "dual-chain" for short). In other words, the term "dual-chain" in the title of this paper actually has two meanings: with the first being the two chains of dual filters, and the second being the duality of the top–down and bottom–up chains.

For a given integer $k \in \mathbb{Z}$, ρ_k and λ_k are the two integers uniquely determined by the relation $k = \mathsf{d}\lambda_k + \rho_k$ with $0 \leqslant \rho_k \leqslant \mathsf{d}-1$; that is, $\lambda_k = \lfloor k/\mathsf{d} \rfloor$ and $\rho_k = k - \mathsf{d}\lambda_k$, where $\lfloor \cdot \rfloor$ is the floor operator. For an interval $[m, n]$ with $m \leqslant n$, we define $\min\{[m,n]\} = m$ and $\max\{[m,n]\} = n$.

Algorithm 1. Dual-chain algorithm for the construction of dual filter systems (without symmetry constraint).

(a) **Input:** $(a, \tilde{a})$, a pair of d-dual filters.
(b) **Top-down chain:** This part of the algorithm produces a top–down chain $a_0 \to a_1 \to \cdots \to a_{r-1} \to a_r$, see Fig. 1.3 (left) or Fig. 1.2.

In order to construct a top–down chain with as few number of filters as possible, swap a and $\tilde{a}$, if necessary, such that $|\mathrm{suppintv}(a)| \geqslant |\mathrm{suppintv}(\tilde{a})|$.
$(a_0, a_1) \leftarrow (a, \tilde{a})$. $j \leftarrow 1$.
while $|\mathrm{suppintv}(a_j)| \neq 0$ **do**
  $a_{j+1} \leftarrow a_{j-1}$.
  $m \leftarrow \min\{\mathrm{suppintv}(a_j)\}$. $n \leftarrow \max\{\mathrm{suppintv}(a_j)\}$. $\tilde{m} \leftarrow \min\{\mathrm{suppintv}(a_{j+1})\}$.
  while $\tilde{m} < m$ **do**
    Construct b through its symbol $\mathsf{b}(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \mathsf{b}^{[\gamma]}(z^\mathsf{d})$, where

$$\mathsf{b}^{[\gamma]}(z) := \begin{cases} \frac{a_{j+1}(\tilde{m})}{\overline{a_j(n)}} \mathsf{a}_j^{[\rho_{\tilde{m}}]}(z)^\star \cdot z^{\lambda_{\tilde{m}}+\lambda_n}, & \gamma = \rho_n; \\ -\frac{a_{j+1}(\tilde{m})}{\overline{a_j(n)}} \mathsf{a}_j^{[\rho_n]}(z)^\star \cdot z^{\lambda_{\tilde{m}}+\lambda_n}, & \gamma = \rho_{\tilde{m}}; \\ 0, & \text{otherwise.} \end{cases}$$

    Here, $a_j = \{a_j(k)\}_{k\in\mathbb{Z}}$, $a_{j+1} = \{a_{j+1}(k)\}_{k\in\mathbb{Z}}$, and $\mathsf{a}_j(z)$ is the symbol of a_j.
    $a_{j+1} \leftarrow a_{j+1} + b$.
    $\tilde{m} \leftarrow \min\{\mathrm{suppintv}(a_{j+1})\}$.
  end while
  $\tilde{n} \leftarrow \max\{\mathrm{suppintv}(a_{j+1})\}$.
  while $\tilde{n} \geqslant n$ **do**
    Construct b through its symbol $\mathsf{b}(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \mathsf{b}^{[\gamma]}(z^\mathsf{d})$, where

$$\mathsf{b}^{[\gamma]}(z) := \begin{cases} \frac{a_{j+1}(\tilde{n})}{\overline{a_j(m)}} \mathsf{a}_j^{[\rho_{\tilde{n}}]}(z)^\star \cdot z^{\lambda_{\tilde{n}}+\lambda_m}, & \gamma = \rho_m; \\ -\frac{a_{j+1}(\tilde{n})}{\overline{a_j(m)}} \mathsf{a}_j^{[\rho_m]}(z)^\star \cdot z^{\lambda_{\tilde{n}}+\lambda_m}, & \gamma = \rho_{\tilde{n}}; \\ 0, & \text{otherwise.} \end{cases}$$

    $a_{j+1} \leftarrow a_{j+1} + b$.
    $\tilde{n} \leftarrow \max\{\mathrm{suppintv}(a_{j+1})\}$.

    end while
  $j \leftarrow j+1$.
end while

(c) **Bottom-up chain:** This part of the algorithm is for constructing a bottom–up chain of dual filter systems, see Fig. 1.3 (right).

$r \leftarrow j$. Then the symbol $\mathsf{a}_r(z)$ of a_r is a monomial, i.e., $\mathsf{a}_r(z) = t_0 z^{c_0}$ for some $t_0 \in \mathbb{F}$ and some $c_0 \in \mathbb{Z}$. Define the band-pass filters $b_{r,1}, \dots, b_{r,\mathsf{d}-1}$ through their symbols given by $\mathsf{b}_{r,1}(z) := z^{c_0+1}, \dots, \mathsf{b}_{r,\mathsf{d}-1}(z) := z^{c_0+\mathsf{d}-1}$, respectively.
$j \leftarrow r$.
while $j \geqslant 1$ **do**
  Construct the polyphase matrix with respect to $(a_j; b_{j,1}, \dots, b_{j,\mathsf{d}-1})$ as in (1.9).
  Let $\mathsf{a}_{j-1}(z)$ be the symbol of a_{j-1} and define $\mathsf{p}(z) := [\mathsf{a}_{j-1}^{[0]}(z), \dots, \mathsf{a}_{j-1}^{[\mathsf{d}-1]}(z)]$. Then $\mathsf{p}(z)\mathbf{P}_j(z)^\star = [\mathsf{d}^{-1}, q_1(z), \dots, q_{\mathsf{d}-1}(z)]$ for some Laurent polynomials $q_1(z), \dots, q_{\mathsf{d}-1}(z)$.
  Define $V(z)$ as in (1.11) with its first row being given by $[1, 0, \dots, 0]V(z) = [1, -\mathsf{d}q_1(z), \dots, -\mathsf{d}q_{\mathsf{d}-1}(z)]$. Then $\mathsf{p}(z)\mathbf{P}_j(z)^\star V(z) = [\mathsf{d}^{-1}, 0, \dots, 0]$.
  $\mathbf{P}_j^\sharp(z) := V(z)^\star \mathbf{P}_j(z)$ and $\mathbf{P}_{j-1}(z) := [\mathsf{d}\mathbf{P}_j(z)^\star V(z)]^{-1}$ are of the form:

$$\mathbf{P}_{j-1} = \begin{bmatrix} \mathsf{a}_{j-1}^{[0]}(z) & \cdots & \mathsf{a}_{j-1}^{[\mathsf{d}-1]}(z) \\ \mathsf{b}_{j-1,1}^{[0]}(z) & \cdots & \mathsf{b}_{j-1,1}^{[\mathsf{d}-1]}(z) \\ \vdots & \ddots & \vdots \\ \mathsf{b}_{j-1,\mathsf{d}-1}^{[0]}(z) & \cdots & \mathsf{b}_{j-1,\mathsf{d}-1}^{[\mathsf{d}-1]}(z) \end{bmatrix}, \qquad \mathbf{P}_j^\sharp(z) = \begin{bmatrix} \mathsf{a}_{j}^{\sharp[0]}(z) & \cdots & \mathsf{a}_{j}^{\sharp[\mathsf{d}-1]}(z) \\ \mathsf{b}_{j,1}^{\sharp[0]}(z) & \cdots & \mathsf{b}_{j,1}^{\sharp[\mathsf{d}-1]}(z) \\ \vdots & \ddots & \vdots \\ \mathsf{b}_{j,\mathsf{d}-1}^{\sharp[0]}(z) & \cdots & \mathsf{b}_{j,\mathsf{d}-1}^{\sharp[\mathsf{d}-1]}(z) \end{bmatrix}.$$

  Construct band-pass filters $b_{j-1,1}, \dots, b_{j-1,\mathsf{d}-1}$ and $b^\sharp_{j,1}, \dots, b^\sharp_{j,\mathsf{d}-1}$ such that their symbols are given by $\mathsf{b}^\sharp_{j,1}(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \mathsf{b}_{j,1}^{\sharp[\gamma]}(z^\mathsf{d}), \dots, \mathsf{b}^\sharp_{j,\mathsf{d}-1}(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \mathsf{b}_{j,\mathsf{d}-1}^{\sharp[\gamma]}(z^\mathsf{d})$ and $\mathsf{b}_{j-1,1}(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \mathsf{b}_{j-1,1}^{[\gamma]}(z^\mathsf{d}), \dots, \mathsf{b}_{j-1,\mathsf{d}-1}(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \mathsf{b}_{j-1,\mathsf{d}-1}^{[\gamma]}(z^\mathsf{d})$, respectively, from rows of $\mathbf{P}_j^\sharp(z)$ and $\mathbf{P}_{j-1}(z)$.
  $j \leftarrow j-1$.
end while

(d) **Output:** a chain of dual filter systems with the perfect reconstruction property, see Fig. 1.3.

2.2. Dual-chain with symmetry constraint

In Theorem 1 and Algorithm 1, we have not considered the symmetry issue in the top–down and bottom–up procedures (see Fig. 1.3). Thus, symmetry of the output wavelet filter systems is not guaranteed. In the following, we shall discuss the notion of symmetry and develop an algorithm, namely Algorithm 2, for constructing a corresponding dual wavelet filter systems that preserve the symmetry property of the given dual filter pair.

A finite sequence $u = \{u(k)\}_{k\in\mathbb{Z}}$ is said to be *symmetric* (*about* $c/2$), if

$$u(c-k) = u(k), \quad \forall k \in \mathbb{Z}, \tag{2.1}$$

and *anti-symmetric or skew symmetric* (*about* $c/2$), if

$$u(c-k) = -u(k), \quad \forall k \in \mathbb{Z}, \tag{2.2}$$

for some integer $c \in \mathbb{Z}$, where $c/2$, which could be a half-integer, is called the *symmetry center* of u. Furthermore, two filters u and v are said to have the *same symmetry pattern*, if u and v are both symmetric as in (2.1), or else both anti-symmetric as in (2.2), about the same symmetry center $c/2$ for some $c \in \mathbb{Z}$.

Let $\mathsf{u}(z)$ be the symbol of u. Then by using the symmetry operator S as defined in [20] by

$$\mathsf{Su}(z) := \frac{\mathsf{u}(z)}{\mathsf{u}(1/z)}, \quad z \in \mathbb{C}\backslash\{0\},$$

it is clear that the sequence u is symmetric about $c/2$ if and only if $\mathsf{Su}(z) = z^c$, and anti-symmetric about $c/2$ if and only if $\mathsf{Su}(z) = -z^c$. Moreover, we see that two filters u and v have the same symmetry pattern, if and only if their corresponding symbols $\mathsf{u}(z)$ and $\mathsf{v}(z)$ satisfy the identity $\mathsf{Su}(z) = \mathsf{Sv}(z) = \epsilon z^c$ on the unit circle, for some $\epsilon \in \{1, -1\}$ and some $c \in \mathbb{Z}$.

A family $\{a_j \mid j = 0, \dots, r\}$ of finite sequences is said to constitute a *chain of consecutive* d*-dual filters with certain symmetry pattern* (or *dual-chain with symmetry*, for short), denoted by: $a_0 \to a_1 \to \cdots \to a_{r-1} \to a_r$, if the chain satisfies the following conditions:

(0) all filters $a_0, \dots, a_r$ have the same symmetry pattern: $\mathsf{Sa}_0(z) = \cdots = \mathsf{Sa}_r(z) = \epsilon z^c$ for some $\epsilon \in \{-1, 1\}$ and some $c \in \mathbb{Z}$;
(1) each (a_j, a_{j+1}) is a pair of d-dual filters, i.e., (a_j, a_{j+1}) satisfies (1.4) with $u = a_j$ and $\tilde{u} = a_{j+1}$, for $j = 0, \dots, r-1$;

(2) $\mathrm{suppintv}(a_{j+1}) \subsetneq \mathrm{suppintv}(a_j)$ for all $j = 1, \ldots, r-1$;
(3) a_r has no more than two nontrivial polyphase components. Precisely, a_r satisfies one of the following:
  (i) a_r has a single filter tap, namely: $\mathsf{a}_r(z) = t_0 z^{c/2}$ for some nonzero constant $t_0 \in \mathbb{F}$;
  (ii) a_r has exactly two nontrivial polyphase components $\mathsf{a}^{[\beta]}(z)$ and $\mathsf{a}^{[\gamma]}(z)$ such that $\mathsf{a}_r(z) = z^\beta \mathsf{a}^{[\beta]}(z^{\mathsf{d}}) + z^\gamma \mathsf{a}^{[\gamma]}(z^{\mathsf{d}})$.

The second theorem in this section, to be stated as Theorem 2 below, assures that all the filters in the top–down dual-chain $a_0 \to a_1 \to \cdots \to a_r$ with $(a_0, a_1) = (a, \tilde{a})$ preserve the symmetry property (of being either symmetric or anti-symmetric) of the given dual filter pair $(a, \tilde{a})$. This result will be applied to preserve the symmetry property of $(a, \tilde{a})$, by the output dual wavelet filter systems $(b_1, \ldots, b_{\mathsf{d}-1})$, $(\tilde{b}_1, \ldots, \tilde{b}_{\mathsf{d}-1})$, by applying the bottom–up algorithm, to be described in Algorithm 2 below.

Theorem 2. *Let $(a, \tilde{a})$ be a pair of* d*-dual filters such that a and $\tilde{a}$ have the same symmetry pattern*: $\mathsf{S}\mathsf{a}(z) = \mathsf{S}\tilde{\mathsf{a}}(z) = \epsilon z^c$ *for some $\epsilon \in \{-1, 1\}$ and some $c \in \mathbb{Z}$. Then there exists a sequence of finitely supported filters $a_0, a_1, \ldots, a_r$ such that* $\mathsf{S}\mathsf{a}_0(z) = \cdots = \mathsf{S}\mathsf{a}_r(z) = \epsilon z^c$ *and $a_0 \to a_1 \to \cdots \to a_{r-1} \to a_r$ is a chain of consecutive* d*-dual filters with symmetry, where $(a_0, a_1) = (a, \tilde{a})$.*

We remark that in Theorem 2, if either c is an odd integer, or c is an even integer but with $\epsilon = -1$, then a_r must satisfy (ii) of Item (3). Moreover, the degrees of $\mathsf{a}^{[\beta]}(z)$ and $\mathsf{a}^{[\gamma]}(z)$ in (ii) of Item (3) could be arbitrarily large. On the other hand, if the integer c is even and $\epsilon = 1$, then a_r could satisfy either (i) or (ii) of Item (3).

