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SYMMETRIC CANONICAL QUINCUNX TIGHT FRAMELETS WITH HIGH VANISHING MOMENTS AND SMOOTHNESS

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ABSTRACT. In this paper, we propose an approach to construct a family of two-dimensional compactly supported real-valued quincunx tight framelets $\{\phi; \psi_1, \psi_2, \psi_3\}$ in $L_2(\mathbb{R}^2)$ with *symmetry property* and arbitrarily high orders of vanishing moments. Such quincunx tight framelets are associated with quincunx tight framelet filter banks $\{a; b_1, b_2, b_3\}$ having increasing orders of vanishing moments, possessing symmetry property, and enjoying the additional double canonical properties:

$$\begin{aligned} b_1(k_1, k_2) &= (-1)^{1+k_1+k_2} a(1-k_1, -k_2), \\ b_3(k_1, k_2) &= (-1)^{1+k_1+k_2} b_2(1-k_1, -k_2), \end{aligned} \qquad \forall\, k_1, k_2 \in \mathbb{Z}.$$

Moreover, the supports of all the high-pass filters b_1, b_2, b_3 are no larger than that of the low-pass filter a. For a low-pass filter a which is not a quincunx orthogonal wavelet filter, we show that a quincunx tight framelet filter bank $\{a; b_1, \dots, b_L\}$ with b_1 taking the above canonical form must have $L \geq 3$ high-pass filters. Thus, our family of double canonical quincunx tight framelets with symmetry property has the minimum number of generators. Numerical calculation indicates that this family of double canonical quincunx tight framelets with symmetry property can be arbitrarily smooth. Using one-dimensional filters having linear-phase moments, in this paper we also provide a second approach to construct multiple canonical quincunx tight framelets with symmetry property. In particular, the second approach yields a family of 6-multiple canonical real-valued quincunx tight framelets in $L_2(\mathbb{R}^2)$ and a family of double canonical complex-valued quincunx tight framelets in $L_2(\mathbb{R}^2)$ such that both of them have symmetry property and arbitrarily increasing orders of smoothness and vanishing moments. Several examples are provided to illustrate our general construction and theoretical results on canonical quincunx tight framelets in $L_2(\mathbb{R}^2)$ with symmetry property, high vanishing moments, and smoothness. Quincunx tight framelets with symmetry property constructed by both approaches in this paper are of particular interest for their applications in computer graphics and image processing due to their polynomial preserving property, full symmetry property, short support, and high smoothness and vanishing moments.

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1. Introduction and motivations

In this paper we study quincunx tight framelets having full symmetry property, short support, high vanishing moments and smoothness. We say that a $d\times d$ matrix M is *a dilation matrix* if M is an integer matrix having all of its eigenvalues greater than one in modulus. In dimension two, typical and important dilation matrices M include

$$2I_2 := \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \qquad M_{\sqrt{2}} := \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad N_{\sqrt{2}} := \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}, \tag{1.1}$$

where $M_{\sqrt{2}}$ and $N_{\sqrt{2}}$ are called *quincunx dilation matrices.* For functions ϕ, ψ_1, …, ψ_L in $L_2(\mathbb{R}^d)$, we say that $\{\phi;\psi_1,\dots,\psi_L\}$ is *a tight M-framelet* for $L_2(\mathbb{R}^d)$ if the affine system $\mathrm{AS}(\{\phi;\psi_1,\dots,\psi_L\})$ is a normalized tight frame of $L_2(\mathbb{R}^d)$; that is, for all $f\in L_2(\mathbb{R}^d)$,

$$\|f\|^2_{L_2(\mathbb{R}^d)} = \sum_{k\in\mathbb{Z}^d} |\langle f,\phi(\cdot-k)\rangle|^2 + \sum_{j=0}^{\infty}\sum_{\ell=1}^{L}\sum_{k\in\mathbb{Z}^d} |\langle f, |\det(M)|^{j/2}\psi_\ell(M^j\cdot-k)\rangle|^2, \tag{1.2}$$

where the affine system generated by the functions $\phi,\psi_1,\dots,\psi_L$ is defined to be

$$\begin{aligned}\mathrm{AS}(\{\phi;\psi_1,\dots,\psi_L\}) := \{&\phi(\cdot-k)\ :\ k\in\mathbb{Z}^d\} \\ &\cup \{|\det(M)|^{j/2}\psi_\ell(M^j\cdot-k)\ :\ j\in\mathbb{N}\cup\{0\}, k\in\mathbb{Z}^d, 1\le\ell\le L\},\end{aligned}$$

$\langle f,g\rangle := \int_{\mathbb{R}^d} f(x)\overline{g(x)}dx$ is the inner product, and $\|f\|_{L_2(\mathbb{R}^d)} := \sqrt{\langle f,f\rangle}$ is the L_2-norm. If $\mathrm{AS}(\{\phi;\psi_1,\dots,\psi_L\})$ is an orthonormal basis of $L_2(\mathbb{R}^d)$, then the set $\{\phi;\psi_1,\dots,\psi_L\}$ of functions is called *an orthonormal M-wavelet.* It is known in [27, Proposition 4] that if $\mathrm{AS}(\{\phi;\psi_1,\dots,\psi_L\})$ is a normalized tight frame (or an orthonormal basis) for $L_2(\mathbb{R}^d)$, then the homogeneous affine system $\mathrm{AS}(\{\psi_1,\dots,\psi_L\})$ must be a normalized tight frame (or an orthonormal basis) for $L_2(\mathbb{R}^d)$ as well, where

$$\mathrm{AS}(\{\psi_1,\dots,\psi_L\}) := \{|\det(M)|^{j/2}\psi_\ell(M^j\cdot-k)\ :\ j\in\mathbb{Z}, k\in\mathbb{Z}^d, 1\le\ell\le L\}. \tag{1.3}$$

Tight M-framelets and orthonormal M-wavelets are often derived from M-refinable functions. By $l_0(\mathbb{Z}^d)$ we denote the set of all finitely supported sequences $u=\{u(k)\}_{k\in\mathbb{Z}^d}$ on $\mathbb{Z}^d$. For $u\in l_0(\mathbb{Z}^d)$, its Fourier series (or symbol) $\widehat{u}$ is a $2\pi\mathbb{Z}^d$-periodic trigonometric polynomial defined by $\widehat{u}(\boldsymbol{\omega}) := \sum_{k\in\mathbb{Z}^d} u(k)e^{-ik\cdot\boldsymbol{\omega}}$, $\boldsymbol{\omega}\in\mathbb{R}^d$. For $a,b_1,\dots,b_L\in l_0(\mathbb{Z}^d)$ such that $\widehat{a}(0)=\sum_{k\in\mathbb{Z}^d} a(k)=1$, the following functions

$$\widehat{\phi}(\boldsymbol{\omega}) := \prod_{j=1}^{\infty}\widehat{a}((M^{-\top})^j\boldsymbol{\omega}), \quad \widehat{\psi_\ell}(\boldsymbol{\omega}) := \widehat{b_\ell}(M^{-\top}\boldsymbol{\omega})\widehat{\phi}(M^{-\top}\boldsymbol{\omega}), \quad \ell=1,\dots,L, \tag{1.4}$$

for $\boldsymbol{\omega}\in\mathbb{R}^d$, are well defined ([8]). In the spatial domain, ϕ satisfies the following *refinement equation*

$$\phi = |\det(M)|\sum_{k\in\mathbb{Z}^d} a(k)\phi(M\cdot-k)$$

and ϕ is called *the M-refinable function/distribution* associated with the *filter/mask* a. For the functions ϕ, ψ_1, …, ψ_L defined in (1.4) through the filters $a,b_1,\dots,b_L\in l_0(\mathbb{Z}^d)$ satisfying $\widehat{a}(0)=1$, $\{\phi;\psi_1,\dots,\psi_L\}$ is a tight M-framelet for $L_2(\mathbb{R}^d)$ if and only if $\{a; b_1, \dots, b_L\}$ is *a tight M-framelet filter bank*; that is,

$$\begin{cases} |\widehat{a}(\boldsymbol{\omega})|^2 + \sum_{\ell=1}^L |\widehat{b_\ell}(\boldsymbol{\omega})|^2 = 1, \\ \overline{\widehat{a}(\boldsymbol{\omega})}\widehat{a}(\boldsymbol{\omega}+2\pi\xi) + \sum_{\ell=1}^L \overline{\widehat{b_\ell}(\boldsymbol{\omega})}\widehat{b_\ell}(\boldsymbol{\omega}+2\pi\xi) = 0, \end{cases} \qquad \xi \in \Omega_M\backslash\{0\}, \tag{1.5}$$

where Ω_M is a set of representatives of the distinct cosets of the quotient group $[(M^\top)^{-1}\mathbb{Z}^d]/\mathbb{Z}^d$ and is given by

$$\Omega_M := [(M^\top)^{-1}\mathbb{Z}^d] \cap [0,1)^d. \tag{1.6}$$

As observed in [19, 22], the equations in (1.5) for a tight M-framelet filter bank only depend on the lattice $M\mathbb{Z}^d$ instead of M itself. That is, for two $d \times d$ integer matrices M and N satisfying

$$M\mathbb{Z}^d = N\mathbb{Z}^d, \tag{1.7}$$

$\{a; b_1, \dots, b_L\}$ is a tight M-framelet filter bank if and only if it is a tight N-framelet filter bank. This simple observation in [19, 22] comes from the fact that (1.7) is equivalent to $M = NE$ for some integer matrix E with $|\det(E)| = 1$, which trivially implies $(M^\top)^{-1}\mathbb{Z}^d = (N^\top)^{-1}\mathbb{Z}^d$. For example, the two quincunx dilation matrices in (1.1) satisfy $M_{\sqrt{2}}\mathbb{Z}^2 = N_{\sqrt{2}}\mathbb{Z}^2$, which is the quincunx lattice $\{(j,k) \in \mathbb{Z}^2 \, : \, j+k \text{ is even}\}$.

When (1.5) holds, it was proved in [45] that the corresponding homogeneous affine system $\mathrm{AS}(\{\psi_1, \dots, \psi_L\})$ forms a normalized tight frame in $L_2(\mathbb{R}^d)$, which is called the unitary extension principle. Under various conditions on $\phi, \psi_1, \dots, \psi_L$ and $a, b_1, \dots, b_L$, tight framelets have been studied in [6, 9, 18, 45] and references therein. Under the natural and necessary condition $\widehat{a}(0) = 1$, the above one-to-one correspondence between a tight M-framelet $\{\phi; \psi_1, \dots, \psi_L\}$ and a tight M-framelet filter bank $\{a; b_1, \dots, b_L\}$ has been presented in [22, Lemma 2.1, Theorems 2.2 and 2.3] or more generally, [27, Corollary 12 and Theorem 17] for fully nonstationary tight framelets. In particular, if $\{a; b_1, \dots, b_L\}$ is a tight M-framelet filter bank with $\widehat{a}(0) = 1$, then the functions $\phi, \psi_1, \dots, \psi_L$ defined in (1.4) must be square integrable functions in $L_2(\mathbb{R}^d)$ (see [22, Lemma 2.1]). Due to this one-to-one correspondence between tight M-framelets and tight M-framelet filter banks, in this paper we shall concentrate on tight M-framelet filter banks. Wavelets and framelets using the quincunx dilation matrices in (1.1) are called quincunx wavelets or quincunx framelets in this paper.

For some applications such as computer graphics and computer aided geometric design, symmetry property of framelets and wavelets is highly desired. There are many different types of symmetries for filters and functions in multiple dimensions. Let us now discuss the general symmetry property of a filter. Let G be a finite set of $d \times d$ integer matrices that forms a group under the usual matrix multiplication. We say that a filter $a \in l_0(\mathbb{Z}^d)$ is *G-symmetric about a point* $\mathbf{c} \in \mathbb{R}^d$ if

$$a(E(k-\mathbf{c})+\mathbf{c}) = a(k), \qquad \forall k \in \mathbb{Z}^d \text{ and } \forall E \in G. \tag{1.8}$$

Similarly, we say that a filter $a \in l_0(\mathbb{Z}^d)$ is *G-antisymmetric about a point* $\mathbf{c} \in \mathbb{R}^d$ if

$$a(E(k-\mathbf{c})+\mathbf{c}) = -a(k), \qquad \forall k \in \mathbb{Z}^d \text{ and } \forall E \in G. \tag{1.9}$$

Generally, for simplicity, we say that a filter a is symmetric (or antisymmetric) if (1.8) (or (1.9)) holds for a nontrivial group G (i.e., $G \neq \{I_d\}$). Quite often we do not want to tell/specify whether a filter is symmetric or antisymmetric. Therefore, for convenience of discussion in this paper, we say that a filter has symmetry property if it is either symmetric or antisymmetric. We say that a filter bank or a set of filters has symmetry property if each of its elements is either symmetric or antisymmetric.