We also remark that a dual filter $\tilde{a} = \{\tilde{a}(k)\}_{k\in\mathbb{Z}}$ of a given symmetric or anti-symmetric primal filter $a = \{a(k)\}_{k\in\mathbb{Z}}$, in general, does not necessarily have any symmetry pattern as a. However, if a and its dual $\tilde{a}$ have certain symmetry, say, $\mathsf{S}\mathsf{a}(z) = \epsilon z^c$ and $\mathsf{S}\tilde{\mathsf{a}}(z) = \tilde{\epsilon} z^{\tilde{c}}$ for some $\epsilon, \tilde{\epsilon} \in \{-1, 1\}$ and some $c, \tilde{c} \in \mathbb{Z}$, then it is quite natural to assume that $\tilde{\epsilon} = \epsilon$ and $\tilde{c} = c$, namely, $\mathsf{S}\mathsf{a} = \mathsf{S}\tilde{\mathsf{a}}$. This point of view could be argued as follows. Let u be the sequence with symbol $\mathsf{a}(z)\tilde{\mathsf{a}}(z)^\star$. Since $(a, \tilde{a})$ is a dual pair, u is interpolatory, so that $u(0) = \mathsf{d}^{-1}$ and $u(\mathsf{d}k) = 0$ for all $k \in \mathbb{Z}\backslash\{0\}$. In other words, $u^{[0]} = \delta$. By $\mathsf{S}\mathsf{a}(z) = \epsilon z^c$ and $\mathsf{S}\tilde{\mathsf{a}}(z) = \tilde{\epsilon} z^{\tilde{c}}$, it follows directly from $\mathsf{u}(z) = \mathsf{a}(z)\tilde{\mathsf{a}}(z)^\star$ that $\mathsf{S}\mathsf{u}(z) = \epsilon\tilde{\epsilon} z^{c-\tilde{c}}$, that is,

$$u(c - \tilde{c} - k) = \epsilon\tilde{\epsilon}u(k), \quad \forall k \in \mathbb{Z}. \tag{2.3}$$

We consider two cases. If $c - \tilde{c}$ is a multiple of d, then since u is interpolatory, it follows directly from (2.3) that we must have $\tilde{c} = c$ and $\tilde{\epsilon} = \epsilon$. Consequently, for this case, we must have $\mathsf{S}\mathsf{a} = \mathsf{S}\tilde{\mathsf{a}}$.

If $c - \tilde{c}$ is not a multiple of d, then since u is interpolatory (or $u^{[0]} = \delta$), it follows directly from (2.3) that $u^{[c-\tilde{c}]} = \epsilon\tilde{\epsilon}\delta$. Since $u^{[0]}$ and $u^{[c-\tilde{c}]}$ are two distinct coset sequences with a single tap, u does not satisfy the sum-rule condition with order greater than 1. This implies that not both a and its dual $\tilde{a}$ could satisfy the sum-rule condition at all. In other words, for the filter pair to satisfy the sum-rule condition and certain symmetry constraint, u must satisfy the sum-rule condition of order at least 2; and hence, $\mathsf{S}\mathsf{a} = \mathsf{S}\tilde{\mathsf{a}}$.

On the other hand, if a primal filter a satisfies the symmetry property: $\mathsf{S}\mathsf{a}(z) = \epsilon z^c$, but its dual filter $\tilde{a}$ so constructed is not symmetric or anti-symmetric, then we can employ the simple symmetrization scheme

$$\tilde{a}^{sym}(k) = \big(\tilde{a}(k) + \epsilon\tilde{a}(c-k)\big)/2, \quad k \in \mathbb{Z}, \tag{2.4}$$

since it is clear that $\tilde{a}^{sym}$ is also a d-dual of a. Furthermore, it follows from the equivalent formulation $\tilde{\mathsf{a}}^{sym}(z) = (\tilde{\mathsf{a}}(z) + \epsilon z^c \tilde{\mathsf{a}}(1/z))/2$ of (2.4) that $\mathsf{S}\tilde{\mathsf{a}}^{sym}(z) = \mathsf{S}\mathsf{a}(z) = \epsilon z^c$; that is, $\tilde{a}^{sym}$ has the same symmetry pattern as a.

Hence, in the development of our dual-chain algorithm with the symmetry constraint, we always assume that both filters of a given dual pair have the same symmetry pattern. In the following algorithm, we build the dual top–down and bottom–up chains that meet the symmetry constraint requirement.

Algorithm 2. Dual-chain algorithm for the construction of dual filter systems (with symmetry constraint).

(a) **Input:** $(a, \tilde{a})$, a pair of d-dual filters such that a and $\tilde{a}$ have the same symmetry pattern: $\mathsf{S}\mathsf{a}(z) = \mathsf{S}\tilde{\mathsf{a}}(z) = \epsilon z^c$ for some $\epsilon \in \{-1, 1\}$ and some $c \in \mathbb{Z}$.
(b) **Top–down chain:** This part of the algorithm produces a top–down chain $a_0 \to a_1 \to \cdots \to a_r$, see Fig. 1.3 (left). Each a_j, $j = 2, \ldots, r$ preserves the symmetry pattern of a and $\tilde{a}$: $\mathsf{S}\mathsf{a}_j(z) = \epsilon z^c$ for $j = 2, \ldots, r$ with $\mathsf{a}_j(z)$ being the symbol of a_j.

1: In order to construct a top–down chain with as few number of filters as possible, swap a and $\tilde{a}$, if necessary, such that $|\mathrm{suppintv}(a)| \geqslant |\mathrm{suppintv}(\tilde{a})|$.
2: $(a_0, a_1) \leftarrow (a, \tilde{a})$. $j \leftarrow 1$.
3: Set $\Gamma := \emptyset$ to be the empty set. Then the cardinality $\#(\Gamma) = 0$.
4: **for** $\gamma = 0$ to $\mathsf{d} - 1$ **do** **if** $\mathsf{a}_j^{[\gamma]}(z) \neq 0$ **then** $\Gamma \leftarrow \Gamma \cup \{\gamma\}$. **end if** **end for**
5: **while** $(|\mathrm{suppintv}(a_j)| \geqslant 2$ and $\#(\Gamma) > 2)$ **do**
6: $a_{j+1} \leftarrow a_{j-1}$.
  $m \leftarrow \min\{\mathrm{suppintv}(a_j)\}$. $n \leftarrow \max\{\mathrm{suppintv}(a_j)\}$. $\tilde{m} \leftarrow \min\{\mathrm{suppintv}(a_{j+1})\}$.
7: **while** $\tilde{m} < m$ **do** Perform Line 6 and Line 7 of Algorithm 1. **end while**

8: $\tilde{n} \leftarrow \max\{\mathrm{suppintv}(a_{j+1})\}$.

9: **while** $\tilde{n} \geqslant n$ **do** Perform Line 11 and Line 12 of Algorithm 1. **end while**

10: $\tilde{m} \leftarrow \min\{\mathrm{suppintv}(a_{j+1})\}$. $\tilde{n} \leftarrow \max\{\mathrm{suppintv}(a_{j+1})\}$.

11: **if** $\tilde{m} = m$ **then**

12: Choose any $\ell \in \mathbb{Z}$ such that $\rho_\ell \in \Gamma$ and $\rho_\ell \notin \{\rho_m, \rho_n\}$. Let m_1, n_1, ℓ_1 be the lowest degrees of the Laurent polynomials $z^{\rho_m}\mathsf{a}_j^{[\rho_m]}(z^{\mathsf{d}})$, $z^{\rho_n}\mathsf{a}_j^{[\rho_n]}(z^{\mathsf{d}})$, $z^{\rho_\ell}\mathsf{a}_j^{[\rho_\ell]}(z^{\mathsf{d}})$, respectively. Let m_2, n_2, ℓ_2 be the highest degrees of the Laurent polynomials $z^{\rho_m}\mathsf{a}_j^{[\rho_m]}(z^{\mathsf{d}})$, $z^{\rho_n}\mathsf{a}_j^{[\rho_n]}(z^{\mathsf{d}})$, $z^{\rho_\ell}\mathsf{a}_j^{[\rho_\ell]}(z^{\mathsf{d}})$, respectively (see (4.2)).

13: Define $\tilde{v}_1$ and $\tilde{v}_2$ through their symbols $\tilde{\mathsf{v}}_1(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \tilde{\mathsf{v}}_1^{[\gamma]}(z^{\mathsf{d}})$ and $\tilde{\mathsf{v}}_2(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \tilde{\mathsf{v}}_2^{[\gamma]}(z^{\mathsf{d}})$, where

$$\tilde{\mathsf{v}}_1^{[\gamma]}(z) := \begin{cases} \mathsf{a}_j^{[\rho_n]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}}, & \gamma = \rho_m; \\ -\mathsf{a}_j^{[\rho_m]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}}, & \gamma = \rho_n; \\ 0, & \text{otherwise}, \end{cases}$$

$$\tilde{\mathsf{v}}_2^{[\gamma]}(z) := \begin{cases} \frac{\overline{a_j(m_1)}}{\overline{a_j(\ell_1)}} \cdot \mathsf{a}_j^{[\rho_\ell]}(z)^\star \cdot z^{\lambda_{\ell_1}+\lambda_{n_2}}, & \gamma = \rho_n; \\ -\frac{\overline{a_j(m_1)}}{\overline{a_j(\ell_1)}} \cdot \mathsf{a}_j^{[\rho_n]}(z)^\star \cdot z^{\lambda_{\ell_1}+\lambda_{n_2}}, & \gamma = \rho_\ell; \\ 0, & \text{otherwise}. \end{cases}$$

14: $b \leftarrow \tilde{v}_1 + \tilde{v}_2$. $\alpha \leftarrow \rho_{\ell_1 - n_1}$. $k_0 \leftarrow \lambda_{\ell_1 - n_1}$.

15: **for** $\kappa = 0$ to k_0 **do**

16: Construct $\tilde{v}$ through its symbol $\tilde{\mathsf{v}}(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \tilde{\mathsf{v}}^{[\gamma]}(z^{\mathsf{d}})$ with

$$\tilde{\mathsf{v}}^{[\gamma]}(z) := \begin{cases} C_\kappa z^{-\kappa} \mathsf{a}_j^{[\rho_\ell]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}-\lambda_{n_1}+\lambda_{\ell_1}}, & \gamma = \rho_m; \\ -C_\kappa z^{-\kappa} \mathsf{a}_j^{[\rho_m]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}-\lambda_{n_1}+\lambda_{\ell_1}}, & \gamma = \rho_\ell; \\ 0, & \text{otherwise}, \end{cases}$$

where $C_\kappa := \frac{b(n_2+\ell_1-n_1-\mathsf{d}\kappa)}{\overline{a_j(m_1)}}$.

17: $b \leftarrow b + \tilde{v}$.

18: **end for**

19: $a_{j+1} \leftarrow a_{j+1} - \frac{a_{j+1}(m)}{b(m)} b$.

20: **end if**

21: Let $\mathsf{a}_{j+1}(z)$ be the symbol of a_{j+1} and construct a_{j+1}^{sym} whose symbol $\mathsf{a}_{j+1}^{sym}(z)$ is defined to be $\mathsf{a}_{j+1}^{sym}(z) := (\mathsf{a}_{j+1}(z) + \epsilon z^c \mathsf{a}_{j+1}(1/z))/2$.

22: $a_{j+1} \leftarrow a_{j+1}^{sym}$. $\Gamma := \{\}$.

23: **for** $\gamma = 0$ to $\mathsf{d}-1$ **do if** $\mathsf{a}_{j+1}^{[\gamma]}(z) \neq 0$ **then** $\Gamma \leftarrow \Gamma \cup \{\gamma\}$. **end if end for**

24: $j \leftarrow j+1$.

25: **end while**

(c) **Bottom–up chain:** This part of the algorithm constructs a bottom–up chain of dual filter systems with symmetry, see Fig. 1.3 (right).

26: $r \leftarrow j$.

Either a_r has only one tap: $\mathsf{a}_r(z) = t_0 z^{c/2}$ for some nonzero constant $t_0 \in \mathbb{F}$; or a_r has exactly two nontrivial polyphase components: $\mathsf{a}_r(z) = z^\beta \mathsf{a}_r^{[\beta]}(z^{\mathsf{d}}) + z^\gamma \mathsf{a}_r^{[\gamma]}(z^{\mathsf{d}})$ for some $0 \leqslant \beta < \gamma \leqslant \mathsf{d}-1$.

27: $W \leftarrow I_{\mathsf{d}}$. I_{d} is the $\mathsf{d} \times \mathsf{d}$ identity matrix.

28: **for** $\gamma = 0$ to $\mathsf{d}-1$ **do**

29: **if** $\gamma < \rho_{c-\gamma}$ **then**

30:

$$[W]_{\{\gamma+1, \rho_{c-\gamma}+1\}, \{\gamma+1, \rho_{c-\gamma}+1\}} := \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

where $[T]_{\{j_1, j_2\}, \{j_1, j_2\}} := M$ is a 2×2 submatrix of T such that $[M]_{k,\ell} = [T]_{j_k, j_\ell}$ for $k, \ell = 1, 2$. $[T]_{j,k}$ is the (j,k)-entry of the matrix T.

31: **end if**

32: **end for**

33: **if** $\mathsf{a}_r(z) = t_0 z^{c/2}$ **then**

34: $W_0(z) \leftarrow W$ and $[W_0(z)]_{\rho_{c/2}, \rho_{c/2}} \leftarrow \mathsf{a}_r^{[\rho_{c/2}]}(z)$

35: **else**

36: $W_0(z) \leftarrow W$ and

$$\big[W_0(z)\big]_{\{\beta,\gamma\},\{\beta,\gamma\}} \leftarrow \frac{1}{2}\begin{bmatrix} \mathsf{a}_r^{[\beta]}(z) & \mathsf{a}_r^{[\gamma]}(z) \\ \mathsf{a}_{r-1}^{[\gamma]}(z)^\star & -\mathsf{a}_{r-1}^{[\beta]}(z)^\star \end{bmatrix},$$

where $\mathsf{a}_{r-1}(z)$ is the symbol of a_{r-1}.