However, the symmetry property of a low-pass filter a does not automatically guarantee the symmetry property of the M-refinable function ϕ defined in (1.4). As discussed in [19, 20, 24], some compatibility condition is needed. We say that a dilation matrix M is *compatible* with a symmetry group G if $MEM^{-1} \in G$ for all $E \in G$. If M is compatible with a symmetry group G, then ϕ in (1.4) is G-symmetric about $c_\phi := (M - I_d)^{-1}\mathbf{c}$ (i.e., $\phi(E(\cdot - c_\phi) + c_\phi) = \phi$ for all $E \in G$) if and only if a is G-symmetric about $\mathbf{c}$ (see [24, Proposition 2.1] and [19, 20]). One of the commonly used two-dimensional symmetry groups in computer graphics is the dihedral group D_4 given by

$$D_4 := \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \right\}. \tag{1.10}$$

Note that $M_{\sqrt{2}}$ is compatible with the symmetry group D_4 and its subgroup $\{I_2, -I_2\}$, but it is not compatible with the symmetry group

$$D_4^+ := \{\pm \mathrm{diag}(1, 1), \pm \mathrm{diag}(1, -1)\}.$$

A matrix N is *G-equivalent* to M if $N = EMF$ for some $E, F \in G$. Note that $N_{\sqrt{2}}$ in (1.1) is D_4-equivalent to $M_{\sqrt{2}}$. It is of interest to point out here that [23, Theorem 2] shows that every 2×2 matrix M compatible with D_4 must be D_4-equivalent to either $M = cI_2$ or $M = cM_{\sqrt{2}}$ for some $c \in \mathbb{Z}$. This makes the quincunx dilation matrices $M_{\sqrt{2}}$ and $N_{\sqrt{2}}$ particularly interesting for constructing tight framelets having the full symmetry D_4. For a low-pass D_4-symmetric filter a, since $N_{\sqrt{2}}$ is D_4-equivalent to $M_{\sqrt{2}}$, we shall see in this paper that the $N_{\sqrt{2}}$-refinable function is just a shifted version of the $M_{\sqrt{2}}$-refinable function. However, the $M_{\sqrt{2}}$-refinable function and the $N_{\sqrt{2}}$-refinable function associated with a low-pass filter a without symmetry could be completely different ([7, 19]). Because we are mainly interested in quincunx tight framelet filter banks with symmetric low-pass filters, as a consequence, there are no essential differences for using either $M_{\sqrt{2}}$ or $N_{\sqrt{2}}$. Therefore, for simplicity, we mainly discuss the dilation matrix $M_{\sqrt{2}}$ in this paper.

A tight M-framelet filter bank $\{a; b_1, \dots, b_L\}$ with $L = |\det(M)| - 1$ is called *an orthogonal M-wavelet filter bank*. It is a simple consequence of the equations in (1.5) (by rewriting the equations in (1.5) in a matrix form) that the low-pass filter a in a tight M-framelet filter bank must satisfy

$$\sum_{\xi \in \Omega_M} |\widehat{a}(\boldsymbol{\omega} + 2\pi\xi)|^2 \le 1, \qquad \forall\, \boldsymbol{\omega} \in \mathbb{R}^d. \tag{1.11}$$

If the above inequality becomes an identity for all $\boldsymbol{\omega} \in \mathbb{R}^d$, then the low-pass filter a is called *an orthogonal M-wavelet filter*. If $\{a; b_1, \dots, b_L\}$ is an orthogonal M-wavelet filter bank, then a must be an orthogonal M-wavelet filter and its corresponding $\{\phi; \psi_1, \dots, \psi_L\}$ in (1.4) is a tight M-framelet for $L_2(\mathbb{R}^d)$ but it may fail to be an orthonormal M-wavelet for $L_2(\mathbb{R}^d)$ ([8]). For a filter bank $\{a; b_1, \dots, b_L\}$ with $L = |\det(M)| - 1$ and $\widehat{a}(0) = 1$, $\{\phi; \psi_1, \dots, \psi_L\}$ in (1.4) is an orthonormal M-wavelet for $L_2(\mathbb{R}^d)$ if and only if $\{a; b_1, \dots, b_L\}$ is an orthogonal M-wavelet filter bank and $\mathrm{sm}(a, M) > 0$, where the technical quantity $\mathrm{sm}(a, M)$ is defined in (2.5). See [1, 7, 8, 20, 21, 24, 26, 43, 44] and references therein for orthonormal wavelets. For a $d \times d$ dilation matrix M, it is trivial to see that $|\det(M)| \ge 2$. For $|\det(M)| = 2$, an orthogonal M-wavelet filter bank $\{a; b_1, \dots, b_L\}$ with $L = |\det(M)| - 1$ has only one high-pass filter b_1 which is derived from the low-pass filter a by

$$\widehat{b_1}(\boldsymbol{\omega}) := e^{-i\boldsymbol{\omega}\cdot\gamma}\overline{\widehat{a}(\boldsymbol{\omega}+2\pi\xi)}, \quad \boldsymbol{\omega} \in \mathbb{R}^d \text{ with } \gamma \in \mathbb{Z}^d\backslash[M\mathbb{Z}^d],\ \xi \in \Omega_M\backslash\{0\}. \tag{1.12}$$

Therefore, for a dilation matrix M with $|\det(M)| = 2$, an orthonormal M-wavelet $\{\phi; \psi_1, \dots, \psi_L\}$ with $L = |\det(M)| - 1$ has only one wavelet function ψ_1. Hence, it is of interest in both theory and application to consider dilation matrices M with $|\det(M)| = 2$. This is another motivation for us to consider the quincunx dilation matrices in (1.1).

Due to the importance of high dimensional problems, multivariate wavelets and framelets have been studied for many years now. For example, quincunx orthonormal wavelets have been investigated in [7, 19] and quincunx biorthogonal wavelets have been studied in [7, 33, 38]. Using the dilation matrix $M_{\sqrt{2}}$ and perturbation of the Daubechies orthonormal wavelets, a family of quincunx orthonormal wavelets with arbitrarily smoothness orders has been reported in [1]. However, compactly supported continuous quincunx orthonormal wavelets cannot have symmetry properties (see [7] and [24, Proposition 2.2]). Moreover, it still remains unknown so far whether there exists a C^1 compactly supported orthonormal $N_{\sqrt{2}}$-refinable function ([7] and [19, Example 3.6]). In fact, if the dilation matrix $M_{\sqrt{2}}$ is changed into $N_{\sqrt{2}}$ for the family of quincunx wavelet filter banks in [1], as a known phenomenon observed in [7], their smoothness orders are no more than one and decrease to zero. The quincunx biorthogonal wavelets constructed in some literature such as [33, 38] have nice smoothness and/or full D_4-symmetry. However the biorthogonal wavelets usually have large supports and the corresponding wavelet transforms have large condition numbers. Pairs of quincunx dual frames have been obtained in [15, Corollary 3.4] having only three wavelet functions without symmetry property. Due to the difficulty in constructing multivariate wavelets with desirable properties such as symmetry property, short support and high vanishing moments (see [7, 8, 19, 24, 26] and references therein), the current interest has been focusing on the construction of tight M-framelets with various dilation matrices and properties. Tight M-framelets have been studied and constructed in many articles. For example, the topic of wavelet frames has been investigated in [6, 8, 9, 18, 27, 44, 45] and references therein. The theory and construction of one-dimensional tight 2-framelets are quite complete so far; for example, see [5, 6, 9, 12, 13, 29, 31, 32, 34, 36, 42, 45] and many references therein. In particular, if a is $\{1, -1\}$-symmetric, the construction of 2-framelet filter bank $\{a; b_1, \dots, b_L\}$ with $L = 2$ or $L = 3$ having symmetry property and short support have been completely solved in [29, 31, 34] with efficient algorithms. The construction of multivariate tight framelets has been reported in [17–19, 22, 30, 37, 39, 41, 46] and references therein. The applications of tight framelets to various applications such as image restoration have been investigated in [13, 35, 47, 48]. Recently, wavelet frames have been used for surface processing [10, 40]. Furthermore, the connections of the wavelet frame based, especially the spline tight wavelet frame based, approach for image restoration to PDE based methods have been established in [2] for the total variational method and extended in [11] for the nonlinear diffusion partial differential equation based methods, as well as in [3] for variational models on the space of piecewise smooth functions.

We now explain our motivations to study quincunx tight framelets and quincunx tight framelet filter banks. From the viewpoint of theory and application for particular areas such as computer aided geometric design and image processing, the following are some key desirable features of a tight M-framelet filter bank $\{a; b_1, \dots, b_L\}$:

(i) The high-pass filters $b_1, \dots, b_L$ have desired high orders of vanishing moments.
(ii) The low-pass filter a has full symmetry property and all the high-pass filters $b_1, \dots, b_L$ possess desired symmetry property.
(iii) The number L of high-pass filters should be relatively small for computational efficiency.
(iv) The low-pass filter a should have short support, while the supports of all high-pass filters $b_1, \dots, b_L$ should not be larger than the support of the low-pass filter a.
(v) The smoothness exponent $\mathrm{sm}(a, M)$ (see (2.5)) can be arbitrarily large.

Let $\{\phi; \psi_1, \dots, \psi_L\}$ be its associated tight M-framelet for $L_2(\mathbb{R}^d)$, where ϕ, ψ_1, …, ψ_L are defined in (1.4). Item (i) implies that all the wavelet generators $\psi_1, \dots, \psi_L$ have high orders of vanishing moments. The high order of vanishing moments in item (i) is closely related to sparse approximation by tight framelets and necessarily requires that the low-pass filter a should have high order of sum rules. Item (v) implies that the smoothness exponents of all the functions $\phi, \psi_1, \dots, \psi_L$ can be arbitrarily large since $\mathrm{sm}(\phi) \ge \mathrm{sm}(a, M)$ and $\mathrm{sm}(\psi_1) = \cdots = \mathrm{sm}(\psi_L) = \mathrm{sm}(\phi)$ (see (2.3)). The definitions of vanishing moments $\mathrm{vm}(a)$, sum rules $\mathrm{sr}(a, M)$, and smoothness exponents $\mathrm{sm}(\phi)$ and $\mathrm{sm}(a, M)$ will be defined in Section 2. High orders of vanishing moments in item (i) and smoothness in item (v) are of theoretical interest and importance for characterizing function spaces by framelets. Item (ii) implies that all the functions $\phi, \psi_1, \dots, \psi_L$ have symmetry property. The symmetry property in item (ii) is indispensable for applications of tight framelets to certain areas such as computer graphics and is often strongly desired in areas such as image processing for better visual quality. Item (iv) implies that all $\phi, \psi_1, \dots, \psi_L$ have shortest possible support. Items (iii) and (iv) are important in applications for computational efficiency. We also point out here that because different applications require different desirable properties of framelets and wavelets, it is not surprising that the above outlined desirable properties in items (i)–(v) may not be needed or should be changed accordingly for a particular application. For example, instead of high orders of vanishing moments in item (i), consecutive orders of vanishing moments starting from vanishing moment one are found to be very useful in image processing [2, 13, 47]. To achieve directionality in [28, 35] for applications of complex tight framelets in image/video denoising, symmetry property of the high-pass filters in item (ii) is sacrificed (but the low-pass filter is symmetric and the high-pass filters have pairwise symmetry). Nevertheless, the outlined properties in items (i)–(v) are highly desired for applications in computer graphics, computer aided geometric design as well as other applications.

Despite numerous efforts by many researchers on constructions of multivariate tight M-framelets and tight M-framelet filter banks in many papers, none of them can really achieve all the above desirable properties in items (i)–(v). For example, tight M-framelet filter banks with short supports have been constructed in [17, 46] from a special class of almost separable low-pass filters. For a d-dimensional filter $a \in l_0(\mathbb{Z}^d)$, we say that a is *an almost separable filter* if its symbol is a finite product of symbols of one-dimensional filters as follows:

$$\widehat{a}(\boldsymbol{\omega})=\prod_{\ell=1}^{K}\widehat{a_\ell}(\gamma_\ell\cdot\boldsymbol{\omega}),\qquad \boldsymbol{\omega}\in\mathbb{R}^d\quad\text{with}\quad a_\ell\in l_0(\mathbb{Z}),\gamma_\ell\in\mathbb{Z}^d. \tag{1.13}$$

Because the one-dimensional filters a_ℓ used in [17,46] are Haar type low-pass filters with sum rule order one, it is not surprising that all the constructed tight framelets in [17,46] have only one vanishing moment. For every $d\times d$ dilation matrix M, tight M-framelet filter banks with arbitrarily high vanishing moments have been reported in [19,22] by employing the simple observation in (1.7) on the role of a dilation matrix M in a tight M-framelet filter bank. Note that every dilation matrix M can be written as $M=E\Lambda F$ (see [19,22]), where E,Λ,F are integer matrices such that $|\det(E)|=|\det(F)|=1$ and Λ is diagonal. This allows [22, Theorem 1.1 and Lemma 3.1] and [19, Corollary 3.4] to trivially have a tight Λ-framelet (or orthonormal Λ-wavelet) filter bank $\{a;b_1,\dots,b_L\}$ with arbitrarily high vanishing moments and short support through tensor product of one-dimensional ones and consequently, $\{a(E\cdot);b_1(E\cdot),\dots,b_L(E\cdot)\}$ is a tight M-framelet (or orthogonal M-wavelet) filter bank. Note that $a(E\cdot)$ is an almost separable filter by $\widehat{a(E\cdot)}(\boldsymbol{\omega})=\widehat{a}((E^\top)^{-1}\boldsymbol{\omega})$. Tight M-framelet filter banks derived from almost separable low-pass filters can be also trivially constructed in [30] through projecting tensor product tight framelet filter banks. In particular, tight $2I_d$-framelet filter banks for every box spline filter having at least order one sum rule can be painlessly constructed (see [30, Theorem 2.5]). In fact, all the constructions in [17,19,22,30,46] can be regarded as various special cases of the projection method developed in [30]. Using sum of squares, for a (two-dimensional) low-pass filter a satisfying (1.11), a general method has been proposed in [4,41]. From any box-spline filter a having at least order one sum rules, recently [16] constructs a tight $2I_d$-framelet filter bank whose high-pass filters have short support as that of the low-pass filter a and the number $L-1$ is equal to the number of nonzero coefficients in a. But all the constructed tight $2I_d$-framelet filter banks in [16] cannot have more than one vanishing moment, since the method in [16] requires a low-pass filter to have nonnegative coefficients. However, all the constructed tight framelets in [4,16,17,19,22,30,41,46] either lack symmetry property or have a very large number L of high-pass filters, while the supports of the constructed high-pass filters in [41] could be much larger than the support of the low-pass filter. Beyond the above constructions of multivariate tight M-framelet filter banks, particular examples of tight M-framelet filter banks have been given in [37,39] and other references. However, it remains unclear whether one can construct a family of tight M-framelet filter banks (in particular, for $M=M_{\sqrt{2}}$ due to [23, Theorem 2] on all dilation matrices compatible with the symmetry group D_4) achieving all the desirable properties in items (i)–(v).

By (1.5), the equations for a tight $M_{\sqrt{2}}$-framelet filter bank $\{a;b_1,\dots,b_L\}$ become

$$|\widehat{a}(\boldsymbol{\omega})|^2+|\widehat{b_1}(\boldsymbol{\omega})|^2+\sum_{\ell=2}^{L}|\widehat{b_\ell}(\boldsymbol{\omega})|^2=1, \tag{1.14}$$

$$\overline{\widehat{a}(\boldsymbol{\omega})}\widehat{a}(\boldsymbol{\omega}+(\pi,\pi))+\overline{\widehat{b_1}(\boldsymbol{\omega})}\widehat{b_1}(\boldsymbol{\omega}+(\pi,\pi))+\sum_{\ell=2}^{L}\overline{\widehat{b_\ell}(\boldsymbol{\omega})}\widehat{b_\ell}(\boldsymbol{\omega}+(\pi,\pi))=0. \tag{1.15}$$

If in addition the following relation (which is a special case of (1.12)) holds:

$$\widehat{b_1}(\boldsymbol{\omega}) = e^{-i\boldsymbol{\omega}\cdot(1,0)}\overline{\widehat{a}(\boldsymbol{\omega}+(\pi,\pi))}, \qquad \boldsymbol{\omega}\in\mathbb{R}^2, \tag{1.16}$$

we call $\{a; b_1, \dots, b_L\}$ *a canonical quincunx tight framelet filter bank.* Moreover, if $\{a; b_1, \dots, b_{2s-1}\}$ is a tight $M_{\sqrt{2}}$-framelet filter bank satisfying (1.16) and

$$\widehat{b_{2\ell+1}}(\boldsymbol{\omega}) = e^{-i\boldsymbol{\omega}\cdot(1,0)}\overline{\widehat{b_{2\ell}}(\boldsymbol{\omega}+(\pi,\pi))}, \qquad \ell=1,\dots,s-1,\ \boldsymbol{\omega}\in\mathbb{R}^2, \tag{1.17}$$

then it is called *an s-multiple canonical quincunx tight framelet filter bank.* In particular, for $s=2$, it is called *a double canonical quincunx tight framelet filter bank.* Note that the particular vector $(1,0)$ in (1.16) and (1.17) can be replaced by any vector from $\mathbb{Z}^2\backslash[M_{\sqrt{2}}\mathbb{Z}^2]$. Also note that (1.16) is equivalent to

$$b_1(k_1,k_2) = (-1)^{1+k_1+k_2}\overline{a(1-k_1,-k_2)}, \qquad k_1,k_2\in\mathbb{Z},$$

and (1.17) is equivalent to

$$b_{2\ell+1}(k_1,k_2) = (-1)^{1+k_1+k_2}\overline{b_{2\ell}(1-k_1,-k_2)}, \qquad k_1,k_2\in\mathbb{Z},\ \ell=1,\dots,s-1.$$

The goal of this paper is to construct a family of quincunx tight framelet filter banks achieving all the above desirable properties in items (i)–(v) with the additional canonical property in (1.16) and (1.17). For an s-multiple canonical quincunx tight framelet filter bank $\{a; b_1, \dots, b_{2s-1}\}$, the conditions in (1.16) and (1.17) automatically imply (1.15) with $L = 2s-1$. Hence, $\{a; b_1, \dots, b_{2s-1}\}$ is an s-multiple canonical quincunx tight framelet filter bank if and only if

$$\sum_{\ell=1}^{s-1}\Big[|\widehat{b_{2\ell}}(\boldsymbol{\omega})|^2 + |\widehat{b_{2\ell}}(\boldsymbol{\omega}+(\pi,\pi))|^2\Big] = 1 - |\widehat{a}(\boldsymbol{\omega})|^2 - |\widehat{a}(\boldsymbol{\omega}+(\pi,\pi))|^2, \tag{1.18}$$

which is simply a problem of sum of squares. If $\{a; b_1, \dots, b_L\}$ is a canonical quincunx tight framelet filter bank satisfying (1.16) and if a is not an orthogonal $M_{\sqrt{2}}$-wavelet filter, then it is quite trivial to show that $L \ge 3$. Indeed, if $L = 1$, then $\{a; b_1\}$ must be an orthogonal $M_{\sqrt{2}}$-wavelet filter bank and, consequently, a must be an orthogonal $M_{\sqrt{2}}$-wavelet filter, which is a contradiction to our assumption on a; hence, $L \ge 2$. Suppose that $L = 2$. By (1.16), the equation in (1.15) with $L = 2$ becomes $\overline{\widehat{b_2}(\boldsymbol{\omega})}\widehat{b_2}(\boldsymbol{\omega}+(\pi,\pi)) = 0$, from which we must have $b_2 = 0$. This implies $L = 1$, a contradiction. Therefore, we must have $L \ge 3$. On the other hand, as shown in [29, 34], there is a very restrictive necessary and sufficient condition for a tight 2-framelet filter bank $\{a; b_1, \dots, b_L\}$ with $L = 2$ and symmetry property. For similar reasons, it is natural that $L = 3$ is the smallest possible number of high-pass filters for a quincunx tight framelet filter bank $\{a; b_1, \dots, b_L\}$ with symmetry property. One of the main goals of this paper is to construct a family of double canonical quincunx tight framelet filter banks $\{a; b_1, b_2, b_3\}$ with symmetry property, short supports, and increasing orders of vanishing moments achieving all the desirable properties in items (i)–(v).

The structure of the paper is as follows. In Section 2, we shall first introduce a family of minimally supported two-dimensional symmetric low-pass filters with arbitrarily high sum rule orders and linear-phase moments. Then we shall employ such symmetric low-pass filters to construct a family of compactly supported tight framelets with double canonical quincunx tight framelet filter banks $\{a; b_1, b_2, b_3\}$ with symmetry property and arbitrarily high orders of vanishing moments. Numerical calculation also indicates that the smoothness exponents of this family of compactly supported tight framelets can be arbitrarily large. In Section 3, we shall generalize the particular construction in Section 2 and propose a general construction of double canonical quincunx tight framelet filter banks with symmetry property and vanishing moments which are derived from one-dimensional filters with linear-phase moments. A few illustrative examples of such double canonical quincunx tight framelet filter banks $\{a; b_1, b_2, b_3\}$ are given in Sections 2 and 3. In Section 4, we shall take another approach by studying multiple canonical quincunx tight framelet filter banks with symmetry property using almost separable low-pass filters. In particular, we present a family of compactly supported 6-multiple canonical real-valued quincunx tight framelets and a family of compactly supported double canonical complex-valued quincunx tight framelets such that both of them have symmetry property and arbitrarily high orders of smoothness exponents and vanishing moments. We complete this paper by providing a detailed proof to Theorems 2.1 and 4.2 in Appendix A.

2. Double canonical quincunx tight framelets with symmetry property and minimal support

In this section we shall first discuss how to construct a family of minimally supported symmetric low-pass filters with increasing orders of sum rules and linear-phase moments. Such a family of symmetric low-pass filters is of particular interest in their applications to computer graphics and computer aided geometric design, due to their polynomial preservation property, short support and high smoothness. Then we shall use such symmetric low-pass filters to build double canonical quincunx tight framelet filter banks with symmetry property and increasing order of vanishing moments.

For an integer j such that $1 \le j \le d$, by ∂_j we denote the partial derivative with respect to the jth coordinate of $\mathbb{R}^d$. Define $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For any $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}_0^d$, we define $|\mu| := |\mu_1| + \cdots + |\mu_d|$ and ∂^μ the differentiation operator $\partial_1^{\mu_1} \cdots \partial_d^{\mu_d}$. For a nonnegative integer m and two smooth functions f, g, we shall use the following big $\mathcal{O}$ notation

$$f(\boldsymbol{\omega}) = g(\boldsymbol{\omega}) + \mathcal{O}(\|\boldsymbol{\omega} - \boldsymbol{\omega}_0\|^m), \qquad \boldsymbol{\omega} \to \boldsymbol{\omega}_0, \tag{2.1}$$

to mean the following relation:

$$\partial^\mu f(\boldsymbol{\omega}_0) = \partial^\mu g(\boldsymbol{\omega}_0), \qquad \forall\, \mu \in \mathbb{N}_0^d \text{ satisfying } |\mu| < m. \tag{2.2}$$

For smooth functions, as shown in [26, Lemma 1], using the big $\mathcal{O}$ notation in (2.1) to mean (2.2) agrees with the commonly accepted big $\mathcal{O}$ notation in the literature.