37: **end if**

38: Let E be a permutation matrix such that the first row of $EW_0(z)$ is the polyphase vector $[\mathsf{a}_r^{[0]}, \ldots, \mathsf{a}_r^{[\mathsf{d}-1]}]$ of a_r. $\mathbf{P}_r(z) \leftarrow EW_0(z)$. Then $\mathbf{P}_r(z)$ is of the form:

$$\mathbf{P}_r(z) := \begin{bmatrix} \mathsf{a}_r^{[0]}(z) & \cdots & \mathsf{a}_r^{[\mathsf{d}-1]}(z) \\ \mathsf{b}_{r,1}^{[0]}(z) & \cdots & \mathsf{b}_{r,1}^{[\mathsf{d}-1]}(z) \\ \vdots & \ddots & \vdots \\ \mathsf{b}_{r,\mathsf{d}-1}^{[0]}(z) & \cdots & \mathsf{b}_{r,\mathsf{d}-1}^{[\mathsf{d}-1]}(z) \end{bmatrix}.$$

Define band-pass filters $b_{r,1}, \ldots, b_{r,\mathsf{d}-1}$ so that their symbols are given by $\mathsf{b}_{r,1}(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \mathsf{b}_{r,1}^{[\gamma]}(z^\mathsf{d}), \ldots, \mathsf{b}_{r,\mathsf{d}-1}(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \mathsf{b}_{r,\mathsf{d}-1}^{[\gamma]}(z^\mathsf{d})$, respectively.

39: $j \leftarrow r$.

40: **while** $j \geqslant 1$ **do** Perform Lines 19–24 of Algorithm 1. **end while**

(d) **Output:** a chain of dual filter systems with the perfect reconstruction property and with symmetry, see Fig. 1.3.

In both Algorithms 1 and 2, we have presented a step-by-step procedure in Step (b) for the construction of a desirable a_{j+1} from a_j and a_{j-1}. We remark, however, that this step can be replaced by solving an associated system of linear equations, and the desirable filter a_{j+1} (with or without the symmetry constraint) so obtained not only satisfies the requirement that a_{j+1} is d-dual of a_j and $\operatorname{suppintv}(a_{j+1}) \subsetneq \operatorname{suppintv}(a_j)$, but its filter length $|\operatorname{suppintv}(a_{j+1})|$ is also the smallest. This system of linear equations is guaranteed to have a solution in view of the existence results in Theorems 1 and 2. As a consequence of replacing Step (b) with this alternative step, the shortest dual chain is achieved.

We also point out the matrix $V(z)$ in Step (c) of Algorithms 1 and 2 is not uniquely determined and there are many different choices of such $V(z)$. Consequently, various choices of $V(z)$ lead to band-pass filters with varying length and the particular simple choice of $V(z)$ in Step (c) of Algorithms 1 and 2 may not lead to band-pass filters with the shortest possible support intervals. To have band-pass filters with short support intervals, some simple tuning on the structure of $V(z)$ might be needed; or we can perform some simple postprocessing (such as simple linear combinations) on the newly constructed wavelet filter systems $((a_j; \vec{b}_j), (a_{j+1}; \vec{b}_{j+1}))$ in Step (c) of Algorithms 1 and 2 so that their support intervals are balanced and short.

2.3. Biorthogonal wavelets in $L_2(\mathbb{R})$

For a pair $((a; b_1, \ldots, b_{\mathsf{d}-1}), (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_{\mathsf{d}-1}))$ of biorthogonal d-wavelet filter systems that satisfies $\sum_{k\in\mathbb{Z}} a(k) = \sum_{k\in\mathbb{Z}} \tilde{a}(k) = 1$, it is shown in [19] that the pair is always associated with an underlying pair of frequency-based dual d-framelets in the distribution space. In this subsection, we shall discuss biorthogonal wavelets in $L_2(\mathbb{R})$ that are associated with biorthogonal wavelet filter systems. Let us first recall some notations and definitions. For a function $f : \mathbb{R} \to \mathbb{C}$,

$$f_{\lambda;k} := |\lambda|^{1/2} f(\lambda \cdot - k), \quad \lambda \in \mathbb{R}\backslash\{0\},\ k \in \mathbb{R}.$$

Let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. The square-integrable functions $\phi, \psi^1, \ldots, \psi^s$ in $L_2(\mathbb{R})$ are said to generate a *nonhomogeneous* d-*wavelet system*:

$$\mathsf{WS}_0\big(\phi; \psi^1, \ldots, \psi^s\big) := \big\{\phi(\cdot - k) \,\big|\, k \in \mathbb{Z}\big\} \cup \big\{\psi^\ell_{\mathsf{d}^j;k} \,\big|\, j \in \mathbb{N}_0,\ k \in \mathbb{Z},\ \ell = 1, \ldots, s\big\}$$

(see [19]). For square-integrable functions $\phi, \psi^1, \ldots, \psi^s$ and $\tilde{\phi}, \tilde{\psi}^1, \ldots, \tilde{\psi}^s$, we say that

$$\big(\{\phi; \psi^1, \ldots, \psi^s\}, \{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\}\big) \tag{2.5}$$

generates a pair of biorthogonal d-wavelet bases in $L_2(\mathbb{R})$, if

$$\big(\mathsf{WS}_0\big(\phi; \psi^1, \ldots, \psi^s\big), \mathsf{WS}_0\big(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s\big)\big)$$

is a pair of biorthogonal bases in $L_2(\mathbb{R})$; that is, each of the systems $\mathsf{WS}_0(\phi; \psi^1, \ldots, \psi^s)$ and $\mathsf{WS}_0(\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^s)$ is a Riesz basis of $L_2(\mathbb{R})$ and the two systems are biorthogonal to each other in $L_2(\mathbb{R})$. Consequently, the following identity holds:

$$\langle f, g\rangle = \sum_{k\in\mathbb{Z}} \big\langle f, \phi(\cdot - k)\big\rangle\big\langle \tilde{\phi}(\cdot - k), g\big\rangle + \sum_{j=0}^{\infty}\sum_{\ell=1}^{s}\sum_{k\in\mathbb{Z}} \big\langle f, \psi^\ell_{\mathsf{d}^j;k}\big\rangle\big\langle \tilde{\psi}^\ell_{\mathsf{d}^j;k}, g\big\rangle, \quad f, g \in L_2(\mathbb{R}).$$

It has been shown in [19] that if the pair in (2.5) generates a pair of biorthogonal d-wavelet bases in $L_2(\mathbb{R})$, then we must have $s = \mathsf{d} - 1$. See [1,7,10,11,13,14,16,18,21,25,29,32,33] for biorthogonal wavelets in $L_2(\mathbb{R})$.

For $0 < \alpha \leqslant 1$ and $1 \leqslant p \leqslant \infty$, we say that $f \in \mathrm{Lip}(\alpha, L_p(\mathbb{R}))$ if there is a constant C such that $\|f - f(\cdot - h)\|_{L_p(\mathbb{R})} \leqslant Ch^\alpha$ for all $h > 0$. The smoothness of a function f in $L_p(\mathbb{R})$ is measured by

$$\nu_p(f) := \sup\bigl\{n + \alpha \bigm| n \in \mathbb{N}_0,\ 0 < \alpha \leqslant 1,\ f^{(n)} \in \mathrm{Lip}\bigl(\alpha, L_p(\mathbb{R})\bigr)\bigr\},$$

where $f^{(n)}$ denotes the nth derivative of f.

In order to state the result on biorthogonal d-wavelets in $L_2(\mathbb{R})$ associated with biorthogonal d-wavelet filter banks, we also need to recall a quantity $\nu_p(a, \mathsf{d})$ for a low-pass filter a and $1 \leqslant p \leqslant \infty$. Since the symbol a of a is a Laurent polynomial, we can write $\mathsf{a}(z) = (1 + z + \cdots + z^{\mathsf{d}-1})^m \mathsf{Q}(z)$ for some Laurent polynomial Q such that $(1 + z + \cdots + z^{\mathsf{d}-1}) \nmid \mathsf{Q}(z)$. Following [15, p. 61 and Proposition 7.2], we may define

$$\nu_p(a, \mathsf{d}) := 1/p - 1 - \log_{\mathsf{d}}\Bigl(\limsup_{n\to\infty} \|\mathsf{Q}_n\|_{\ell_p(\mathbb{Z})}^{1/n}\Bigr), \quad 1 \leqslant p \leqslant \infty,$$

where $\|\mathsf{Q}_n\|_{\ell_p(\mathbb{Z})}^p := \sum_{k\in\mathbb{Z}} |\mathsf{Q}_n(k)|^p$ and $\sum_{k\in\mathbb{Z}} \mathsf{Q}_n(k) z^k := \mathsf{Q}(z)\mathsf{Q}(z^{\mathsf{d}}) \cdots \mathsf{Q}(z^{\mathsf{d}^{n-1}})$. It has been proved in [15, Theorem 4.3] that the cascade algorithm with some mask (low-pass filter) a and a dilation factor d converges in $L_p(\mathbb{R})$ (as well as $C(\mathbb{R})$ when $p = \infty$) if and only if $\nu_p(a, \mathsf{d}) > 0$. Let ϕ be the compactly supported normalized d-refinable distribution with mask a and dilation d such that $\hat{\phi}(\xi) := \prod_{j=1}^\infty \mathsf{a}(e^{-i\mathsf{d}^{-j}\xi})$. In general, we have $\nu_p(a, \mathsf{d}) \leqslant \nu_p(\phi)$. If the integer shifts of ϕ form a Riesz system, then $\nu_p(a, \mathsf{d}) = \nu_p(\phi)$. The quantity $\nu_p(a, \mathsf{d})$ plays an important role in the study of the convergence of cascade algorithms and smoothness of refinable functions, see [15,16] and the references therein on these topics. Moreover, when $p = 2$, we also have

$$\nu_2(a, \mathsf{d}) = -1/2 - \log_{\mathsf{d}} \sqrt{\rho(a, \mathsf{d})}, \tag{2.6}$$

where $\rho(a, \mathsf{d})$ denotes the spectral radius of the square matrix $(u(\mathsf{d}j - k))_{-K \leqslant j,k \leqslant K}$, where $K := \lceil \frac{N}{\mathsf{d}-1} \rceil$ and $\mathsf{Q}(z)\mathsf{Q}(z)^\star =: \sum_{k=-N}^{k=N} u(k) z^k$ (see [12, Theorem 2.1]).

In what follows, let

$$\begin{aligned}(a; b_1, \ldots, b_{\mathsf{d}-1}) &:= (a_0; b_{0,1}, \ldots, b_{0,\mathsf{d}-1}),\\ (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_{\mathsf{d}-1}) &:= \bigl(a_1; b_{1,1}^\sharp, \ldots, b_{1,\mathsf{d}-1}^\sharp\bigr),\end{aligned} \tag{2.7}$$

where $((a_0; b_{0,1}, \ldots, b_{0,\mathsf{d}-1}), (a_1; b_{1,1}^\sharp, \ldots, b_{1,\mathsf{d}-1}^\sharp))$ is computed by applying the bottom–up procedure in Algorithm 1 or 2 (see also Fig. 1.3). Define a pair of generators $(\{\phi; \psi^1, \ldots, \psi^{\mathsf{d}-1}\}, \{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^{\mathsf{d}-1}\})$ of distributions associated with $(a; b_1, \ldots, b_{\mathsf{d}-1})$ and $(\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_{\mathsf{d}-1})$ as follows.

$$\hat{\phi}(\xi) := \prod_{j=1}^\infty \mathsf{a}\bigl(e^{-i\mathsf{d}^{-j}\xi}\bigr) \quad\text{and}\quad \hat{\tilde{\phi}}(\xi) := \prod_{j=1}^\infty \tilde{\mathsf{a}}\bigl(e^{-i\mathsf{d}^{-j}\xi}\bigr), \quad \xi \in \mathbb{R} \tag{2.8}$$

and

$$\psi^\ell(x) := \mathsf{d}\sum_{k\in\mathbb{Z}} b_\ell(k)\phi(\mathsf{d}x - k) \quad\text{and}\quad \tilde{\psi}^\ell(x) := \mathsf{d}\sum_{k\in\mathbb{Z}} \tilde{b}_\ell(k)\tilde{\phi}(\mathsf{d}x - k), \tag{2.9}$$

for $\ell = 1, \ldots, \mathsf{d} - 1$. Then, we have the following result.