In the following we introduce several quantities that are frequently used in this paper, in particular, sum rule order $\mathrm{sr}(a, M)$, vanishing moment order $\mathrm{vm}(a)$, linear-phase moment order $\mathrm{lpm}(a)$, smoothness exponents $\mathrm{sm}(a, M), \mathrm{sm}_p(a, M)$ and $\mathrm{sm}(\phi)$.

Let $a \in l_0(\mathbb{Z}^d)$ be a filter. We say that the filter a has *order m sum rules* with respect to a dilation matrix M if $\widehat{a}(0) = 1$ and $\widehat{a}(\boldsymbol{\omega} + 2\pi\xi) = \mathcal{O}(\|\boldsymbol{\omega}\|^m)$ as $\boldsymbol{\omega} \to 0$ for all $\xi \in \Omega_M \backslash \{0\}$. In particular, we define $\mathrm{sr}(a, M) := m$ with m being the largest such integer. We say that the filter a has *order n vanishing moments* if $\widehat{a}(\boldsymbol{\omega}) = \mathcal{O}(\|\boldsymbol{\omega}\|^n)$ as $\boldsymbol{\omega} \to 0$. In particular, we define $\mathrm{vm}(a) := n$ with n being the largest such integer. We say that a filter $a \in l_0(\mathbb{Z}^d)$ has *order n linear-phase moments with phase* $\mathbf{c} \in \mathbb{R}^d$ if $\widehat{a}(\boldsymbol{\omega}) = e^{-i\mathbf{c}\cdot\boldsymbol{\omega}} + \mathcal{O}(\|\boldsymbol{\omega}\|^n)$ as $\boldsymbol{\omega} \to 0$. In particular, we define $\mathrm{lpm}(a) := n$ with n being the largest such integer. The notion of linear-phase moments has been introduced in [25] for studying symmetric complex orthonormal 2-wavelets and plays a central role in the construction of complex symmetric orthonormal wavelets, subdivision schemes with polynomial preservation property in computer graphics, and symmetric tight framelets with vanishing moments (see [12, 14, 25, 26, 28]).

For a function $\phi \in L_2(\mathbb{R}^d)$, its Sobolev *smoothness exponent* $\mathrm{sm}(\phi)$ is defined to be

$$\mathrm{sm}(\phi) := \sup\Big\{\tau \in \mathbb{R} \;:\; \int_{\mathbb{R}^d} |\widehat{\phi}(\boldsymbol{\omega})|^2(1+\|\boldsymbol{\omega}\|^2)^\tau d\boldsymbol{\omega} < \infty\Big\}. \tag{2.3}$$

If ϕ is an M-refinable function associated with a filter $a \in l_0(\mathbb{Z}^d)$, then the smoothness exponent $\mathrm{sm}(\phi)$ is closely linked to a quantity $\mathrm{sm}(a, M)$ introduced in [21]. For $u \in l_0(\mathbb{Z}^d)$ and $\mu = (\mu_1, \dots, \mu_d) \in \mathbb{N}_0^d$, we define

$$\nabla_k u := u - u(\cdot - k), \qquad k \in \mathbb{Z}^d \quad \text{and} \quad \nabla^\mu := \nabla_{e_1}^{\mu_1} \cdots \nabla_{e_d}^{\mu_d}, \tag{2.4}$$

where $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{R}^d$ has its only nonzero entry 1 at the jth coordinate. By δ we denote *the Dirac sequence* such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \in \mathbb{Z}^d \backslash \{0\}$. For $a \in l_0(\mathbb{Z}^d)$ and a $d \times d$ dilation matrix M, let $m := \mathrm{sr}(a, M)$. For $1 \le p \le \infty$, the *smoothness exponent* $\mathrm{sm}_p(a, M)$ (see [21]) is defined to be

$$\mathrm{sm}_p(a, M) := \tfrac{d}{p} - d \log_{|\det(M)|} \rho_m(a, M)_p \text{ and } \mathrm{sm}(a, M) := \mathrm{sm}_2(a, M), \tag{2.5}$$

where

$$\rho_m(a, M)_p := \sup\Big\{\lim_{n\to\infty} \|\nabla^\mu \mathcal{S}_{a,M}^n \delta\|_{l_p(\mathbb{Z}^d)}^{1/n} \;:\; \mu \in \mathbb{N}_0^d, |\mu| = m\Big\} \tag{2.6}$$

and the subdivision operator $\mathcal{S}_{a,M}$ is defined to be

$$[\mathcal{S}_{a,M} v](n) := |\det(M)| \sum_{k \in \mathbb{Z}^d} v(k) a(n - Mk), \qquad n \in \mathbb{Z}^d. \tag{2.7}$$

The quantity $\mathrm{sm}(a, M)$ can be computed by [20, Algorithm 2.1]. We say that M is *isotropic* if M is similar to a diagonal matrix $\mathrm{diag}(\lambda_1, \dots, \lambda_d)$ with $|\lambda_1| = \cdots = |\lambda_d|$. Note that the two quincunx matrices $M_{\sqrt{2}}$ and $N_{\sqrt{2}}$ in (1.1) are isotropic. For an isotropic dilation matrix M, we have $\mathrm{sm}(\phi) \ge \mathrm{sm}(a, M)$ and if in addition the integer shifts of ϕ are stable (i.e., $\sum_{k\in\mathbb{Z}^d} |\widehat{\phi}(\boldsymbol{\omega} + 2\pi k)|^2 \ne 0$ for all $\boldsymbol{\omega} \in \mathbb{R}^d$), then $\mathrm{sm}(\phi) = \mathrm{sm}(a, M)$ (e.g., see [20, 21] and many references therein).

Suppose that $\{a; b_1, \dots, b_L\}$ is a tight M-framelet filter bank. Through the equations in (1.5) and assuming that $\widehat{a}(0) = 1$, it is shown (see e.g. [9, 28]) that

$$\min(\mathrm{vm}(b_1), \dots, \mathrm{vm}(b_L)) = \min(\mathrm{sr}(a, M), \tfrac{1}{2} \mathrm{lpm}(a * a^\star)), \tag{2.8}$$

where $\widehat{a * a^\star}(\boldsymbol{\omega}) := |\widehat{a}(\boldsymbol{\omega})|^2$. It is straightforward to see that $\mathrm{lpm}(a * a^\star) \ge \mathrm{lpm}(a)$. If the low-pass filter a is symmetric about a point $\mathbf{c} \in \mathbb{R}^d$: $a(2\mathbf{c} - k) = \overline{a(k)}$ for all $k \in \mathbb{Z}^d$, it has been shown in [28, Proposition 5.3] that $\mathrm{lpm}(a * a^\star) = \mathrm{lpm}(a)$ and for a tight M-framelet filter bank $\{a; b_1, \dots, b_L\}$ with $\widehat{a}(0) = 1$,

$$\min(\mathrm{vm}(b_1), \dots, \mathrm{vm}(b_L)) = \min(\mathrm{sr}(a, M), \tfrac{1}{2} \mathrm{lpm}(a)). \tag{2.9}$$

Therefore, to construct quincunx tight framelet filter banks with symmetry property and high vanishing moments, it is necessary to have low-pass filters having high orders of sum rules and linear-phase moments.

The following result presents a family of minimally supported D_4-symmetric low-pass filters having increasing orders of sum rules and linear-phase moments.

Theorem 2.1. *For every positive integer n, there exists a unique two-dimensional filter $a^{2D}_{2n,2n}$ such that $a^{2D}_{2n,2n}$ is supported inside $[1-n,n]^2\cap\mathbb{Z}^2$, has order $2n$ sum rules with respect to $M_{\sqrt{2}}$ and order $2n$ linear-phase moments with phase $\mathbf{c}:=(1/2,1/2)$. Moreover,*

(i) *the filter $a^{2D}_{2n,2n}$ is real-valued and is given by*

$$\widehat{a^{2D}_{2n,2n}}(\omega_1,\omega_2)=\frac{1}{2}[\widehat{u}(\omega_1+\omega_2)+\widehat{u}(\omega_1-\omega_2)e^{-i\omega_2}], \tag{2.10}$$

where $\widehat{u}(\omega):=(\widehat{a^I_{2n}}(\omega/2)-\widehat{a^I_{2n}}(\omega/2+\pi))e^{-i\omega/2}$ and a^I_{2n} is the interpolatory 2-wavelet filter given by

$$\widehat{a^I_{2n}}(\omega):=\cos^{2n}(\omega/2)\sum_{j=0}^{n-1}\binom{n-1+j}{j}\sin^{2j}(\omega/2),\quad \omega\in\mathbb{R}; \tag{2.11}$$

(ii) *the filter $a^{2D}_{2n,2n}$ is D_4-symmetric about the point $\mathbf{c}=(1/2,1/2)$;*

(iii) *$\phi^{N_{\sqrt{2}}}=\phi^{M_{\sqrt{2}}}(\cdot+(1,1))$ and $\phi^{M_{\sqrt{2}}}$ is real-valued with the following symmetry property:*

$$\phi^{M_{\sqrt{2}}}(E(\cdot-\mathbf{c}_\phi)+\mathbf{c}_\phi)=\phi^{M_{\sqrt{2}}},\qquad \forall\, E\in D_4, \tag{2.12}$$

with $\mathbf{c}_\phi:=(M_{\sqrt{2}}-I_2)^{-1}\mathbf{c}=(3/2,1/2)$, where $\phi^{M_{\sqrt{2}}}$ and $\phi^{N_{\sqrt{2}}}$ are the refinable functions associated with the filter a and the dilation matrices $M_{\sqrt{2}},N_{\sqrt{2}}$ in (1.1), respectively, and are defined in the frequency domain through

$$\begin{cases}\widehat{\phi^{M_{\sqrt{2}}}}(\boldsymbol{\omega}):=\prod_{j=1}^\infty \widehat{a^{2D}_{2n,2n}}((M^\top_{\sqrt{2}})^{-j}\boldsymbol{\omega}),\\ \widehat{\phi^{N_{\sqrt{2}}}}(\boldsymbol{\omega}):=\prod_{j=1}^\infty \widehat{a^{2D}_{2n,2n}}((N^\top_{\sqrt{2}})^{-j}\boldsymbol{\omega}),\end{cases}\qquad \boldsymbol{\omega}\in\mathbb{R}^2. \tag{2.13}$$

The proof of Theorem 2.1 is given in Appendix A. We now derive double canonical quincunx tight framelet filter banks with symmetry property from the low-pass filters $a^{2D}_{2n,2n}$ constructed in Theorem 2.1.

Theorem 2.2. *Let $a=a^{2D}_{2n,2n}$ with $n\in\mathbb{N}$ be the filter constructed in (2.10) of Theorem 2.1. Define a high-pass filter b_2 by*

$$\widehat{b_2}(\omega_1,\omega_2):=\frac{1}{2}[\widehat{v}(\omega_1+\omega_2)+\widehat{v}(\omega_1-\omega_2)e^{-i\omega_2}] \tag{2.14}$$

with $\widehat{v}(\omega):=2\widehat{a^D_n}(\omega/2)\widehat{a^D_n}(\omega/2+\pi)$, and define high-pass filters b_1,b_3 as in (1.16) and (1.17), where $a^D_n\in l_0(\mathbb{Z})$ is a real-valued Daubechies orthogonal 2-wavelet filter satisfying $|\widehat{a^D_n}(\omega)|^2=\widehat{a^I_{2n}}(\omega)$. Then $\{a;b_1,b_2,b_3\}$ is a double canonical quincunx tight framelet filter bank satisfying

(i) *all high-pass filters b_1,b_2,b_3 have real coefficients and the following symmetry property:*

$$b_1(E(k-\mathring{\mathbf{c}})+\mathring{\mathbf{c}})=\det(E)b_1(k),\ \forall\, k\in\mathbb{Z}^2, E\in D_4 \text{ with } \mathring{\mathbf{c}}:=(1/2,-1/2) \tag{2.15}$$

and

$$b_2(k_1,1-k_2)=b_2(k_1,k_2) \text{ and } b_3(k_1,-1-k_2)=-b_3(k_1,k_2),\ \forall\, k_1,k_2\in\mathbb{Z}; \tag{2.16}$$

(ii) *all high-pass filters b_1,b_2,b_3 have at least order n vanishing moments;*

(iii) *the supports of b_1,b_2,b_3 are no larger than that of the low-pass filter a.*

Moreover, $\{\phi^{M_{\sqrt{2}}};\psi_1,\psi_2,\psi_3\}$ *is a tight* $M_{\sqrt{2}}$*-framelet in* $L_2(\mathbb{R}^2)$ *such that* $\phi^{M_{\sqrt{2}}}$ *has the symmetry property in* (2.12),

$$\psi_1(E(\cdot-\mathbf{c}_1)+\mathbf{c}_1)=\det(E)\psi_1, \qquad \forall\, E\in D_4 \quad \textit{with} \quad \mathbf{c}_1:=(1,1), \tag{2.17}$$

and

$$\psi_2(x_2+1,x_1-1)=\psi_2(x_1,x_2), \qquad \psi_3(x_2,x_1)=-\psi_3(x_1,x_2), \tag{2.18}$$

where $\phi^{M_{\sqrt{2}}}$ *is defined in* (2.13) *and* $\widehat{\psi_\ell}(\boldsymbol{\omega}):=\widehat{b_\ell}((M_{\sqrt{2}}^\top)^{-1}\boldsymbol{\omega})\widehat{\phi^{M_{\sqrt{2}}}}((M_{\sqrt{2}}^\top)^{-1}\boldsymbol{\omega})$ *for* $\ell=1,2,3$.

Proof. Let $\widehat{u}(\omega)=(\widehat{a^I_{2n}}(\omega/2)-\widehat{a^I_{2n}}(\omega/2+\pi))e^{-i\omega/2}=(2\widehat{a^I_{2n}}(\omega/2)-1)e^{-i\omega/2}$, where we use $\widehat{a^I_{2n}}(\omega/2)+\widehat{a^I_{2n}}(\omega/2+\pi)=1$. By the definition of $a=a^{2D}_{2n,2n}$ in (2.10), we have

$$|\widehat{a}(\omega_1,\omega_2)|^2+|\widehat{a}(\omega_1+\pi,\omega_2+\pi)|^2=\frac{1}{2}\Big[|\widehat{u}(\omega_1+\omega_2)|^2+|\widehat{u}(\omega_1-\omega_2)|^2\Big]. \tag{2.19}$$

Similarly, by the definition of b_2, we have

$$|\widehat{b_2}(\omega_1,\omega_2)|^2+|\widehat{b_2}(\omega_1+\pi,\omega_2+\pi)|^2=\frac{1}{2}\Big[|\widehat{v}(\omega_1+\omega_2)|^2+|\widehat{v}(\omega_1-\omega_2)|^2\Big]. \tag{2.20}$$

Since $|\widehat{a^D_n}(\omega)|^2=\widehat{a^I_{2n}}(\omega)$, we have

$$\begin{aligned}1-|\widehat{u}(\omega)|^2&=1-|2\widehat{a^I_{2n}}(\omega/2)-1|^2=1-4(\widehat{a^I_{2n}}(\omega/2))^2+4\widehat{a^I_{2n}}(\omega/2)-1\\&=4\widehat{a^I_{2n}}(\omega/2)(1-\widehat{a^I_{2n}}(\omega/2))=4\widehat{a^I_{2n}}(\omega/2)\widehat{a^I_{2n}}(\omega/2+\pi)=|\widehat{v}(\omega)|^2.\end{aligned}$$

Consequently, (1.18) holds with $s=2$. Therefore, $\{a;b_1,b_2,b_3\}$ is a double canonical quincunx tight framelet filter bank.

Since a is D_4-symmetric about the point $\mathbf{c}=(1/2,1/2)$, (1.8) is equivalent to

$$\widehat{a}(E^\top\boldsymbol{\omega})=e^{i(I_2-E)\mathbf{c}\cdot\boldsymbol{\omega}}\widehat{a}(\boldsymbol{\omega}), \quad \boldsymbol{\omega}\in\mathbb{R}^2, E\in D_4. \tag{2.21}$$

For $E\in D_4$, we have $(I-E)(1,1)\in 2\mathbb{Z}^2$ and by the definition of b_1,

$$\begin{aligned}\widehat{b_1}(E^\top\boldsymbol{\omega})&=e^{-i\boldsymbol{\omega}\cdot E(1,0)}\overline{\widehat{a}(E^\top\boldsymbol{\omega}+(\pi,\pi))}=e^{-i\boldsymbol{\omega}\cdot E(1,0)}\overline{\widehat{a}(E^\top(\boldsymbol{\omega}+(\pi,\pi)))}\\&=e^{-i\boldsymbol{\omega}\cdot E(1,0)}e^{-i(I-E)\mathbf{c}\cdot(\boldsymbol{\omega}+(\pi,\pi))}\overline{\widehat{a}(\boldsymbol{\omega}+(\pi,\pi))}=\det(E)e^{i(I-E)\hat{\mathbf{c}}\cdot\boldsymbol{\omega}}\widehat{b_1}(\boldsymbol{\omega}).\end{aligned}$$

This proves (2.15). By the definitions of b_2 and $\widehat{b_3}(\boldsymbol{\omega})=e^{-i\omega_1}\overline{\widehat{b_2}(\boldsymbol{\omega}+(\pi,\pi))}$, we have

$$\widehat{b_2}(\omega_1,-\omega_2)=\widehat{b_2}(\omega_1,\omega_2)e^{i\omega_2} \quad \text{and} \quad b_3(\omega_1,-\omega_2)=-\widehat{b_3}(\omega_1,\omega_2)e^{-i\omega_2},$$

which are equivalent to (2.16). Therefore, item (i) holds.

Item (ii) follows directly from

$$\min(\mathrm{vm}(b_1),\mathrm{vm}(b_2),\mathrm{vm}(b_3))=\min(\mathrm{sr}(a,M_{\sqrt{2}}),\frac{1}{2}\,\mathrm{lpm}(a))=n$$

due to $\mathrm{sr}(a,M_{\sqrt{2}})=\mathrm{lpm}(a)=2n$. Item (iii) can be directly checked.

By [24, Proposition 2.1], the identity in (2.17) follows directly from (2.12) and (2.15), while the identities in (2.18) follow directly from (2.12) and (2.16). $\square$

TABLE 1. The smoothness exponents of the quincunx low-pass filters $a^{2D}_{2n,2n}$ in (2.10) and of interpolatory 2-wavelet filters a^I_{2n} in (2.11) for $n = 1, \dots, 10$, computed by [20, Algorithm 2.1]. Note that $\mathrm{sm}(a^{2D}_{2n,2n}, N_{\sqrt{2}}) = \mathrm{sm}(a^{2D}_{2n,2n}, M_{\sqrt{2}})$.

n	1	2	3	4	5	6	7	8	9	10
$\mathrm{sm}(a^{2D}_{2n,2n}, M_{\sqrt{2}})$	2.0	3.037	3.546	4.027	4.497	4.966	5.435	5.904	6.371	6.837
$\mathrm{sm}(a^I_{2n}, 2)$	1.5	2.441	3.175	3.793	4.344	4.862	5.363	5.853	6.335	6.812

The smoothness exponents $\mathrm{sm}(a^{2D}_{2n,2n}, M_{\sqrt{2}})$ and $\mathrm{sm}(a^I_{2n}, 2)$ for $n = 1, \dots, 10$ in Table 1 are calculated by [20, Algorithm 2.1] using D_4 symmetry group. Note that $\mathrm{sm}(a, N_{\sqrt{2}}) = \mathrm{sm}(a, M_{\sqrt{2}})$ since a is D_4-symmetric.

We complete this section by presenting two examples to illustrate the results in Theorems 2.1 and 2.2.

Example 2.1. Take $n = 1$ in Theorems 2.1 and 2.2. Then $a = a^{2D}_{2,2}$ in (2.10) with $n = 1$ is given by

$$\widehat{a}(\omega_1, \omega_2) = \frac{1}{4}(1 + e^{-i\omega_1})(1 + e^{-i\omega_2})$$

and

$$\widehat{b_1}(\boldsymbol{\omega}) := e^{-i\omega_1}\overline{\widehat{a}(\boldsymbol{\omega} + (\pi, \pi))} = \frac{1}{4}(1 - e^{-i\omega_1})(e^{i\omega_2} - 1).$$

By $\widehat{a^D_1}(\omega) := \frac{1}{2}(1 + e^{-i\omega})$, we have $\widehat{v}(\omega) := 2\widehat{a^D_1}(\omega/2)\widehat{a^D_1}(\omega/2 + \pi) = \frac{1}{2}(1 - e^{-i\omega})$. Then

$$\widehat{b_2}(\omega_1, \omega_2) := \frac{1}{2}(\widehat{v}(\omega_1 + \omega_2) + \widehat{v}(\omega_1 - \omega_2)e^{-i\omega_2}) = \frac{1}{4}(1 - e^{-i\omega_1})(1 + e^{-i\omega_2})$$

and

$$\widehat{b_3}(\boldsymbol{\omega}) := e^{-i\omega_1}\overline{\widehat{b_2}(\boldsymbol{\omega} + (\pi, \pi))} = \frac{1}{4}(1 + e^{-i\omega_1})(1 - e^{i\omega_2}).$$

The double canonical quincunx tight framelet filter bank $\{a; b_1, b_2, b_3\}$ is given by

$$a = \frac{1}{4}\begin{bmatrix} 1 & 1 \\ \boxed{1} & 1 \end{bmatrix}_{[0,1]^2}, \qquad b_1 = \frac{1}{4}\begin{bmatrix} \boxed{-1} & 1 \\ 1 & -1 \end{bmatrix}_{[0,1]\times[-1,0]},$$
$$b_2 = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ \boxed{1} & -1 \end{bmatrix}_{[0,1]^2}, \quad b_3 = \frac{1}{4}\begin{bmatrix} \boxed{1} & 1 \\ -1 & -1 \end{bmatrix}_{[0,1]\times[-1,0]}.$$

Note that $\mathrm{sr}(a, M_{\sqrt{2}}) = 2$, $\mathrm{lpm}(a) = 2$, and $\mathrm{sm}(a, M_{\sqrt{2}}) = \mathrm{sm}(a, N_{\sqrt{2}}) = 2$. The filter a is D_4-symmetric about $(\frac{1}{2}, \frac{1}{2})$, while b_1 has the symmetry property in (2.15) and b_2, b_3 have the symmetry property in (2.16) with $\mathrm{vm}(b_1) = 2$ and $\mathrm{vm}(b_2) = \mathrm{vm}(b_3) = 1$. Let $\phi, \psi_1, \psi_2, \psi_3$ be defined as in (1.4) with $M = M_{\sqrt{2}}$, $L = 3$ and $a = a^{2D}_{2,2}$. Then $\{\phi; \psi_1, \psi_2, \psi_3\}$ is a tight $M_{\sqrt{2}}$-framelet in $L_2(\mathbb{R}^2)$ such that $\phi, \psi_1, \psi_2, \psi_3$ have symmetry property as in (2.12), (2.17), and (2.18).