Theorem 3. *Let $(a, \tilde{a})$ be a pair of* d*-dual filters. Then the filter systems $(a; b_1, \ldots, b_{\mathsf{d}-1})$ and $(\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_{\mathsf{d}-1})$ defined in* (2.7) *satisfy the duality property*:

$$\mathbf{P}(z)\tilde{\mathbf{P}}(z)^\star = \mathsf{d}^{-1} I_{\mathsf{d}}, \tag{2.10}$$

where $\mathbf{P}(z)$ and $\tilde{\mathbf{P}}(z)$ are defined as in (1.13)*. In addition, if a and $\tilde{a}$ have the same symmetry pattern*:

$$\mathsf{S}\mathsf{a}(z) = \mathsf{S}\tilde{\mathsf{a}}(z) = \epsilon_0 z^{c_0}, \quad \epsilon_0 \in \{1, -1\};\ c_0 \in \mathbb{Z}, \tag{2.11}$$

then all band-pass filters b_j and $\tilde{b}_j$ have the symmetry patterns:

$$\mathsf{S}\mathsf{b}_1(z) = \mathsf{S}\tilde{\mathsf{b}}_1(z) = \epsilon_1 z^{c_1}, \quad \ldots, \quad \mathsf{S}\mathsf{b}_{\mathsf{d}-1}(z) = \mathsf{S}\tilde{\mathsf{b}}_{\mathsf{d}-1}(z) = \epsilon_{\mathsf{d}-1} z^{c_{\mathsf{d}-1}} \tag{2.12}$$

for some $\epsilon_j \in \{1, -1\}$ and some $c_j \in \mathbb{Z}$, $j = 1, \ldots, \mathsf{d} - 1$. Furthermore, if $\sum_{k\in\mathbb{Z}} a(k) = \sum_{k\in\mathbb{Z}} \tilde{a}(k) = 1$ (that is, a and $\tilde{a}$ are low-pass filters) and $\nu_2(a, \mathsf{d}) > 0$, $\nu_2(\tilde{a}, \mathsf{d}) > 0$, then the pair

$$\bigl(\bigl\{\phi; \psi^1, \ldots, \psi^{\mathsf{d}-1}\bigr\}, \bigl\{\tilde{\phi}; \tilde{\psi}^1, \ldots, \tilde{\psi}^{\mathsf{d}-1}\bigr\}\bigr) \tag{2.13}$$

defined by (2.8) *and* (2.9) *generates a pair of biorthogonal* d*-wavelet bases in* $L_2(\mathbb{R})$. *Moreover, if* (2.11) *and* (2.12) *are satisfied, then the biorthogonal system* (2.13) *associated with the dual filter bank system* $((a; b_1, \ldots, b_{\mathrm{d}-1}), (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_{\mathrm{d}-1}))$ *has the following symmetry property*:

$$\phi = \phi\left(\frac{c_0}{\mathrm{d}-1} - \cdot\right), \qquad \tilde{\phi} = \tilde{\phi}\left(\frac{c_0}{\mathrm{d}-1} - \cdot\right) \tag{2.14}$$

and

$$\begin{aligned} &\psi^1 = \epsilon_1 \psi^1\left(\frac{c_1}{\mathrm{d}} + \frac{c_0}{\mathrm{d}(\mathrm{d}-1)} - \cdot\right), \quad \ldots, \quad \psi^{\mathrm{d}-1} = \epsilon_{\mathrm{d}-1} \psi^{\mathrm{d}-1}\left(\frac{c_{\mathrm{d}-1}}{\mathrm{d}} + \frac{c_0}{\mathrm{d}(\mathrm{d}-1)} - \cdot\right), \\ &\tilde{\psi}^1 = \epsilon_1 \tilde{\psi}^1\left(\frac{c_1}{\mathrm{d}} + \frac{c_0}{\mathrm{d}(\mathrm{d}-1)} - \cdot\right), \quad \ldots, \quad \tilde{\psi}^{\mathrm{d}-1} = \epsilon_{\mathrm{d}-1} \tilde{\psi}^{\mathrm{d}-1}\left(\frac{c_{\mathrm{d}-1}}{\mathrm{d}} + \frac{c_0}{\mathrm{d}(\mathrm{d}-1)} - \cdot\right). \end{aligned} \tag{2.15}$$

Proof. The duality property in (2.10) is a consequence of our bottom–up construction. In fact, for each $j = 1, \ldots, r$, the polyphase matrices $\mathbf{P}_{j-1}(z)$ and $\mathbf{P}_j^\sharp(z)$ (see Fig. 1.3) for the filter systems $(a_{j-1}; b_{j-1,1}, \ldots, b_{j-1,\mathrm{d}-1})$ and $(a_j; b_{j,1}^\sharp, \ldots, b_{j,\mathrm{d}-1}^\sharp)$, respectively, satisfy the identities $\mathbf{P}_{j-1}(z)\mathbf{P}_j^\sharp(z)^\star = \mathrm{d}^{-1} I_{\mathrm{d}}$, thereby $\mathbf{P}(z) = \mathbf{P}_0(z)$ and $\tilde{\mathbf{P}}(z) = \mathbf{P}_1^\sharp(z)$ satisfy (2.10). Since $(a, \tilde{a})$ is a pair of d-dual filters, by applying [16, Theorem 3.1] or [15], both ϕ and $\tilde{\phi}$ are compactly supported refinable functions in $L_2(\mathbb{R})$ and their integer shifts are biorthogonal to each other. By [16, Theorem 3.1], (2.13) generates a pair of biorthogonal d-wavelet bases in $L_2(\mathbb{R})$. The proof of symmetry property of the dual filter system $((a; b_1, \ldots, b_{\mathrm{d}-1}), (\tilde{a}; \tilde{b}_1, \ldots, \tilde{b}_{\mathrm{d}-1}))$ follows from [12, Lemmas 4.1 and 4.2] or [22, Proofs of Lemma 1 and Algorithm 1]. □

Note that the time-domain formulation of the matrix identity in (2.10) is given by:

$$\begin{aligned} &\sum_{k\in\mathbb{Z}} \overline{a(k)}\tilde{a}(\mathrm{d}j+k) = \mathrm{d}^{-1}\delta(j), \quad j \in \mathbb{Z}; \\ &\sum_{k\in\mathbb{Z}} \overline{b_\ell(k)}\tilde{b}_{\ell'}(\mathrm{d}j+k) = \mathrm{d}^{-1}\delta(j)\delta(\ell-\ell'), \quad j \in \mathbb{Z};\ \ell, \ell' = 1, \ldots, \mathrm{d}-1; \\ &\sum_{k\in\mathbb{Z}} \overline{a(k)}\tilde{b}_\ell(\mathrm{d}j+k) = 0, \quad j \in \mathbb{Z},\ \ell = 1, \ldots, \mathrm{d}-1; \\ &\sum_{k\in\mathbb{Z}} \overline{b_\ell(k)}\tilde{a}(\mathrm{d}j+k) = 0, \quad j \in \mathbb{Z},\ \ell = 1, \ldots, \mathrm{d}-1. \end{aligned}$$

The interested reader is referred to [12,17,22] for general discussion on the symmetry property of filter banks. For more references on biorthogonal and orthogonal wavelets in $L_2(\mathbb{R})$, see [1,2,5,6,8,9,12,17,22,23,26–28,30,32,33].

3. Illustrative examples

In this section, we will illustrate our algorithms and results stated in Section 2 by examples of wavelet filter systems for cardinal B-splines as well as by applying the CBC algorithm to construct d-dual filter pairs. We will only consider dilation factors $\mathrm{d} = 3$ and $\mathrm{d} = 4$. Since the length of the coefficient support interval of the dual $\tilde{a}$ is generally longer than that of a, we will choose the initial dual pair $(a_0, a_1) = (\tilde{a}, a)$ for the top–down dual chain to minimize the number of iterative steps.

In the first two examples, a will denote the d-refinement sequence of the (centered) cardinal B-splines under consideration, with the desirable dual filter denoted by $\tilde{a}$. In terms of their symbols $\mathrm{a}(z)$ and $\tilde{\mathrm{a}}(z)$, we select a dual filter $\tilde{a}$ such that $\mathrm{a}(z)\tilde{\mathrm{a}}(z)^\star = \mathrm{a}_{2n}^I(z)$ defined in (1.3) for some suitable integer n.

Example 1. Let the dilation factor be $\mathrm{d} = 3$ and consider the refinement sequence a of the centered cardinal linear B-spline $N_2(\cdot + 1)$. By splitting the mask a_4^I in (1.3) with $n = 2$ as described above, we obtain the desired dual pair $(a, \tilde{a})$, with symbols $\mathrm{a}(z), \tilde{\mathrm{a}}(z)$ given by

$$\mathrm{a}(z) := \left(\frac{1+z+z^2}{3}\right)^2 z^{-2}, \qquad \tilde{\mathrm{a}}(z) := \left(\frac{1+z+z^2}{3}\right)^2 z^{-2}\tilde{\mathrm{p}}_0(z)$$

where $\tilde{\mathrm{p}}_0(z) = \frac{1}{3}(-4z + 11 - 4z^{-1})$. Note that the coefficient support interval of a is $[-2, 2]$, and that of $\tilde{a}$ is $[-3, 3]$. Both a and $\tilde{a}$ are symmetric about 0: $\mathrm{S}a = \mathrm{S}\tilde{a} = 1$.

By Lemma 2 in Section 4 (or the top–down chain in Algorithm 2), we obtain a chain of consecutive 3-dual filters $a_0 \to a_1 \to a_2$ with $a_0 = \tilde{a}$, $a_1 = a$, and $a_2 = \delta$.

Since $a_2 = \delta$, it is easy to construct the band-pass filters $b_{2,1}, b_{2,2}$ by the following unimodular polyphase matrix $\mathbf{P}_2(z)$:

$$\mathbf{P}_2(z) = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & -1 \end{bmatrix};$$

that is, the symbols of $b_{2,1}$ and $b_{2,2}$ are $\mathsf{b}_{2,1}(z) = \frac{1}{2}(z+z^2)$ and $\mathsf{b}_{2,2}(z) = \frac{1}{2}(z-z^2)$. Note that $b_{2,1}$ is symmetric about $3/2$: $\mathsf{Sb}_{2,1}(z) = z^3$, while $b_{2,2}$ is anti-symmetric about $3/2$: $\mathsf{Sb}_{2,2}(z) = -z^3$.

Next by applying Algorithm 2 to construct the bottom–up chain, we can derive the band-pass filters for the given pair $(a, \tilde{a})$, such that the pair $((a; b_1, b_2), (\tilde{a}; \tilde{b}_1, \tilde{b}_2))$ constitutes a dual filter system with the perfect reconstruction property. The symbols $\mathsf{b}_1(z), \mathsf{b}_2(z)$ and $\tilde{\mathsf{b}}_1(z), \tilde{\mathsf{b}}_2(z)$ of b_1, b_2 and $\tilde{b}_1, \tilde{b}_2$ are given by

$$\mathsf{b}_1(z) = \mathsf{q}_1(z) + z^3\mathsf{q}_1\big(z^{-1}\big), \qquad \mathsf{b}_2(z) = \mathsf{q}_2(z) - z^3\mathsf{q}_2\big(z^{-1}\big),$$
$$\tilde{\mathsf{b}}_1(z) = \tilde{\mathsf{q}}_1(z) + z^3\tilde{\mathsf{q}}_1\big(z^{-1}\big), \qquad \tilde{\mathsf{b}}_2(z) = \tilde{\mathsf{q}}_2(z) - z^3\tilde{\mathsf{q}}_2\big(z^{-1}\big),$$

with

$$\mathsf{q}_1(z) = \frac{1}{27}\big(6z^2 - 3z^3 - 2z^4 - z^5\big), \qquad \mathsf{q}_2(z) = \frac{1}{81}\big(-26z^2 + 3z^3 + 2z^4 + z^5\big),$$
$$\tilde{\mathsf{q}}_1(z) = -\frac{1}{2}(1-z), \qquad \tilde{\mathsf{q}}_2(z) = -\frac{1}{6}(1-3z).$$

Note that b_1 and $\tilde{b}_1$ are both symmetric about $3/2$: $\mathsf{Sb}_1(z) = \mathsf{S}\tilde{\mathsf{b}}_1(z) = z^3$, while b_2 and $\tilde{b}_2$ are both anti-symmetric about $3/2$: $\mathsf{Sb}_2(z) = \mathsf{S}\tilde{\mathsf{b}}_2(z) = -z^3$. Also, $\mathrm{suppintv}(b_1) = \mathrm{suppintv}(b_2) = [-2, 5]$ and $\mathrm{suppintv}(\tilde{b}_1) = \mathrm{suppintv}(\tilde{b}_2) = [0, 3]$.

Moreover, by calculation (see (2.6)), we have $\nu_2(a, 3) = -1/2 - \log_3(\sqrt{1/81}) = 1.5$ and $\nu_2(\tilde{a}, 3) = -1/2 - \log_3(\sqrt{17/81}) \approx 0.2105$. Hence, it follows from Theorem 3 that the pair $(\{\phi; \psi^1, \psi^2\}, \{\tilde{\phi}; \tilde{\psi}^1, \tilde{\psi}^2\})$, associated with the dual filter system $((a; b_1, b_2), (\tilde{a}; \tilde{b}_1, \tilde{b}_2))$, generates a biorthogonal 3-wavelet basis of $L_2(\mathbb{R})$. The symmetry patterns of the functions are specified in (2.14) and (2.15) with $c_0 = 1$, $\epsilon_1 = 1$, $c_1 = 3$, and $\epsilon_2 = -1$, $c_2 = 3$.

Example 2. Let the dilation factor be $\mathsf{d} = 4$ and consider the refinement sequence a of the centered cardinal cubic B-spline $N_4(\cdot + 2)$. Again, by splitting the mask a_6^I in (1.3) with $n = 3$, we can obtain the desired dual $(a, \tilde{a})$ with symbols $\mathsf{a}(z)$ and $\tilde{\mathsf{a}}(z)$ given by

$$\mathsf{a}(z) := \left(\frac{1+z+z^2+z^3}{4}\right)^4 z^{-6}, \qquad \tilde{\mathsf{a}}(z) := \left(\frac{1+z+z^2+z^3}{4}\right)^2 z^{-3}\tilde{\mathsf{p}}_0(z),$$

where $\tilde{\mathsf{p}}_0(z) = \frac{1}{8}(63(z^{-2}+z^2) - 282(z^{-1}+z) + 446)$. Note that $\mathrm{suppintv}(a) = [-6, 6]$, $\mathrm{suppintv}(\tilde{a}) = [-5, 5]$, and both filters a and $\tilde{a}$ are symmetric about 0: $\mathsf{Sa} = \mathsf{S}\tilde{\mathsf{a}} = 1$.

By the same procedure as described in the above example, we obtain a chain of consecutive 4-dual filters $a_0 \to a_1 \to a_2 \to a_3 = \frac{1}{24}\delta$ with $a_0 = \tilde{a}$ and $a_1 = a$. The symbol $\mathsf{a}_2(z)$ of a_2 is given by $\mathsf{a}_2(z) = \frac{1}{12}(-5z^{-1} + 12 - 5z)$.