Example 2.2. Take $n=2$ in Theorems 2.1 and 2.2. Then $a=a^{2D}_{4,4}$ in (2.10) with $n=2$ is given by

$$\widehat{a}(\omega_1,\omega_2)=\frac{1}{32}\Big(9+9e^{-i\omega_1}+9e^{-i\omega_2}+9e^{-i(\omega_1+\omega_2)}-e^{i(\omega_1+\omega_2)}-e^{i(\omega_2-2\omega_1)}\\ -e^{i(\omega_1-2\omega_2)}-e^{-i2(\omega_1+\omega_2)}\Big).$$

By $\widehat{b_1}(\boldsymbol{\omega}):=e^{-i\omega_1}\overline{\widehat{a}(\boldsymbol{\omega}+(\pi,\pi))}$, the filters a and b_1 are given by

$$a=\frac{1}{32}\begin{bmatrix}-1&0&0&-1\\0&9&9&0\\0&\boxed{9}&9&0\\-1&0&0&-1\end{bmatrix}_{[-1,2]^2},\quad b_1=\frac{1}{32}\begin{bmatrix}1&0&0&-1\\0&\boxed{-9}&9&0\\0&9&-9&0\\-1&0&0&1\end{bmatrix}_{[-1,2]\times[-2,1]}.$$

Let a^D_2 be the Daubechies orthogonal 2-wavelet filter given by

$$\widehat{a^D_2}(\omega)=\frac{1}{8}\Big((1-\sqrt{3})e^{i\omega}+(3-\sqrt{3})+(3+\sqrt{3})e^{-i\omega}+(1+\sqrt{3})e^{-i2\omega}\Big). \tag{2.22}$$

Define $\widehat{v}(\omega):=2\widehat{a^D_2}(\omega/2)\widehat{a^D_2}(\omega/2+\pi)$. Then $\widehat{b_2}(\omega_1,\omega_2):=\frac{1}{2}(\widehat{v}(\omega_1+\omega_2)+\widehat{v}(\omega_1-\omega_2)e^{-i\omega_2})$ is given by

$$b_2=\frac{1}{32}\begin{bmatrix}\sqrt{3}-2&0&0&2+\sqrt{3}\\0&-\sqrt{3}+6&-\sqrt{3}-6&0\\0&\boxed{-\sqrt{3}+6}&-\sqrt{3}-6&0\\\sqrt{3}-2&0&0&2+\sqrt{3}\end{bmatrix}_{[-1,2]^2}.$$

By $\widehat{b_3}(\boldsymbol{\omega}):=e^{-i\omega_1}\overline{\widehat{b_2}(\boldsymbol{\omega}+(\pi,\pi))}$, the filter b_3 is given by

$$b_3=\frac{1}{32}\begin{bmatrix}-2-\sqrt{3}&0&0&\sqrt{3}-2\\0&\sqrt{3}+6&\boxed{-\sqrt{3}+6}&0\\0&-\sqrt{3}-6&\sqrt{3}-6&0\\2+\sqrt{3}&0&0&2-\sqrt{3}\end{bmatrix}_{[-1,2]\times[-2,1]}.$$

Note that $\mathrm{sr}(a,M_{\sqrt{2}})=4$, $\mathrm{lpm}(a)=4$, and $\mathrm{sm}(a,M_{\sqrt{2}})=\mathrm{sm}(a,N_{\sqrt{2}})\approx 3.03654$. Hence $\phi^{M_{\sqrt{2}}},\phi^{N_{\sqrt{2}}}\in C^2(\mathbb{R}^2)$. The filter a is D_4-symmetric about $(1/2,1/2)$, while b_1 has the symmetry property in (2.15) and b_2,b_3 have the symmetry property in (2.16) with $\mathrm{vm}(b_1)=4$ and $\mathrm{vm}(b_2)=\mathrm{vm}(b_3)=2$. The filter bank $\{a;b_1,b_2,b_3\}$ is a double canonical quincunx tight frame filter bank. Let $\phi,\psi_1,\psi_2,\psi_3$ be defined in (1.4) with $M=M_{\sqrt{2}}, L=3$ and $a=a^{2D}_{4,4}$. Then $\{\phi;\psi_1,\psi_2,\psi_3\}$ is a tight $M_{\sqrt{2}}$-framelet in $L_2(\mathbb{R}^2)$ such that all $\phi,\psi_1,\psi_2,\psi_3$ have symmetry property as in (2.12), (2.17), and (2.18).

3. Double canonical quincunx tight framelets with symmetry property derived from one-dimensional filters

Motivated by the special form in (2.10) for the two-dimensional quincunx low-pass filters $a^{2D}_{2n,2n}$, we now further generalize the construction and results in Section 2 for building double canonical quincunx tight framelets with symmetry property from one-dimensional filters.

Theorem 3.1. *Let $u \in l_0(\mathbb{Z})$ be a one-dimensional finitely supported filter with $\widehat{u}(0) = 1$. Define a two-dimensional filter a^{2D} by*

$$\widehat{a^{2D}}(\omega_1, \omega_2) = \frac{1}{2}[\widehat{u}(\omega_1 + \omega_2) + \widehat{u}(\omega_1 - \omega_2)e^{-i\omega_2}]. \tag{3.1}$$

Then

(i) *a^{2D} has order n sum rules with respect to $M_{\sqrt{2}}$ if and only if u has n linear-phase moments with phase $1/2$, i.e.,*

$$\widehat{u}(\omega) = e^{-i\omega/2} + \mathcal{O}(|\omega|^n), \qquad \omega \to 0. \tag{3.2}$$

(ii) *a^{2D} has order n linear-phase moments with phase $(1/2, 1/2)$ if and only if u has n linear-phase moments with phase $1/2$, i.e., (3.2) holds.*

(iii) *a^{2D} is D_4-symmetric about the point $(1/2, 1/2)$ if and only if u is symmetric about the point $1/2$, that is, $u(1-k) = u(k)$ for all $k \in \mathbb{Z}$.*

Proof. The claim in item (iii) can be directly checked. We now prove items (i) and (ii). If (3.2) holds, then

$$\begin{aligned}\widehat{a^{2D}}(\boldsymbol{\omega} + (\pi, \pi)) &= \frac{1}{2}(\widehat{u}(\omega_1 + \omega_2) - \widehat{u}(\omega_1 - \omega_2)e^{-i\omega_2}) \\ &= \frac{1}{2}(e^{-i(\omega_1+\omega_2)/2} - e^{-i(\omega_1-\omega_2)/2}e^{-i\omega_2}) + \mathcal{O}(\|\boldsymbol{\omega}\|^n) \\ &= \mathcal{O}(\|\boldsymbol{\omega}\|^n)\end{aligned}$$

as $\boldsymbol{\omega} \to 0$. Hence, (3.2) implies that a^{2D} has order n sum rules with respect to $M_{\sqrt{2}}$.

Conversely, suppose that a^{2D} has order n sum rules with respect to $M_{\sqrt{2}}$. Then

$$\frac{1}{2}(\widehat{u}(\omega_1 + \omega_2) - \widehat{u}(\omega_1 - \omega_2)e^{-i\omega_2}) = \widehat{a^{2D}}(\boldsymbol{\omega} + (\pi, \pi)) = \mathcal{O}(\|\boldsymbol{\omega}\|^n), \qquad \boldsymbol{\omega} \to 0,$$

from which we deduce that

$$\widehat{v}(\omega_1 + \omega_2) = \widehat{v}(\omega_1 - \omega_2) + \mathcal{O}(\|\boldsymbol{\omega}\|^n), \quad \boldsymbol{\omega} \to 0 \quad \text{with} \quad \widehat{v}(\omega) := \widehat{u}(\omega)e^{i\omega/2}. \tag{3.3}$$

By $\widehat{u}(0) = 1$, we have $\widehat{v}(0) = 1$. Now (3.3) implies

$$\widehat{v}^{(j)}(0) = \partial_2^j[\widehat{v}(\omega_1 + \omega_2)]|_{\omega_1=0,\omega_2=0} = \partial_2^j[\widehat{v}(\omega_1 - \omega_2)]|_{\omega_1=0,\omega_2=0} = (-1)^j \widehat{v}^{(j)}(0),$$

for all $0 \le j \le n-1$ and

$$\begin{aligned}\widehat{v}^{(j+1)}(0) &= \partial_1\partial_2^j[\widehat{v}(\omega_1 + \omega_2)]|_{\omega_1=0,\omega_2=0} \\ &= \partial_1\partial_2^j[\widehat{v}(\omega_1 - \omega_2)]|_{\omega_1=0,\omega_2=0} \\ &= (-1)^j \widehat{v}^{(j+1)}(0)\end{aligned}$$

for all $0 \le j \le n-2$. From the above identities, it is easy to deduce that we must have $\widehat{v}(0) = 1$ and $\widehat{v}^{(j)}(0) = 0$ for all $j = 1, \dots, n-1$. That is, $\widehat{v}(\omega) = 1 + \mathcal{O}(|\omega|^n)$ as $\omega \to 0$. Consequently, by $\widehat{v}(\omega) = \widehat{u}(\omega)e^{i\omega/2}$, (3.2) must hold. This proves item (i).

Similarly, if (3.2) holds, then

$$\begin{aligned}\widehat{a^{2D}}(\boldsymbol{\omega}) &= \frac{1}{2}(\widehat{u}(\omega_1 + \omega_2) + \widehat{u}(\omega_1 - \omega_2)e^{-i\omega_2}) \\ &= \frac{1}{2}(e^{-i(\omega_1+\omega_2)/2} + e^{-i(\omega_1-\omega_2)/2}e^{-i\omega_2}) + \mathcal{O}(\|\boldsymbol{\omega}\|^n) \\ &= e^{-i(\omega_1+\omega_2)/2} + \mathcal{O}(\|\boldsymbol{\omega}\|^n)\end{aligned}$$

as $\boldsymbol{\omega} \to 0$. Hence, (3.2) implies that a^{2D} has order n linear-phase moments with phase $(1/2, 1/2)$. Conversely, if a^{2D} has order n linear-phase moments with phase $(1/2, 1/2)$, then we must have

$$\widehat{v}(\omega_1 + \omega_2) = -\widehat{v}(\omega_1 - \omega_2) + \mathcal{O}(\|\boldsymbol{\omega}\|^n), \quad \boldsymbol{\omega} \to 0 \quad \text{with} \quad \widehat{v}(\omega) = \widehat{u}(\omega) e^{i\omega/2}.$$

A similar proof as in the proof of item (i) shows that (3.2) must hold. This proves item (ii). $\square$

For the filter u in Theorem 3.1, we also have the following result.

Proposition 3.1. *For a finitely supported filter $u \in l_0(\mathbb{Z})$ with $\widehat{u}(0) = 1$ such that u is symmetric about the point $1/2$, $\operatorname{lpm}(u)$ must be an even integer. Moreover, the filter u is symmetric about the point $1/2$ and u has $2n$ linear-phase moments with phase $1/2$ if and only if u takes the form*

$$\widehat{u}(\omega) = \frac{1 + e^{-i\omega}}{2} \Big(\sin^{2n}(\omega/2) R(\sin^2(\omega/2)) + 1 + \sum_{j=1}^{n-1} \frac{(2j-1)!!}{(2j)!!} \sin^{2j}(\omega/2) \Big) \tag{3.4}$$

for some polynomial R, where $(2j-1)!! = (2j-1)(2j-3)\cdots(3)(1)$ and $(2j)!! = (2j)(2j-2)\cdots(2)$. In particular, the two-dimensional filter a^{2D} defined in (3.1) using the filter u in (3.4) with $R = 0$ is the same filter $a^{2D}_{2n,2n}$ in (2.10).

Proof. Note that u is symmetric about the point $\frac{1}{2}$ if and only if $\widehat{u}(\omega) = e^{-i\omega}\widehat{u}(-\omega)$, that is, $e^{i\omega/2}\widehat{u}(\omega) = e^{-i\omega/2}\widehat{u}(-\omega)$. Moreover, the symmetry of u also implies that $\sum_{k\in\mathbb{Z}} u(k)k = 1/2$. Thus, it is trivial to see that $[e^{i\omega/2}\widehat{u}(\omega)]^{(2j-1)}(0) = 0$ for all $j \in \mathbb{N}$. Consequently, by the definition of linear-phase moments with phase $1/2$, $\operatorname{lpm}(u)$ must be an even integer.

Since u is symmetric about $1/2$, we must have $\widehat{u}(\omega) = 2^{-1}(1+e^{-i\omega})P(\sin^2(\omega/2))$ for some polynomial P. Therefore, $e^{i\omega/2}\widehat{u}(\omega) = \cos(\omega/2)P(\sin^2(\omega/2))$. Now u has order $2n$ linear-phase moments with phase $1/2$ if and only if

$$\cos(\omega/2)P(\sin^2(\omega/2)) = e^{i\omega/2}\widehat{u}(\omega) = 1 + \mathcal{O}(|\omega|^{2n}), \quad \omega \to 0,$$

which, by considering $x = \sin^2(\omega/2)$, is further equivalent to $P(x) = (1-x)^{-1/2} + \mathcal{O}(x^n)$ as $x \to 0$. Considering the Taylor expansion of $(1-x)^{-1/2}$ at $x = 0$, we must have

$$P(x) = x^n R(x) + \sum_{j=0}^{n-1} \binom{-1/2}{j} (-x)^j = x^n R(x) + 1 + \sum_{j=1}^{n-1} \frac{(2j-1)!!}{(2j)!!} x^j,$$

for some polynomial R.