An analogous application of the bottom–up procedure in Algorithm 2 yields the band-pass filters for the pair $(a, \tilde{a})$ of 4-dual filters, such that $((a; b_1, b_2, b_3), (\tilde{a}; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3))$ constitutes a dual filter system with the perfect reconstruction property, and the symbols $\mathsf{b}_1(z), \mathsf{b}_2(z), \mathsf{b}_3(z)$ and $\tilde{\mathsf{b}}_1(z), \tilde{\mathsf{b}}_2(z), \tilde{\mathsf{b}}_3(z)$ of b_1, b_2, b_3 and $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$ are given as follows:

$$\mathsf{b}_1(z) = \mathsf{q}_1(z) + z^4\mathsf{q}_1\big(z^{-1}\big), \qquad \mathsf{b}_2(z) = \mathsf{q}_2(z) + \mathsf{q}_2\big(z^{-1}\big), \qquad \mathsf{b}_3(z) = \mathsf{q}_3(z) - \mathsf{q}_3\big(z^{-1}\big),$$
$$\tilde{\mathsf{b}}_1(z) = \tilde{\mathsf{q}}_1(z) + z^4\tilde{\mathsf{q}}_1\big(z^{-1}\big), \qquad \tilde{\mathsf{b}}_2(z) = \tilde{\mathsf{q}}_2(z) + \tilde{\mathsf{q}}_2\big(z^{-1}\big), \qquad \tilde{\mathsf{b}}_3(z) = \tilde{\mathsf{q}}_3(z) - \tilde{\mathsf{q}}_3\big(z^{-1}\big),$$

with

$$\mathsf{q}_1(z) = \frac{1}{4}\big(z^6 + 4z^5 + 10z^4 + 20z^3 - 35z^2\big),$$
$$\mathsf{q}_2(z) = \frac{1}{448}\big(-23z^6 - 92z^5 - 230z^4 - 460z^3 + 695z^2 + 104 + 6\big),$$
$$\mathsf{q}_3(z) = \frac{1}{512}\big(z^6 + 4z^5 + 10z^4 + 20z^3 - 55z^2 - 16z\big),$$
$$\tilde{\mathsf{q}}_1(z) = \frac{1}{64}\big(3z^5 - 8z^4 + 5z^3\big),$$
$$\tilde{\mathsf{q}}_2(z) = \frac{1}{512}\big(63z^5 - 156z^4 + 71z^3 + 16z^2 + 218z - 212\big),$$
$$\tilde{\mathsf{q}}_3(z) = \frac{1}{28}\big(-63z^5 + 156z^4 - 71z^3 - 16z^2 - 64z\big).$$

Note that b_1 and $\tilde{b}_1$ are both symmetric about 2: $\mathsf{Sb}_1(z) = \mathsf{S}\tilde{\mathsf{b}}_1(z) = z^4$, b_2 and $\tilde{b}_2$ are both symmetric about 0: $\mathsf{Sb}_2(z) = \mathsf{S}\tilde{\mathsf{b}}_2(z) = z^4$, and b_3 and $\tilde{b}_3$ are both anti-symmetric about 0: $\mathsf{Sb}_3(z) = \mathsf{S}\tilde{\mathsf{b}}_3(z) = -z^4$. Also, $\mathrm{suppintv}(b_1) = [-2, 6]$, $\mathrm{suppintv}(b_2) = \mathrm{suppintv}(b_3) = [-6, 6]$, and $\mathrm{suppintv}(\tilde{b}_1) = [-1, 5]$, $\mathrm{suppintv}(\tilde{b}_2) = \mathrm{suppintv}(\tilde{b}_3) = [-5, 5]$.

By calculation, $\nu_2(a,4) = -1/2 - \log_4(\sqrt{1/65536}) = 3.5$ and $\nu_2(\tilde{a},4) = -1/2 - \log_4(\sqrt{21.9813\ldots}) \approx -1.6146$. The pair $(\{\phi;\psi^1,\psi^2,\psi^3\},\{\tilde{\phi};\tilde{\psi}^1,\tilde{\psi}^2,\tilde{\psi}^3\})$ associated with $((a;b_1,b_2,b_3),(\tilde{a};\tilde{b}_1,\tilde{b}_2,\tilde{b}_3))$ is a pair of biorthogonal d-wavelet bases in a pair of dual Sobolev spaces $(H^\tau(\mathbb{R}),H^{-\tau}(\mathbb{R}))$ for all $1.6146 < \tau < 3.5$ (see [21] for detail).

As illustrated by the above two examples, it is easy to obtain pairs $(a,\tilde{a})$ of d-dual filters by splitting the interpolatory mask a^I_{2n} in (1.3). Unfortunately, this simple approach does not, in general, generate sufficiently smoothness exponents of $\tilde{a}$ to assure continuity of the corresponding dual scaling functions, and hence, of the associated biorthogonal wavelets. In what follows, we shall employ the CBC algorithm in [1,13,14] to construct pairs $(a,\tilde{a})$ of d-dual filters with larger smoothness exponents. In order to construct continuous biorthogonal wavelet bases, we need to construct pairs $(a,\tilde{a})$ of d-dual filters such that both $\nu_2(a,\mathrm{d}) > 0.5$ and $\nu_2(\tilde{a},\mathrm{d}) > 0.5$ (see [15]).

Example 3. Let the dilation factor be $\mathrm{d}=3$. Using the CBC algorithm, we obtain a pair $(a,\tilde{a})$ of 3-dual filters as follows:

$$\mathrm{a}(z) := \left(\frac{1+z+z^2}{3}\right)^3 z^{-3}\mathrm{p}_0(z), \qquad \tilde{\mathrm{a}}(z) := \left(\frac{1+z+z^2}{3}\right)^2 z^{-2}\tilde{\mathrm{p}}_0(z),$$

where $\mathrm{p}_0(z) = \frac{1}{16}(-7z^{-1}+15+15z-7z^2)$ and $\tilde{\mathrm{p}}_0(z) = \frac{1}{1920}(553z^{-3} - 1580z^{-2} + 422z^{-1} + 1565 + 1565z + 422z^2 - 1580z^3 + 533z^4))$, such that $\mathrm{suppintv}(a) = [-4,5]$, $\mathrm{suppintv}(\tilde{a}) = [-5,6]$, and $\mathrm{Sa} = \mathrm{S\tilde{a}} = z$.

By Lemma 2 in Section 4 (or the top–down chain in Algorithm 2), we obtain a chain of consecutive 3-dual filters $a_0 \to a_1 \to a_2 \to a_3 \to a_4$ with $a_0 = \tilde{a}$, $a_1 = a$, where the symbols $\mathrm{a}_2(z)$, $\mathrm{a}_3(z)$, and $\mathrm{a}_4(z)$ of a_2, a_3, and a_4 are given by:

$$\mathrm{a}_2(z) = \frac{1}{80}\left(-7z^{-2} + 6z^{-1} + 41 + 41z + 6z^2 - 7z^3\right),$$
$$\mathrm{a}_3(z) = \frac{1}{108}\left(35z^{-1} + 30 + 30z + 35z^2\right),$$
$$\mathrm{a}_4(z) = \frac{3}{5}(1+z).$$

The same application of the bottom–up chain in Algorithm 2 yields the band-pass filters corresponding to the low-pass filter pair $(a,\tilde{a})$. That is, we have obtained the 3-dual filter system $((a;b_1,b_2),(\tilde{a};\tilde{b}_1,\tilde{b}_2))$ with the perfect reconstruction property. The symbols $\mathrm{b}_1(z),\mathrm{b}_2(z)$ and $\tilde{\mathrm{b}}_1(z),\tilde{\mathrm{b}}_2(z)$ of b_1,b_2 and $\tilde{b}_1,\tilde{b}_2$ are given by

$$\mathrm{b}_1(z) = \mathrm{q}_1(z) - z\mathrm{q}_1\left(z^{-1}\right), \qquad \mathrm{b}_2(z) = \mathrm{q}_2(z) + z^4\mathrm{q}_2\left(z^{-1}\right),$$
$$\tilde{\mathrm{b}}_1(z) = \tilde{\mathrm{q}}_1(z) - z\tilde{\mathrm{q}}_1\left(z^{-1}\right), \qquad \tilde{\mathrm{b}}_2(z) = \tilde{\mathrm{q}}_2(z) + z^4\tilde{\mathrm{q}}_2\left(z^{-1}\right),$$

with

$$\mathrm{q}_1(z) = \frac{1}{528}\left(-480z + 115z^2 - 162z^3 + 54z^4 + 63z^5\right),$$
$$\mathrm{q}_2(z) = \frac{1}{41778}\left(11483z^2 - 14226z^3 + 1266z^4 + 1477z^5\right),$$
$$\tilde{\mathrm{q}}_1(z) = \frac{1}{1399680}\left(-339423z - 149706z^2 + 145089z^3 + 11869z^4 + 5214z^5 - 6083z^6\right),$$
$$\tilde{\mathrm{q}}_2(z) = \frac{1}{933120}\left(267126z^2 - 56126z^3 - 227669z^4 - 100014z^5 + 116683z^6\right).$$

Note that b_1 and $\tilde{b}_1$ are both anti-symmetric about 1/2: $\mathrm{Sb}_1(z) = \mathrm{S\tilde{b}}_1(z) = -z$, and b_2 and $\tilde{b}_2$ are both symmetric about 2: $\mathrm{Sb}_2(z) = \mathrm{S\tilde{b}}_2(z) = z^4$. Also, $\mathrm{suppintv}(b_1) = [-4,5]$, $\mathrm{suppintv}(b_2) = [-1,5]$, $\mathrm{suppintv}(\tilde{b}_1) = [-5,6]$, and $\mathrm{suppintv}(\tilde{b}_2) = [-2,6]$.

By calculation, we have $\nu_2(a,3) = -1/2 - \log_3(\sqrt{0.00294\ldots}) \approx 2.1520$ and $\nu_2(\tilde{a},3) = -1/2 - \log_3(\sqrt{0.04000\ldots}) \approx 0.9649$. By Theorem 3, the pair $(\{\phi;\psi^1,\psi^2\},\{\tilde{\phi};\tilde{\psi}^1,\tilde{\psi}^2\})$, associated with the dual filter system $((a;b_1,b_2),(\tilde{a};\tilde{b}_1,\tilde{b}_2))$, generates a (continuous) biorthogonal 3-wavelet basis of $L_2(\mathbb{R})$. The symmetry patterns of the functions are specified in (2.14) and (2.15) with $c_0 = 1$, $\epsilon_1 = -1$, $c_1 = 1$, and $\epsilon_2 = 1$, $c_2 = 4$. See Fig. 3.1 for graphs of the pair $(\{\phi;\psi^1,\psi^2\},\{\tilde{\phi};\tilde{\psi}^1,\tilde{\psi}^2\})$.

Example 4. Let the dilation factor be $\mathrm{d}=4$. Using the CBC algorithm, we have a pair $(a,\tilde{a})$ of 4-dual filters as follows:

$$\mathrm{a}(z) := \left(\frac{1+z+z^2+z^3}{4}\right)^2 z^{-3}\mathrm{p}_0(z), \qquad \tilde{\mathrm{a}}(z) := \left(\frac{1+z+z^2+z^3}{4}\right)^2 z^{-3}\tilde{\mathrm{p}}_0(z),$$

where $\mathrm{p}_0(z) = \frac{1}{2}(-z^{-1}+4-z)$ and $\tilde{\mathrm{p}}_0(z) = \frac{1}{2}(-z^{-2}+4-z^{-2})$, such that $\mathrm{suppintv}(a) = [-4,4]$, $\mathrm{suppintv}(\tilde{a}) = [-5,5]$, and $\mathrm{Sa} = \mathrm{S\tilde{a}} = 1$.

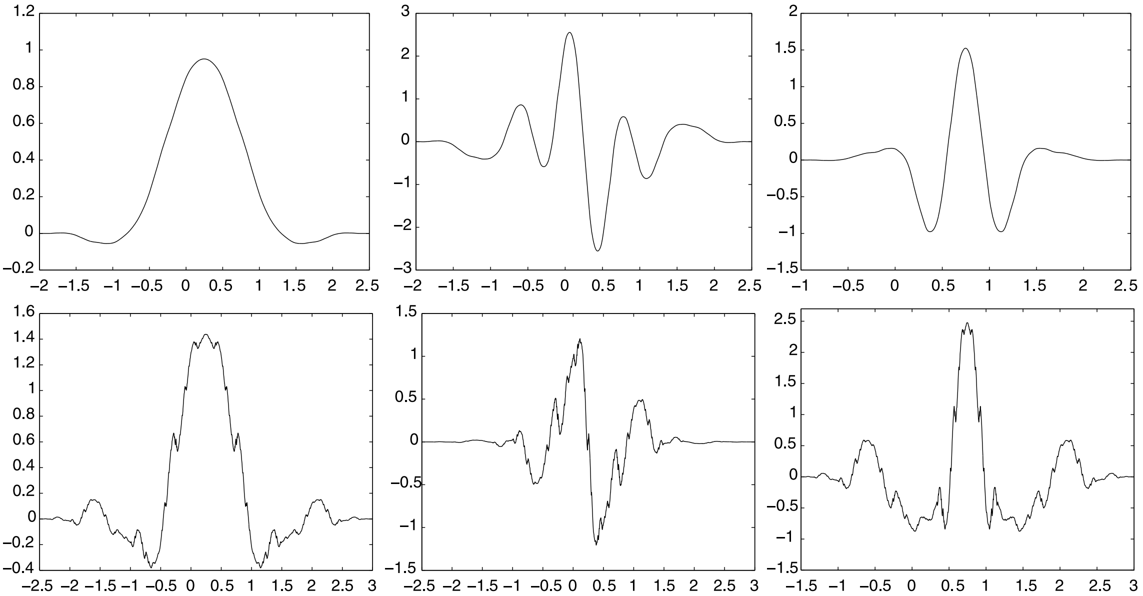

Fig. 3.1. Graphs of ϕ, ψ^1, ψ^2 (top row) and $\tilde{\phi}$, $\tilde{\psi}^1$, $\tilde{\psi}^2$ (bottom row) in Example 3.

By the same procedure as described in the above examples, we obtain a chain of consecutive 4-dual filters $a_0 \to a_1 \to a_2 \to a_3 = \frac{1}{3}\delta$ with $a_0 = \tilde{a}$, $a_1 = a$, and the symbol of a_2 being given by $\mathsf{a}_2(z) = \frac{1}{16}(z^{-3} + 2z^{-2} - z^{-1} + 12 - z + 2z^2 + z^3)$.