When $R = 0$, the filter u in (3.4) is supported inside $[1-n, n]$. Define $\widehat{v}(\omega) := (\widehat{a^I_{2n}}(\omega/2) - \widehat{a^I_{2n}}(\omega/2 + \pi))e^{-i\omega/2}$. Since a^I_{2n} is an interpolatory 2-wavelet filter, it is trivial to see that $\widehat{v}(\omega) = (1 - 2\widehat{a^I_{2n}}(\omega/2 + \pi))e^{-i\omega/2}$. By $\operatorname{sr}(a^I_{2n}, 2) = 2n$, it is trivial to see that

$$\widehat{v}(\omega) = e^{-i\omega/2} + \mathcal{O}(|\omega|^{2n}), \qquad \omega \to 0.$$

That is, $\operatorname{lpm}(v) \geq 2n$. Since a^I_{2n} is supported inside $[1-2n, 2n-1]$, we deduce that v is supported inside $[1-n, n]$. By the uniqueness of u, we must have $v = u$. This proves $a^{2D} = a^{2D}_{2n,2n}$ in (2.10). $\square$

We now construct double canonical quincunx tight framelet filter banks from the low-pass filters in (3.1).

Theorem 3.2. *Let $u \in l_0(\mathbb{Z})$ be a finitely supported filter such that*

$$|\widehat{u}(\omega)| \le 1, \qquad \omega \in \mathbb{R}. \tag{3.5}$$

Define a^{2D} as in (3.1), $\widehat{b_1}(\boldsymbol{\omega}) := e^{-i\omega_1}\overline{\widehat{a^{2D}}(\boldsymbol{\omega}+(\pi,\pi))}$, *and*

$$\widehat{b_2}(\boldsymbol{\omega}) := \frac{1}{2}[\widehat{v}(\omega_1+\omega_2)+\widehat{v}(\omega_1-\omega_2)e^{-i\omega_2}], \quad \widehat{b_3}(\boldsymbol{\omega}) := e^{-i\omega_1}\overline{\widehat{b_2}(\boldsymbol{\omega}+(\pi,\pi))}, \tag{3.6}$$

where $v \in l_0(\mathbb{Z})$ is a filter obtained from Fejér-Riesz lemma satisfying $|\widehat{v}(\omega)|^2 = 1-|\widehat{u}(\omega)|^2$. Then $\{a^{2D}; b_1, b_2, b_3\}$ is a double canonical quincunx tight framelet filter bank.

Proof. By the definitions of $a = a^{2D}$ in (3.1) and b_2 in (3.6), as proved in the proof of Theorem 2.2, (2.19) and (2.20) must hold. Since $|\widehat{u}(\omega)|^2+|\widehat{v}(\omega)|^2 = 1$, it is trivial to see that (1.18) holds with $s = 2$. Hence, $\{a^{2D}; b_1, b_2, b_3\}$ is a double canonical quincunx tight framelet filter bank. □

By the same proof of Theorem 3.2, we have the following generalized result of Theorem 3.2.

Theorem 3.3. *Let $u, v \in l_0(\mathbb{Z})$ be finitely supported filters such that*

$$|\widehat{u}(\omega)|^2 + |\widehat{v}(\omega)|^2 = 1, \qquad \omega \in \mathbb{R}. \tag{3.7}$$

Let M be a $d \times d$ integer matrix such that $|\det(M)| = 2$. Define

$$\widehat{a^{dD}}(\boldsymbol{\omega}) := \frac{1}{2}[\widehat{u}(\gamma_1\cdot\boldsymbol{\omega})+\widehat{u}(\gamma_2\cdot\boldsymbol{\omega})e^{-i\gamma_3\cdot\boldsymbol{\omega}}], \quad \widehat{b_1}(\boldsymbol{\omega}) := e^{-i\gamma_4\cdot\boldsymbol{\omega}}\overline{\widehat{a^{dD}}(\boldsymbol{\omega}+2\pi\xi)}$$

and

$$\widehat{b_2}(\boldsymbol{\omega}) := \frac{1}{2}[\widehat{v}(\gamma_1\cdot\boldsymbol{\omega})+\widehat{v}(\gamma_2\cdot\boldsymbol{\omega})e^{-i\gamma_3\cdot\boldsymbol{\omega}}], \quad \widehat{b_3}(\boldsymbol{\omega}) := e^{-i\gamma_4\cdot\boldsymbol{\omega}}\overline{\widehat{b_2}(\boldsymbol{\omega}+2\pi\xi)}, \tag{3.8}$$

where $\boldsymbol{\omega} \in \mathbb{R}^d$, $\xi \in \Omega_M\backslash\{0\}$, $\gamma_1, \gamma_2 \in M\mathbb{Z}^d\backslash\{0\}$, and $\gamma_3, \gamma_4 \in \mathbb{Z}^d\backslash[M\mathbb{Z}^d]$. Then $\{a^{dD}; b_1, b_2, b_3\}$ is a double canonical tight M-framelet filter bank.

For a real-valued symmetric filter u satisfying

$$u(1-k) = u(k), \qquad \text{for all } k \in \mathbb{Z}, \tag{3.9}$$

it is of interest to ask whether there exists a finitely supported real-valued filter v satisfying (3.7) with certain symmetry property so that the constructed high-pass filters b_2 and b_3 in Theorem 3.2 will have better symmetry property as in Example 2.1, where better symmetry property here means a larger group of integer matrices used in the definition of G-symmetric filters in (1.8) or G-antisymmetric filters in (1.9). This is negatively answered by the following result.

Theorem 3.4. *Let $u, v \in l_0(\mathbb{Z})$ be two finitely supported real-valued filters. Then* (3.7) *and* (3.9) *hold, $\sum_{k\in\mathbb{Z}} u(k) = 1$, and v has some symmetry property (i.e., v is either symmetric or antisymmetric) if and only if*

$$\widehat{u}(\omega) = 2^{-1}(e^{ij\omega}+e^{-i(j+1)\omega}) \quad \text{and} \quad \widehat{v}(\omega) = 2^{-1}e^{-ik\omega}(e^{ij\omega}-e^{-i(j+1)\omega}) \tag{3.10}$$

for some $j, k \in \mathbb{Z}$.

Proof. The sufficient part is trivial, since (3.10) implies (3.7) and v is antisymmetric.

We now prove the necessity part. Since u has the symmetry property in (3.9), we can write $\widehat{u}(\omega) = 2^{-1}(1+e^{-i\omega})P(\sin^2(\omega/2))$ for some polynomial P with real coefficients. Since $\widehat{u}(0) = 1$, we must have $P(0) = 1$. Consequently, we have $|\widehat{u}(\omega)|^2 = \cos^2(\omega/2)(P(\sin^2(\omega/2)))^2 = (1-x)(P(x))^2$ with $x := \sin^2(\omega/2)$.

Since v has some symmetry property and there are essentially four different types of symmetries, we must have

$$
\begin{aligned}
&\widehat{v}(\omega) = e^{-ik\omega} Q(\sin^2(\omega/2)),\\
&\widehat{v}(\omega) = e^{-ik\omega} 2^{-1}(1+e^{-i\omega}) Q(\sin^2(\omega/2)),\\
&\widehat{v}(\omega) = e^{-ik\omega} 2^{-1}(e^{i\omega}-e^{-i\omega}) Q(\sin^2(\omega/2))
\end{aligned}
\tag{3.11}
$$

or

$$
\widehat{v}(\omega) = e^{-ik\omega} 2^{-1}(1-e^{-i\omega}) Q(\sin^2(\omega/2))
\tag{3.12}
$$

for some $k \in \mathbb{Z}$ and some polynomial Q with real coefficients. We now show that v must have the symmetry property in (3.12). Otherwise, v must take one of the three forms in (3.11). Then $|\widehat{v}(\omega)|^2 = (Q(x))^2$, $(1-x)(Q(x))^2$, or $x(1-x)(Q(x))^2$, respectively. Now by $|\widehat{u}(\omega)|^2 + |\widehat{v}(\omega)|^2 = 1$, we will have $(1-x)(P(x))^2 + (Q(x))^2 = 1$, $(1-x)(P(x))^2 + (1-x)(Q(x))^2 = 1$, or $(1-x)(P(x))^2 + x(1-x)(Q(x))^2 = 1$. The last two identities cannot hold due to the factor $1-x$, while the first identity must fail by considering $x \to -\infty$ and noting $P \not\equiv 0$. Thus, v must have the symmetry property in (3.12).

By (3.9) and (3.12), we see that both $e^{i\omega/2}\widehat{u}(\omega)$ and $ie^{i(k+1/2)\omega}\widehat{v}(\omega)$ are real-valued. Therefore,

$$
\begin{aligned}
&e^{i\omega}\left[\widehat{u}(\omega) + e^{ik\omega}\widehat{v}(\omega)\right]\left[\widehat{u}(\omega) - e^{ik\omega}\widehat{v}(\omega)\right]\\
&= \left[e^{i\omega/2}\widehat{u}(\omega) + e^{i(k+1/2)\omega}\widehat{v}(\omega)\right]\left[e^{i\omega/2}\widehat{u}(\omega) - e^{i(k+1/2)\omega}\widehat{v}(\omega)\right]\\
&= |\widehat{u}(\omega)|^2 + |\widehat{v}(\omega)|^2 = 1.
\end{aligned}
$$

Hence, the first two nontrivial factors in the above identities must be monomial, that is,

$$
\widehat{u}(\omega) + e^{ik\omega}\widehat{v}(\omega) = \lambda e^{ij\omega}, \quad \widehat{u}(\omega) - e^{ik\omega}\widehat{v}(\omega) = e^{-i(j+1)\omega}/\lambda
$$

for some $j \in \mathbb{Z}$ and $\lambda \in \mathbb{R}\backslash\{0\}$. From the above identities, we have $\widehat{u}(\omega) = [\lambda e^{ij\omega} + e^{-i(j+1)\omega}/\lambda]/2$. By (3.9), we must have $\lambda = 1$ and (3.10) holds. $\square$

By Theorem 3.4, we can conclude that except for the Haar-type double canonical quincunx tight framelet filter bank that is similar to Example 2.1, there is no other real-valued double canonical quincunx tight framelet filter bank with better symmetry property. Moreover, due to Proposition 3.1, it is quite easy to observe that the real-valued low-pass filter u constructed in (3.10) can have no more than two linear-phase moments and therefore, the tight framelet filter banks constructed in Theorem 3.2 can have no more than one vanishing moment. This shortcoming can be easily remedied by using complex-valued filters. As shown in [25, Theorem 1 and Algorithm 2], there are finitely supported complex-valued low-pass orthogonal 2-wavelet filters a such that $a(1-k) = a(k)$ for all $k \in \mathbb{Z}$ with arbitrarily high orders of sum rules and linear-phase moments. Take $u = a$. Then we can easily obtain complex-valued double canonical quincunx tight framelet filter banks with symmetry property and arbitrarily high orders of vanishing moments. For the convenience of the reader, we provide an example here by combining [25, Algorithm 1] and Proposition 3.1.

Note that $M_{\sqrt{2}}$ is not compatible with the symmetry group

$$D_4^+ := \{\pm\mathrm{diag}(1,1), \pm\mathrm{diag}(1,-1)\}.$$

But we have

$$M_{\sqrt{2}} D_4^+ M_{\sqrt{2}}^{-1} := \left\{ M_{\sqrt{2}} E M_{\sqrt{2}}^{-1} : E \in D_4^+ \right\} = \left\{ \pm \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \pm \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\} =: D_4^-$$

and $M_{\sqrt{2}} D_4^- M_{\sqrt{2}}^{-1} = D_4^+$.