The same application of the bottom–up chain in Algorithm 2 yields the band-pass filters corresponding to the low-pass filter pair $(a, \tilde{a})$. That is, we have obtained the 4-dual filter system $((a; b_1, b_2, b_3), (\tilde{a}; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3))$ with the perfect reconstruction property. The symbols $\mathsf{b}_1(z), \mathsf{b}_2(z), \mathsf{b}_3(z)$ and $\tilde{\mathsf{b}}_1(z), \tilde{\mathsf{b}}_2(z), \tilde{\mathsf{b}}_3(z)$ of b_1, b_2, b_3 and $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$ are given by

$$\mathsf{b}_1(z) = \mathsf{q}_1(z) + z^4\mathsf{q}_1(z^{-1}), \qquad \mathsf{b}_2(z) = \mathsf{q}_2(z) + z^4\mathsf{q}_2(z^{-1}), \qquad \mathsf{b}_3(z) = \mathsf{q}_3(z) - z^4\mathsf{q}_3(z^{-1}),$$
$$\tilde{\mathsf{b}}_1(z) = \tilde{\mathsf{q}}_1(z) + z^4\tilde{\mathsf{q}}_1(z^{-1}), \qquad \tilde{\mathsf{b}}_2(z) = \tilde{\mathsf{q}}_2(z) + z^4\tilde{\mathsf{q}}_2(z^{-1}), \qquad \tilde{\mathsf{b}}_3(z) = \tilde{\mathsf{q}}_3(z) - z^4\tilde{\mathsf{q}}_3(z^{-1}),$$

with

$$\mathsf{q}_1(z) = \frac{1}{768}(-524z^2 + 1000z^3 - 487z^4 + 6z^5 + 4z^6 + 2z^7 - z^8),$$
$$\mathsf{q}_2(z) = z^2 - 2z^3 + z^4,$$
$$\mathsf{q}_3(z) = \frac{1}{4}(-2z^3 + z^4),$$
$$\tilde{\mathsf{q}}_1(z) = \frac{1}{4}(-6z^2 - 3z^3 + 6z^4 + 3z^5),$$
$$\tilde{\mathsf{q}}_2(z) = \frac{1}{32}(-31z^2 - 17z^3 + 32z^4 + 16z^5),$$
$$\tilde{\mathsf{q}}_3(z) = \frac{1}{32}(-7z^3 + 2z^4 + z^5).$$

Note that b_1 and $\tilde{b}_1$ are both symmetric about 2: $\mathsf{S}\mathsf{b}_1(z) = \mathsf{S}\tilde{\mathsf{b}}_1(z) = z^4$, b_2 and $\tilde{b}_2$ are both symmetric about 2: $\mathsf{S}\mathsf{b}_2(z) = \mathsf{S}\tilde{\mathsf{b}}_2(z) = z^4$, and b_3 and $\tilde{b}_3$ are both anti-symmetric about 2: $\mathsf{S}\mathsf{b}_3(z) = \mathsf{S}\tilde{\mathsf{b}}_3(z) = -z^4$. Also, $\operatorname{suppintv}(b_1) = [-4, 8]$, $\operatorname{suppintv}(b_2) = \operatorname{suppintv}(b_3) = [0, 4]$, $\operatorname{suppintv}(\tilde{b}_1) = \operatorname{suppintv}(\tilde{b}_2) = \operatorname{suppintv}(\tilde{b}_3) = [-1, 5]$.

By calculation, we have $\nu_2(a, 4) = -1/2 - \log_4(\sqrt{9/512}) \approx 0.9575$ and $\nu_2(\tilde{a}, 4) = -1/2 - \log_4 \sqrt{9/512} \approx 0.9575$. By Theorem 3, the pair $(\{\phi; \psi^1, \psi^2, \psi^3\}, \{\tilde{\phi}; \tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3\})$, associated with the dual filter system $((a; b_1, b_2, b_3), (\tilde{a}; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3))$, generates a (continuous) biorthogonal 4-wavelet basis in $L_2(\mathbb{R})$. The symmetry patterns of the functions are specified in (2.14) and (2.15) with $c_0 = 0$, $\epsilon_1 = 1$, $c_1 = 4$, $\epsilon_2 = 1$, $c_2 = 4$, and $\epsilon_3 = -1$, $c_3 = 4$. See Fig. 3.2 for graphs of the pair $(\{\phi; \psi^1, \psi^2, \psi^3\}, \{\tilde{\phi}; \tilde{\psi}^1, \tilde{\psi}^2, \tilde{\psi}^3\})$.

Example 5. Let the dilation factor be $\mathsf{d} = 4$ and a be the refinement mask of the centered cardinal cubic B-spline $N_4(\cdot + 2)$. Using the CBC algorithm, we have a pair $(a, \tilde{a})$ of 4-dual filters as follows:

$$\mathsf{a}(z) := \left(\frac{1 + z + z^2 + z^3}{4}\right)^4 z^{-6}, \qquad \tilde{\mathsf{a}}(z) := \left(\frac{1 + z + z^2 + z^3}{4}\right)^4 z^{-6}\tilde{\mathsf{p}}_0(z)$$

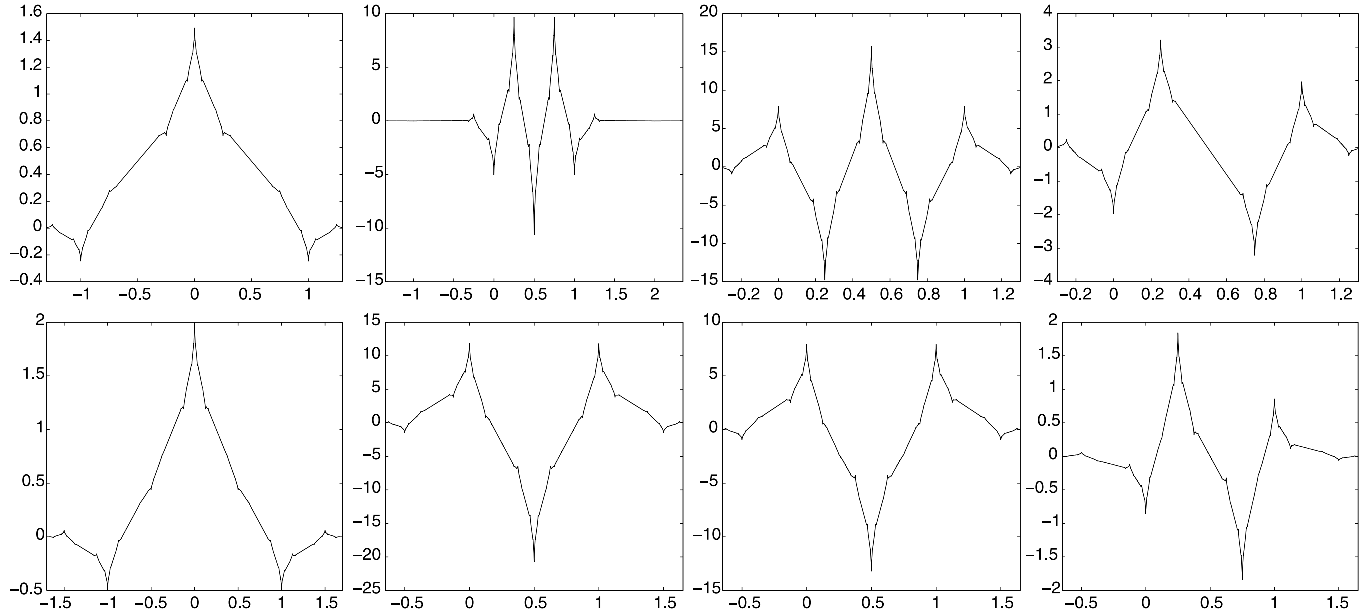

Fig. 3.2. Graphs of ϕ, ψ^1, ψ^2, ψ^3 (top row) and $\tilde{\phi}$, $\tilde{\psi}^1$, $\tilde{\psi}^2$, $\tilde{\psi}^3$ (bottom row) in Example 4.

where $\tilde{\mathsf{p}}_0(z) = \frac{1}{16}(56(z^{-6}+z^6) - 337(z^{-5}+z^5) + 680(z^{-4}+z^4) - 346(z^{-3}+z^3) - 336(z^{-2}+z^2) + 93(z^{-1}+z) + 768)$. Note that both filters a and $\tilde{a}$ are symmetric about 0: $\mathsf{S}a = \mathsf{S}\tilde{a} = 1$, with $\operatorname{suppintv}(a) = [-6, 6]$ and $\operatorname{suppintv}(\tilde{a}) = [-12, 12]$.

By the same procedure as described in the above examples, we obtain a chain of consecutive 4-dual filters $a_0 \to a_1 \to a_2 \to a_3 = \frac{1}{24}\delta$ with $a_0 = \tilde{a}$, $a_1 = a$. The symbol $\mathsf{a}_2(z)$ of a_2 is given by $\mathsf{a}_2(z) = \frac{1}{12}(-5z^{-1} + 12 - 5z)$.

An analogous application of the bottom–up procedure in Algorithm 2 yields the band-pass filters for the pair $(a, \tilde{a})$ of 4-dual filters, such that $((a; b_1, b_2, b_3), (\tilde{a}; \tilde{b}_1, \tilde{b}_2, \tilde{b}_3))$ constitutes a dual filter system with the perfect reconstruction property, and the symbols $\mathsf{b}_1(z), \mathsf{b}_2(z), \mathsf{b}_3(z)$ and $\tilde{\mathsf{b}}_1(z), \tilde{\mathsf{b}}_2(z), \tilde{\mathsf{b}}_3(z)$ of b_1, b_2, b_3 and $\tilde{b}_1, \tilde{b}_2, \tilde{b}_3$ are given as follows:

$$\mathsf{b}_1(z) = \mathsf{q}_1(z) + z^4\mathsf{q}_1(z^{-1}), \qquad \mathsf{b}_2(z) = \mathsf{q}_2(z) + z^4\mathsf{q}_2(z^{-1}), \qquad \mathsf{b}_3(z) = \mathsf{q}_3(z) - z^4\mathsf{q}_3(z^{-1}),$$
$$\tilde{\mathsf{b}}_1(z) = \tilde{\mathsf{q}}_1(z) + z^4\tilde{\mathsf{q}}_1(z^{-1}), \qquad \tilde{\mathsf{b}}_2(z) = \tilde{\mathsf{q}}_2(z) + z^4\tilde{\mathsf{q}}_2(z^{-1}), \qquad \tilde{\mathsf{b}}_3(z) = \tilde{\mathsf{q}}_3(z) - z^4\tilde{\mathsf{q}}_3(z^{-1}),$$

with

$$\begin{aligned}
\mathsf{q}_1(z) = {} & \frac{1}{1042848}\big(-1049z^{10} - 4196z^9 - 10490z^8 - 20980z^7 + 376688z^6 - 566284z^5 \\
& - 314982z^4 + 1504356z^3 - 963063z^2\big), \\
\mathsf{q}_2(z) = {} & \frac{1}{271044608}\big(1759806z^{14} + 7039224z^{13} + 17598060z^{12} + 35196120z^{11} \\
& + 55637885z^{10} + 16328348z^9 - 29571370z^8 - 87299636z^7 \\
& - 339321518z^6 - 116787908z^5 - 78940386z^4 - 33706356z^3 + 552067731z^2\big), \\
\mathsf{q}_3(z) = {} & \frac{1}{271044608}\big(-68121z^{14} - 272484z^{13} - 681210z^{12} - 1362420z^{11} \\
& - 2139871z^{10} - 576712z^9 + 1283060z^8 + 3656056z^7 + 136178521z^6 \\
& - 189826004z^5 - 153145242z^4 + 349120380z^3\big),
\end{aligned}$$

$$\begin{aligned}
\tilde{\mathsf{q}}_1(z) = {} & \frac{1}{2168356864}\big(1824984z^{12} - 3682557z^{11} - 3519612z^{10} + 4041036z^9 + 2205108z^8 \\
& + 16712307z^7 + 6389928z^6 + 14472885z^5 - 183909596z^4 + 148985129z^3 - 3519612z^2\big), \\
\tilde{\mathsf{q}}_2(z) = {} & \frac{1}{2085696}\big(-33768z^8 + 68139z^7 + 65124z^6 - 77295z^5 - 227200z^4 + 74644z^3 + 130356z^2\big), \\
\tilde{\mathsf{q}}_3(z) = {} & \frac{1}{64}\big(z^5 - 4z^4 + 5z^3\big).
\end{aligned}$$

Note that b_1 and $\tilde{b}_1$ are both symmetric about 2: $\mathsf{S}b_1(z) = \mathsf{S}\tilde{b}_1(z) = z^4$, b_2 and $\tilde{b}_2$ are both symmetric about 2: $\mathsf{S}b_2(z) = \mathsf{S}\tilde{b}_2(z) = z^4$, and b_3 and $\tilde{b}_3$ are both anti-symmetric about 2: $\mathsf{S}b_3(z) = \mathsf{S}\tilde{b}_3(z) = -z^4$. We also have $\operatorname{suppintv}(b_1) = [-6, 10]$, $\operatorname{suppintv}(b_2) = \operatorname{suppintv}(b_3) = [-10, 14]$, $\operatorname{suppintv}(\tilde{b}_1) = [-8, 12]$, $\operatorname{suppintv}(\tilde{b}_2) = [-4, 8]$, $\operatorname{suppintv}(\tilde{b}_3) = [-1, 5]$.

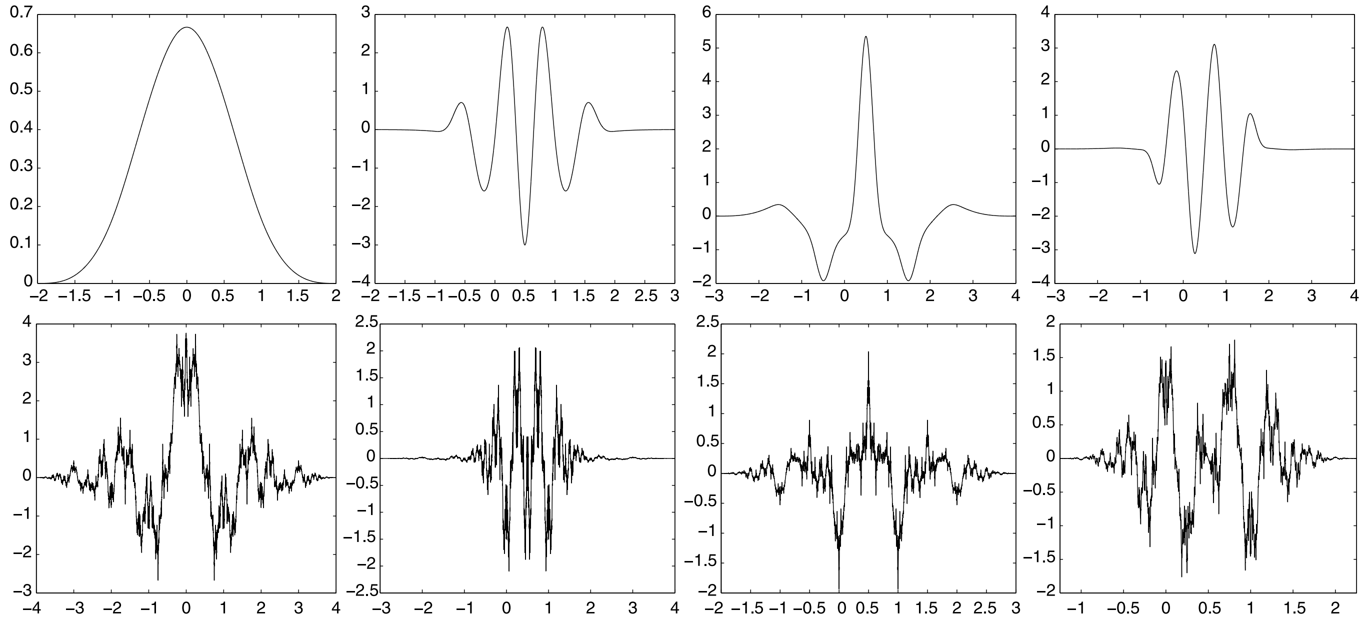

Fig. 3.3. Graphs of ϕ, ψ^1, ψ^2, ψ^3 (top row) and $\tilde{\phi}$, $\tilde{\psi}^1$, $\tilde{\psi}^2$, $\tilde{\psi}^3$ (bottom row) in Example 5.

By calculation, $\nu_2(a,4) = -1/2 - \log_4(\sqrt{1/65536}) = 3.5$ and $\nu_2(\tilde{a},4) = -1/2 - \log_4(\sqrt{0.06172\ldots}) \approx 0.5045$. By Theorem 3, the pair $(\{\phi;\psi^1,\psi^2,\psi^3\},\{\tilde{\phi};\tilde{\psi}^1,\tilde{\psi}^2,\tilde{\psi}^3\})$, associated with the dual filter system $((a;b_1,b_2,b_3),(\tilde{a};\tilde{b}_1,\tilde{b}_2,\tilde{b}_3))$, generates a (continuous) biorthogonal 4-wavelet basis in $L_2(\mathbb{R})$. The symmetry patterns of the functions are specified in (2.14) and (2.15) with $c_0 = 9$, $\epsilon_1 = -1$, $c_1 = 9$, $\epsilon_2 = 1$, $c_2 = 5$, and $\epsilon_3 = -1$, $c_3 = 5$. See Fig. 3.3 for graphs of the pair $(\{\phi;\psi^1,\psi^2,\psi^3\},\{\tilde{\phi};\tilde{\psi}^1,\tilde{\psi}^2,\tilde{\psi}^3\})$.

4. Proofs of main results

In this section we shall prove the main results that have been stated in Section 2.

Two filters $u = \{u(k)\}_{k\in\mathbb{Z}}$ and $v = \{v(k)\}_{k\in\mathbb{Z}}$ are said to be d-orthogonal, if

$$\sum_{k\in\mathbb{Z}} u(k)\overline{v(\mathsf{d}j+k)} = 0, \quad j\in\mathbb{Z},$$

or equivalently, in their polyphase formulation,

$$\sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{u}^{[\gamma]}(z)\mathsf{v}^{[\gamma]}(z)^{\star} = 0,$$

where $\mathsf{u}^{[\gamma]}(z), \mathsf{v}^{[\gamma]}(z)$ are the polyphase components of u, v, as defined in (1.6). Observe the analogy between d-orthogonality and d-duality as in (1.4) or (1.7).

We say that a finite filter u is d-*dual reducible* if there is a d-dual $\tilde{u}$ of u such that $\operatorname{suppintv}(\tilde{u}) \subsetneq \operatorname{suppintv}(u)$; that is, $\tilde{u}$ has a strictly smaller coefficient support interval than that of u.

The following result shows that any finite filter u with more than one filter tap that has at least one finitely supported d-dual filter must be d-dual reducible.

Lemma 1. *Let $(u,\tilde{u})$ be a pair of d-dual filters with $\operatorname{suppintv}(u) = [m,n]$ and $n-m \geqslant 1$. Then there exists a filter $\tilde{u}^{new}$ such that $(u,\tilde{u}^{new})$ is a pair of d-dual filters and either $\operatorname{suppintv}(\tilde{u}^{new}) \subseteq [m+1,n]$ or $\operatorname{suppintv}(\tilde{u}^{new}) \subseteq [m,n-1]$.*

Proof. Let $\tilde{u} =: \{\tilde{u}(k)\}_{k\in\mathbb{Z}}$ and $\operatorname{suppintv}(\tilde{u}) := [\tilde{m},\tilde{n}]$. If $\operatorname{suppintv}(\tilde{u}) \subseteq [m+1,n]$ or $\operatorname{suppintv}(\tilde{u}) \subseteq [m,n-1]$, by defining $\tilde{u}^{new} := \tilde{u}$, we are done. Hence, we shall assume $[\tilde{m},\tilde{n}] \nsubseteq [m,n-1]$ and $[\tilde{m},\tilde{n}] \nsubseteq [m+1,n]$. In what follows, we shall construct a desired filter $\tilde{u}^{new}$ such that $\operatorname{suppintv}(\tilde{u}^{new}) \subseteq [m+1,n]$. A similar approach applies for the construction of $\tilde{u}^{new}$ such that $\operatorname{suppintv}(\tilde{u}^{new}) \subseteq [m,n-1]$.

The idea of our construction is as follows. First observe that if a filter v is d-orthogonal to u, then $(u,\tilde{u}+v)$ is also a pair of d-dual filters. To reduce the support interval of $\tilde{u}$ by using a new dual filter $\tilde{u}+v$, we construct a filter v, which is d-orthogonal to u and has only two nontrivial polyphase components, so that the support interval of $\tilde{u}+v$ is shorter than that of $\tilde{u}$. Then we can apply the same strategy repeatedly to obtain a desired dual filter $\tilde{u}^{new}$.

Let $\mathsf{u}(z)$ be the symbol of $u = \{u(k)\}_{k\in\mathbb{Z}}$. If we already have $\tilde{m} \geqslant m+1$, then we set $s_1 := 0$; otherwise, we shall construct filters $v_1, \dots, v_{s_1}$ that are d-orthogonal to u such that $\min\{\mathrm{suppintv}(\tilde{u}+v_1+\cdots+v_{s_1})\} \geqslant m+1$. Recall that $\min\{[t_1,t_2]\} = t_1$ and $\max\{[t_1,t_2]\} = t_2$.

We first construct a filter sequence $v_1 = \{v_1(k)\}$ with polyphase representation of its symbol $\mathsf{v}_1(z) = \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \mathsf{v}_1^{[\gamma]}(z^{\mathsf{d}})$ such that the polyphase components of v_1 are given by

$$\mathsf{v}_1^{[\gamma]}(z) := \begin{cases} \frac{\tilde{u}(\tilde{m})}{\overline{u(n)}} \mathsf{u}^{[\rho_{\tilde{m}}]}(z)^\star \cdot z^{\lambda_{\tilde{m}}+\lambda_n}, & \gamma = \rho_n; \\ -\frac{\tilde{u}(\tilde{m})}{\overline{u(n)}} \mathsf{u}^{[\rho_n]}(z)^\star \cdot z^{\lambda_{\tilde{m}}+\lambda_n}, & \gamma = \rho_{\tilde{m}}; \\ 0, & \text{otherwise}. \end{cases}$$

In view of (1.4), note that $\rho_m \neq \rho_{\tilde{n}}$ and $\rho_{\tilde{m}} \neq \rho_n$. Also, observe that $\sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{u}^{[\gamma]}(z)\mathsf{v}_1^{[\gamma]}(z)^\star = 0$; that is, u and v_1 are d-orthogonal. Moreover, it is easy to see that $|\mathrm{suppintv}(v_1)| \leqslant |\mathrm{suppintv}(u)|$, $|\mathrm{suppintv}(\tilde{u}+v_1)| \leqslant |\mathrm{suppintv}(\tilde{u})| - 1$, and $\min\{\mathrm{suppintv}(\tilde{u}+v_1)\} \geqslant \tilde{m}+1$.

Repeat this procedure iteratively, if necessary, for $j = 1, \dots, s_1$, with $\tilde{u} + \sum_{k=1}^{j-1} v_k$ being replaced by $\tilde{u} + \sum_{k=1}^{j} v_k$ (where $\sum_{k=1}^{0} := 0$), $\tilde{m}_{j-1}$ being replaced by $\tilde{m}_j$, $\rho_{\tilde{m}_{j-1}}$ being replaced by $\rho_{\tilde{m}_j}$, and $\lambda_{\tilde{m}_{j-1}}$ being replaced by $\lambda_{\tilde{m}_j}$, where $\tilde{m}_j := \min\{\mathrm{suppintv}(\tilde{u} + \sum_{k=1}^{j} v_k)\}$, until

$$\min\left\{\mathrm{suppintv}\left(\tilde{u} + \sum_{k=1}^{s_1} u_k\right)\right\} \geqslant \min\bigl\{\mathrm{suppintv}(u)\bigr\} + 1 = m+1.$$

If $\max\{\mathrm{suppintv}(\tilde{u} + \sum_{k=1}^{s_1} v_k)\} \leqslant n$, we are done; otherwise, we next construct filters $v_{s_1+1}, \dots, v_{s_1+s_2}$ such that $\max\{\mathrm{suppintv}(\tilde{u} + \sum_{k=1}^{s_1+s_2} v_k)\} \leqslant n$.

Suppose that $\max\{\mathrm{suppintv}(\tilde{u} + \sum_{k=1}^{s_1} v_k)\} =: \tilde{n}_0 > n$. Then, we may apply the same procedure as for the case $\tilde{m} \leqslant m$ to the sequence pair $(\tilde{u}_{s_1}, u)$, where $\tilde{u}_{s_1} := \tilde{u} + \sum_{k=1}^{s_1} v_k = \{\tilde{u}_{s_1}(k)\}_{k\in\mathbb{Z}}$, in order to construct the filter sequence v_{s_1+1} to satisfy

$$\max\bigl\{\mathrm{suppintv}(\tilde{u}_{s_1} + v_{s_1+1})\bigr\} \leqslant \max\bigl\{\mathrm{suppintv}(\tilde{u}_{s_1})\bigr\} - 1. \tag{4.1}$$

It can be easily shown that (4.1) is satisfied by setting the polyphase components of the symbol of v_{s_1+1} to be

$$\mathsf{v}_{s_1+1}^{[\gamma]}(z) := \begin{cases} \frac{\tilde{u}_{s_1}(\tilde{n}_0)}{\overline{u(m)}} \mathsf{u}^{[\rho_{\tilde{n}_0}]}(z)^\star \cdot z^{\lambda_{\tilde{n}_0}+\lambda_m}, & \gamma = \rho_m; \\ -\frac{\tilde{u}_{s_1}(\tilde{n}_0)}{\overline{u(m)}} \mathsf{u}^{(\rho_m)}(z)^\star \cdot z^{\lambda_{\tilde{n}_0}+\lambda_m}, & \gamma = \rho_{\tilde{n}_0}; \\ 0, & \text{otherwise}. \end{cases}$$

Repeat this procedure iteratively, if necessary, for $j = 1, \dots, s_2$, with $\tilde{u}_{s_1} + \sum_{k=1}^{j-1} v_{s_1+k}$ being replaced by $\tilde{u}_{s_1} + \sum_{k=1}^{j} v_{s_1+k}$, $\tilde{n}_{j-1}$ being replaced by $\tilde{n}_j$, $\rho_{\tilde{n}_{j-1}}$ being replaced by $\rho_{\tilde{n}_j}$, and $\lambda_{\tilde{n}_{j-1}}$ being replaced by $\lambda_{\tilde{n}_j}$, where $\tilde{n}_j := \max\{\mathrm{suppintv}(\tilde{u}_{s_1} + \sum_{k=1}^{j} v_{s_1+k})\}$, until

$$\max\left\{\mathrm{suppintv}\left(\tilde{u} + \sum_{k=1}^{s_2} v_{s_1+k}\right)\right\} \leqslant \max\bigl\{\mathrm{suppintv}(u)\bigr\} = n.$$

We may now set $\tilde{u}^{new} := \tilde{u} + \sum_{k=1}^{s_1+s_2} u_k$. Observe that

$$s_1 + s_2 \leqslant \bigl|\mathrm{suppintv}(\tilde{u})\bigr| - \bigl|\mathrm{suppintv}(u)\bigr| + 1.$$

This completes the proof of the lemma. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. By applying Lemma 1 repeatedly, we obtain a chain of filters $a_0, \dots, a_r$ with consecutive dual pairs, in the sense that (a_{j-1}, a_j) is d-dual for each $j = 1, \dots, r$, such that

$$\mathrm{suppintv}(a_{j+1}) \subsetneq \mathrm{suppintv}(a_j), \quad j = 1, \dots, r-1,$$

where r is determined by the fact that the chain eventually terminates. Since (a_{r-1}, a_r) is a dual pair and a_r is not d-dual reducible, a_r cannot be the zero sequence and therefore must have a single tap (or equivalently $|\mathrm{suppintv}(a_r)| = 0$). □

We say that a finite filter u that satisfies the symmetry condition $\mathsf{S}\mathrm{u}(z) = \epsilon z^c$ for some $\epsilon \in \{-1, 1\}$ and some $c \in \mathbb{Z}$, is d-*dual reducible with symmetry* if there is a d-dual filter $\tilde{u}$ of u such that $\mathsf{S}\tilde{\mathrm{u}}(z) = \mathsf{S}\mathrm{u}(z)$ and $\operatorname{suppintv}(\tilde{u}) \subsetneq \operatorname{suppintv}(u)$; that is, $\tilde{u}$ has the same symmetry pattern as u, and $\tilde{u}$ has a strictly smaller coefficient support interval than that of u.

To illustrate the notion of d-*dual reducible with symmetry*, let us consider the special case of $\mathsf{d} = 2$ and the primal filter a with symbol $\mathrm{a}(z) = z^{-1}(1+z)^3/8$. We see that a has a unique 2-dual $\tilde{a}$ with support $[-1, 2]$ and symmetric about $1/2$. In fact the symbol of this (unique) dual filter $\tilde{a}$ is given by $\tilde{\mathrm{a}}(z) = -1/4z^{-1} + 3/4 + 3/4z - 1/4z^2$. Since the support interval of a is not reduced, it is not 2-dual reducible with symmetry. Observe that a has two nontrivial polyphase components.

In the following we will show that for a symmetric or anti-symmetric finite filter u that has more than two nontrivial polyphase components and has at least one finite d-dual must be d-dual reducible with symmetry. Of course, to consider more than two nontrivial polyphase components, the dilation factor d must be at least 3.

Lemma 2. *Let $(u, \tilde{u})$ be a pair of* d*-dual filters such that u and $\tilde{u}$ have the same symmetry pattern:* $\mathsf{S}\mathrm{u}(z) = \mathsf{S}\tilde{\mathrm{u}}(z) = \epsilon z^c$ *for some $\epsilon \in \{-1, 1\}$ and some $c \in \mathbb{Z}$. Suppose* $\operatorname{suppintv}(u) = [m, n]$ *and u has more than two nontrivial polyphase components. Then there exists a* d*-dual filter $\tilde{u}^{new}$ of u such that* $\mathsf{S}\tilde{\mathrm{u}}^{new}(z) = \epsilon z^c$ *and* $\operatorname{suppintv}(\tilde{u}^{new}) \subseteq [m+1, n-1]$.

Proof. By applying Lemma 1 to the pair $(u, \tilde{u})$ of d-dual filters, we can construct a dual filter $\tilde{u}_1 = \{\tilde{u}_1(k)\}_{k\in\mathbb{Z}}$ (which might not have any symmetry property), such that $\operatorname{suppintv}(\tilde{u}_1) \subseteq [m, n-1]$. If $\operatorname{suppintv}(\tilde{u}_1) \subseteq [m+1, n-1]$, then the dual filter $\tilde{u}^{new}$, defined by $\tilde{\mathrm{u}}^{new}(z) := (\tilde{\mathrm{u}}_1(z) + \epsilon z^c \tilde{\mathrm{u}}_1(1/z))/2$ has the same symmetry pattern as u, and that $\operatorname{suppintv}(\tilde{u}^{new}) \subseteq [m+1, n-1]$, completing the proof of the lemma. So, we may assume $\operatorname{suppintv}(\tilde{u}_1) \subseteq [m, n-1]$ and $\min\{\operatorname{suppintv}(\tilde{u}_1)\} = m$, and construct a dual filter $\tilde{u}_2$ of u such that $\operatorname{suppintv}(\tilde{u}_2) \subseteq [m+1, n-1]$.

The idea is similar to that in the proof of Lemma 1, except that we need three nontrivial polyphase components of u. By using three nontrivial polyphase components, we shall construct a filter $\tilde{v} = \{\tilde{v}(k)\}_{k\in\mathbb{Z}}$ that satisfies the following three conditions:

(i) $\operatorname{suppintv}(\tilde{v}) \subseteq [m, n-1]$,
(ii) $\tilde{v}(m) \neq 0$, and
(iii) $\tilde{v}$ is d-orthogonal to u.

Then the filter $\tilde{u}_2 := \tilde{u}_1 - \frac{\tilde{u}(m)}{\tilde{v}(m)}\tilde{v}$ is a d-dual of u with $\operatorname{suppintv}(\tilde{u}_2) \subseteq [m+1, n-1]$. Hence, the above argument again completes the proof of the lemma.

Observe that we already have two nontrivial polyphase components, $\mathrm{u}^{[\rho_m]}(z)$ and $\mathrm{u}^{[\rho_n]}(z)$. For the third polyphase component, let $\mathrm{u}^{[\rho_\ell]}(z)$ be any one of the other nontrivial ones. Let m_1, n_1, ℓ_1 be the lowest degrees, and m_2, n_2, ℓ_2 the highest degrees, of the Laurent polynomials $z^{\rho_m}\mathrm{u}^{[\rho_m]}(z^{\mathsf{d}})$, $z^{\rho_n}\mathrm{u}^{[\rho_n]}(z^{\mathsf{d}})$, $z^{\rho_\ell}\mathrm{u}^{[\rho_\ell]}(z^{\mathsf{d}})$, respectively. That is, $\mathrm{u}^{[\rho_m]}(z)$, $\mathrm{u}^{[\rho_n]}(z)$, and $\mathrm{u}^{[\rho_\ell]}(z)$ can be written as

$$\begin{aligned}
\mathrm{u}^{[\rho_m]}(z) &= \sum_{k=\lambda_{m_1}}^{\lambda_{m_2}} u(\mathsf{d}k + \rho_m) z^k; \\
\mathrm{u}^{[\rho_n]}(z) &= \sum_{k=\lambda_{n_1}}^{\lambda_{n_2}} u(\mathsf{d}k + \rho_n) z^k; \\
\mathrm{u}^{[\rho_\ell]}(z) &= \sum_{k=\lambda_{\ell_1}}^{\lambda_{\ell_2}} u(\mathsf{d}k + \rho_\ell) z^k.
\end{aligned} \tag{4.2}$$

Obviously, we have $m_1 = m$, $n_2 = n$, and $m_1 < \ell_1 \leqslant \ell_2 < n_2$. Here and thereafter, recall that for $k \in \mathbb{Z}, \lambda_k := \lfloor k/\mathsf{d} \rfloor$ and $\rho_k := k - \mathsf{d}\lambda_k$.

We are now ready to construct the filter $\tilde{v} = \{\tilde{v}(k)\}_{k\in\mathbb{Z}}$ that satisfies the three conditions (i), (ii), and (iii) above, by setting $\tilde{v} = \tilde{v}_1 + \tilde{v}_2$ (for the case $\ell_1 < n_1$), or by setting $\tilde{v} = \tilde{v}_1 + \tilde{v}_2 + \tilde{v}_{3,0} + \cdots + \tilde{v}_{3,k_0}$ (for the case $\ell_1 > n_1$), where all of the filters $\tilde{v}_1, \tilde{v}_2, \tilde{v}_{3,0}, \ldots, \tilde{v}_{3,k_0}$ are d-orthogonal to u, and to be constructed in the following discussion. Here $k_0 = \lambda_{\ell_1 - n_1}$.

Let the symbols $\tilde{\mathrm{v}}_1(z)$ and $\tilde{\mathrm{v}}_2(z)$ of $\tilde{v}_1$ and $\tilde{v}_2$ be given by

$$\tilde{\mathrm{v}}_1(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \tilde{\mathrm{v}}_1^{[\gamma]}(z^{\mathsf{d}}), \qquad \tilde{\mathrm{v}}_2(z) := \sum_{\gamma=0}^{\mathsf{d}-1} z^\gamma \tilde{\mathrm{v}}_2^{[\gamma]}(z^{\mathsf{d}}),$$

where the polyphase components are constructed as follows:

$$\tilde{\mathrm{v}}_1^{[\gamma]}(z) := \begin{cases} \mathrm{u}^{[\rho_n]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}}, & \gamma = \rho_m; \\ -\mathrm{u}^{[\rho_m]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}}, & \gamma = \rho_n; \\ 0, & \text{otherwise,} \end{cases}$$

$$\tilde{\mathrm{v}}_2^{[\gamma]}(z) := \begin{cases} C \cdot \mathrm{u}^{[\rho_\ell]}(z)^\star \cdot z^{\lambda_{\ell_1}+\lambda_{n_2}}, & \gamma = \rho_n; \\ -C \cdot \mathrm{u}^{[\rho_n]}(z)^\star \cdot z^{\lambda_{\ell_1}+\lambda_{n_2}}, & \gamma = \rho_\ell; \\ 0, & \text{otherwise}, \end{cases} \tag{4.3}$$

with $C := \overline{(\frac{u(m_1)}{u(\ell_1)})}$. It is easy to verify that

$$\operatorname{suppintv}(\tilde{v}_1) = [m_1, n_2] \quad \text{and} \quad \operatorname{suppintv}(\tilde{v}_2) = \bigl[\ell_1, \max\bigl\{n_2, n_2 + (\ell_1 - n_1)\bigr\}\bigr].$$

There are only two cases to be discussed, namely: $\ell_1 < n_1$ and $\ell_1 > n_1$.

Case 1. $\ell_1 < n_1$. In this case, $\operatorname{suppintv}(\tilde{v}_2) = [\ell_1, n_2]$. Let $\tilde{v} := \tilde{v}_1 + \tilde{v}_2$, so that $\operatorname{suppintv}(\tilde{v}) \subseteq [m, n-1]$ and $\tilde{v}(m) \neq 0$. Also, in view of our construction of $\tilde{v}_1$ and $\tilde{v}_2$ in (4.3), it follows that $\tilde{v}$ is d-orthogonal to u. Hence, $\tilde{v}$ satisfies the above conditions (i), (ii), and (iii), and the same argument given above completes the proof of the lemma.

Case 2. $\ell_1 > n_1$. In this case, $\operatorname{suppintv}(\tilde{v}_1 + \tilde{v}_2) = [m_1, n_2 + (\ell_1 - n_1)] \supsetneq [m, n-1]$. Let $\ell_1 - n_1 = \mathrm{d}k_0 + \alpha$ with $0 \leqslant k_0 \in \mathbb{N}$ and $\alpha \in \{0, \dots, \mathrm{d}-1\}$ (i.e., $k_0 = \lambda_{\ell_1 - n_1}$ and $\alpha = \rho_{\ell_1 - n_1}$). Define a sequence of filters $\tilde{v}_{3,0}, \dots, \tilde{v}_{3,k_0}$ that are d-orthogonal to u as follows.

Let $\tilde{\mathrm{v}}_{3,j}(z) := \sum_{\gamma=0}^{\mathrm{d}-1} z^\gamma \tilde{\mathrm{v}}_{3,j}^{[\gamma]}(z^{\mathrm{d}})$, $j = 0, \dots, k_0$ be the symbol of $\tilde{\mathrm{v}}_{3,j}$. For $j = 0$, and for each $\gamma = 0, \dots, \mathrm{d}-1$,

$$\tilde{\mathrm{v}}_{3,0}^{[\gamma]}(z) := \begin{cases} C_0 \mathrm{u}^{[\rho_\ell]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}-\lambda_{n_1}+\lambda_{\ell_1}}, & \gamma = \rho_m; \\ -C_0 \mathrm{u}^{[\rho_m]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}-\lambda_{n_1}+\lambda_{\ell_1}}, & \gamma = \rho_\ell; \\ 0, & \text{otherwise}, \end{cases}$$

where $C_0 := -\overline{(\frac{u(n_1)}{u(\ell_1)})}$. Let $\tilde{w}_0 := \tilde{v}_1 + \tilde{v}_2 + \tilde{v}_{3,0} =: \{\tilde{w}_0(k)\}$. Then it is easy to see that $\operatorname{suppintv}(\tilde{w}_0) \subseteq [m_1, n_2 + (\ell_1 - n_1 - \mathrm{d})]$.

For $j = 1, \dots, k_0$, define $\tilde{v}_{3,j}$ through its symbol by

$$\tilde{\mathrm{v}}_{3,j}^{[\gamma]}(z) := \begin{cases} C_j z^{-j} \mathrm{u}^{[\rho_\ell]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}-\lambda_{n_1}+\lambda_{\ell_1}}, & \gamma = \rho_m; \\ -C_j z^{-j} \mathrm{u}^{[\rho_m]}(z)^\star \cdot z^{\lambda_{m_1}+\lambda_{n_2}-\lambda_{n_1}+\lambda_{\ell_1}}, & \gamma = \rho_\ell; \\ 0, & \text{otherwise} \end{cases}$$

and $\tilde{w}_j := \tilde{v}_1 + \tilde{v}_2 + \tilde{v}_{3,0} + \dots + \tilde{v}_{3,j} =: \{\tilde{w}_j(k)\}$ for $j = 1, \dots, k_0$, where C_j, $j = 1, \dots, k_0$ are some constants determined by the following recursive formula:

$$C_j := \frac{\tilde{w}_{j-1}(n_2 + \ell_1 - n_1 - \mathrm{d}j)}{\overline{u(m_1)}}, \quad j = 1, \dots, k_0.$$

Then $\tilde{v} := \tilde{w}_{k_0} = \tilde{v}_1 + \tilde{v}_2 + \tilde{v}_{3,0} + \dots + \tilde{v}_{3,k_0}$ again satisfies (i), (ii), and (iii), and the same argument can be applied to complete the proof of the lemma. □

We are now ready to prove Theorem 2.

Proof of Theorem 2. The proof of Theorem 2 is the same as that of Theorem 1, with the exception that Lemma 2, instead of Lemma 1, is applied repeatedly. The chain of consecutive d-dual filters with symmetry $a_0 \to a_1 \to \dots \to a_{j-1} \to a_j \to \cdots$ terminates at $j = r$ for some $r \geqslant 2$ when either a_r has exactly two nontrivial polyphase components or a_r has only one tap. □

5. Final remarks

(1) The filters a_0 and a_1 that constitute the initial dual pair of the top–down dual-chain in Fig. 1.2 do not have to be low-pass filters in the construction of filter banks.

(2) For dilation $\mathrm{d} = 2$, the unimodular 2×2 polynomial matrix approach in [3,4] is applied to the pair (a, b), instead of $(a, \tilde{a})$ in this paper, where b denotes the synthesis wavelet filter. Since there are $\mathrm{d}-1$ synthesis filters $b_1, \dots, b_{\mathrm{d}-1}$ associated with the low-pass filter a, the approach in [3,4] does not apply to the general integer dilation setting without significant modification.

(3) For the consideration of the mth order cardinal B-splines with d-refinement sequence $a = \{a(k)\}_{k \in \mathbb{Z}}$ given by (1.1)–(1.2) (see also Examples 1 and 2 in Section 3), the symbol $\tilde{\mathrm{a}}(z)$ of its dual $\tilde{a}$ is obtained by applying $\mathrm{a}_\ell^I(z)$ in (1.3), which has an even sum-rule order $\ell := 2n$. A recursive formula for computing $\mathrm{a}_\ell^I(z)$ for odd ℓ is given in [3, Theorem 2.1], but only for the special case $\mathrm{d} = 2$. On the other hand, by applying the general CBC algorithm in [1,13,14], one can construct a desirable (symmetric) dual filter $\tilde{a}$ with any preassigned order of sum rules for a given primal filter a, without relying on the interpolatory d-refinement Laurent polynomial symbol (see Examples 3, 4, and 5 in Section 3). In any case, the algorithms developed in this paper can be applied to derive the associated high-pass or band-pass filter systems. This provides a complete procedure for the construction of univariate biorthogonal wavelets with an arbitrary dilation factor.

(4) The dual-chain approach introduced in this paper is not restricted to the univariate setting, in that both Algorithms 1 and 2 can be generalized to construct multivariate filter systems. However, we have not attempted to extend Theorems 1 and 2 to study the existence of dual-chains starting from an arbitrary initial pair of dual multivariate filters. Our investigation of multivariate filter systems and biorthogonal wavelets will be addressed elsewhere in the future.

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