Example 3.1. Take $n = 3$ and $R = 0$ in (3.4) of Proposition 3.1. Then $P(x) = 1 + \frac{1}{2}x + \frac{3}{8}x^2$ and

$$\widetilde{Q}(x) := \frac{1-(1-x)(P(x))^2}{x} = \frac{x^2(9x^2+15x+40)}{64} \geq 0, \qquad \forall\, x \in \mathbb{R}.$$

Then $Q(x) := \frac{3}{8}x(x + \frac{5+i3\sqrt{15}}{6})$ satisfies $|Q(x)|^2 = \widetilde{Q}(x)$ for all $x \in \mathbb{R}$. Define filters u and v by

$$\widehat{u}(\omega) = 2^{-1}(1+e^{-i\omega})P(\sin^2(\omega/2)), \qquad \widehat{v}(\omega) = 2^{-1}(1-e^{-i\omega})Q(\sin^2(\omega/2)).$$

Then $\mathrm{lpm}(u) = 6$ with phase $c = 1/2$,

$$\widehat{u}(\omega) = \frac{1}{256}(150(1+e^{-i\omega}) - 25(e^{i\omega}+e^{-2i\omega}) + 3(e^{2i\omega}+e^{-3i\omega}))$$

and

$$\widehat{v}(\omega) = \frac{1}{256}((60+i18\sqrt{15})(1-e^{-i\omega}) - (25+i6\sqrt{15})(e^{i\omega}-e^{-2i\omega}) + 3(e^{2i\omega}-e^{-3i\omega})).$$

The filters u and v satisfy $u(k) = u(1-k)$ and $v(k) = -v(1-k)$ for $k \in \mathbb{Z}$; that is, u is symmetric about $1/2$ while v is antisymmetric about $1/2$. Note that the real-valued filter v in Theorem 2.2 defined by $\widehat{v}(\omega) = 2\widehat{a_n^D}(\omega/2)\widehat{a_n^D}(\omega/2+\pi)$ does not have symmetry property.

Define $a = a^{2D}$ as in (3.1). Then, the filter a satisfies $a = a^{2D} = a^{2D}_{6,6}$ in (2.10) due to $P(x) = (1-x)^{-1/2} + \mathcal{O}(x^4)$ and a is supported on $[-2,3]^2$. The canonical high-pass filter b_1 of a is given by $\widehat{b_1}(\boldsymbol{\omega}) = e^{-i\omega_1}\overline{\widehat{a}(\boldsymbol{\omega}+(\pi,\pi))}$. The filters a and b_1 are given by

$$a = \frac{1}{512}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 3 \\ 0 & -25 & 0 & 0 & -25 & 0 \\ 0 & 0 & 150 & 150 & 0 & 0 \\ 0 & 0 & \boxed{150} & 150 & 0 & 0 \\ 0 & -25 & 0 & 0 & -25 & 0 \\ 3 & 0 & 0 & 0 & 0 & 3 \end{bmatrix}_{[-2,3]^2},$$

$$b_1 = \frac{1}{512}\begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 3 \\ 0 & 25 & 0 & 0 & -25 & 0 \\ 0 & 0 & \boxed{-150} & 150 & 0 & 0 \\ 0 & 0 & 150 & -150 & 0 & 0 \\ 0 & -25 & 0 & 0 & 25 & 0 \\ 3 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}_{[-2,3]\times[-3,2]}.$$

Note that $a = a^{2D}_{6,6}$ is real-valued and D_4-symmetric about $\mathbf{c} = (1/2, 1/2)$ while b_1 has the symmetry property given by (2.15). Define high-pass filters b_2 and b_3 by

(3.6). Then, the high-pass filter b_2 is supported on $[-2,3]^2$ and is given by

$$b_2 = \frac{1}{512}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & -3 \\ 0 & -25-6\,i\sqrt{15} & 0 & 0 & 25+6\,i\sqrt{15} & 0 \\ 0 & 0 & 60+18\,i\sqrt{15} & -60-18\,i\sqrt{15} & 0 & 0 \\ 0 & 0 & \boxed{60+18\,i\sqrt{15}} & -60-18\,i\sqrt{15} & 0 & 0 \\ 0 & -25-6\,i\sqrt{15} & 0 & 0 & 6\,25+i\sqrt{15} & 0 \\ 3 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}_{[-2,3]^2}.$$

The canonical high-pass filter b_3 of b_2 is supported on $[-2,3]\times[-3,2]$ and is given by

$$b_3 = \frac{1}{512}\begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 3 \\ 0 & 6\,i\sqrt{15}-25 & 0 & 0 & 6\,i\sqrt{15}-25 & 0 \\ 0 & 0 & \boxed{60-18\,i\sqrt{15}} & 60-18\,i\sqrt{15} & 0 & 0 \\ 0 & 0 & -60+18\,i\sqrt{15} & -60+18\,i\sqrt{15} & 0 & 0 \\ 0 & 25-6\,i\sqrt{15} & 0 & 0 & 25-6\,i\sqrt{15} & 0 \\ -3 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}_{[-2,3]\times[-3,2]}.$$

The high-pass filters b_2 and b_3 are complex-valued and have the following symmetry properties:

$$b_2(E(k-\mathbf{c})+\mathbf{c}) = E_{1,1}b_2(k), \quad \forall k\in\mathbb{Z}^2, E\in D_4^+ \qquad \text{with} \qquad \mathbf{c}=(1/2,1/2),$$
$$b_3(E(k-\mathring{\mathbf{c}})+\mathring{\mathbf{c}}) = E_{2,2}b_3(k), \quad \forall k\in\mathbb{Z}^2, E\in D_4^+ \qquad \text{with} \qquad \mathring{\mathbf{c}}=(1/2,-1/2),$$

where $E_{i,j}$ is the (i,j)-entry of E.

The filter bank $\{a;b_1,b_2,b_3\}$ is a double canonical quincunx tight framelet filter bank with $\mathrm{vm}(b_1)=6$ and $\mathrm{vm}(b_2)=\mathrm{vm}(b_3)=3$. Let $\phi,\psi_1,\psi_2,\psi_3$ be defined in (1.4) with $M=M_{\sqrt{2}}$, $L=3$ and $a=a^{2D}_{6,6}$. Then $\{\phi;\psi_1,\psi_2,\psi_3\}$ is a tight $M_{\sqrt{2}}$-framelet in $L_2(\mathbb{R}^2)$ such that ϕ,ψ_1 have symmetry property as in (2.12), (2.17). ψ_2,ψ_3 are of complex value and have the following symmetry properties:

$$\begin{aligned}\psi_2(E(\cdot-\mathbf{c}_2)+\mathbf{c}_2) &= [MEM^{-1}]_{1,1}\psi_2,\\ \psi_3(E(\cdot-\mathbf{c}_3)+\mathbf{c}_3) &= [MEM^{-1}]_{2,2}\psi_3,\end{aligned} \qquad \forall E\in D_4^-,$$

where $\mathbf{c}_2=(3/2,1/2)$ and $\mathbf{c}_3=(1,1)$.

4. Multiple canonical quincunx tight framelet filter banks with symmetry proprety

In this section we study multiple canonical quincunx tight framelet filter banks with symmetry property derived from one-dimensional filters.

As discussed in Section 1, for every $d\times d$ dilation matrix M, compactly supported tight M-framelets $\{\phi;\psi_1,\ldots,\psi_L\}$ with arbitrarily high vanishing moments and smoothness can be easily constructed (e.g. [22, Theorem 1.1]) but at the cost of a large number L of wavelet/framelet functions. The key idea to construct such and similar compactly supported tight M-framelets in [17, 19, 22, 30, 46] is to use the almost separable low-pass filters in (1.13). For example, for two one-dimensional tight 2-framelet filter banks $\{b_0;b_1,\ldots,b_J\}$ and $\{u_0;u_1,\ldots,u_L\}$ one can trivially verify (see [22, Lemma 3.2] and [46]) that

$$\{b_j\otimes u_k \ :\ 0\le j\le J, 0\le k\le L\}, \tag{4.1}$$

is a quincunx tight framelet filter bank derived from the separable low-pass filter $b_0 \otimes u_0$, where $\widehat{b_j \otimes u_k}(\omega_1, \omega_2) := \widehat{b_j}(\omega_1)\widehat{u_k}(\omega_2)$. Moreover, every one-dimensional tight 2-framelet filter bank $\{b_0; b_1, \ldots, b_J\}$ is automatically a quincunx tight framelet filter bank by identifying $\mathbb{Z}$ with either $\mathbb{Z} \times \{0\}$ or $\{0\} \times \mathbb{Z}$ so that a one-dimensional filter can be regarded as a two-dimensional filter ([19]). Such tight framelet filter banks are particular instances of the tight framelet filter banks constructed via the projection method in [30]. In fact, one can directly check that if $\{b_0; b_1, \ldots, b_J\}$ is a one-dimensional tight 2-framelet filter bank and if the filters $u_0, u_1, \ldots, u_L$ satisfy

$$|\widehat{u_0}(\omega)|^2 + |\widehat{u_1}(\omega)|^2 + \cdots + |\widehat{u_L}(\omega)|^2 = 1, \tag{4.2}$$

then the filter bank in (4.1) is still a quincunx tight framelet filter bank. Note that (4.2) is weaker than requiring $\{u_0; u_1, \ldots, u_L\}$ to be a tight 2-framelet filter bank. For every pair of finitely supported low-pass filters b_0 and u_0 satisfying $|\widehat{b_0}(\omega)|^2 + |\widehat{b_0}(\omega + \pi)|^2 \le 1$ and $|\widehat{u_0}(\omega)|^2 \le 1$, one can always construct ([9]) a finitely supported tight 2-framelet filter bank $\{b_0; b_1, b_2\}$, and by Fejér-Riesz lemma, there always exists a finitely supported filter u_1 such that (4.2) holds with $L = 1$. Consequently, the quincunx tight framelet filter bank in (4.1) with $J = 2$ and $L = 1$ has only five high-pass filters derived from the given low-pass $b_0 \otimes u_0$, and $\{b_0; b_1, b_2\}$ is a quincunx tight framelet filter bank with only two high-pass filters. However, such quincunx tight framelet filter banks often lack symmetry property and are not necessarily a multiple canonical quincunx tight framelet filter bank. By modifying (4.1) slightly, we next show that multiple canonical quincunx tight frame filter banks can be easily obtained from one-dimensional tight framelet filter banks as long as $\{b_0; b_1, \ldots, b_J\}$ has the multiple canonical property.

Theorem 4.1. *Let s, L be positive integers. Suppose that $\{b_0; b_1, \ldots, b_{2s-1}\}$ is a one-dimensional finitely supported s-multiple canonical tight 2-framelet filter bank having the canonical property: $\widehat{b_{2j+1}}(\omega) = e^{-i\omega}\overline{\widehat{b_{2j}}(\omega + \pi)}$, $j = 0, \ldots, s-1$. Suppose that $u_0, u_1, \ldots, u_L \in l_0(\mathbb{Z})$ are one-dimensional filters satisfying (4.2). Then $\{b^{2D}_{j,k} : j = 0, \ldots, 2s-1; k = 0, \ldots, L\}$ is an $s(L+1)$-multiple canonical quincunx tight framelet filter bank, where*

$$\begin{cases} \widehat{b^{2D}_{2j,k}}(\boldsymbol{\omega}) & := \widehat{b_{2j}}(\omega_1)\widehat{u_k}(\omega_2), \\ \widehat{b^{2D}_{2j+1,k}}(\boldsymbol{\omega}) & := \widehat{b_{2j+1}}(\omega_1)\overline{\widehat{u_k}(\omega_2 + \pi)}, \end{cases} \qquad \boldsymbol{\omega} = (\omega_1, \omega_2) \in \mathbb{R}^2, \tag{4.3}$$

for $j = 0, \ldots, s-1$ and $k = 0, \ldots, L$.

Proof. By the canonical property in (1.16) and (1.17), it follows directly from the definition of $b^{2D}_{j,k}$ that the two-dimensional filter bank $\{b^{2D}_{j,k} : j = 0, \ldots, 2s-1; k = 0, \ldots, L\}$ has the desired $s(L+1)$-multiple canonical property. On the other hand, we have

$$\begin{aligned} \sum_{j=0}^{2s-1}\sum_{k=0}^{L} |\widehat{b^{2D}_{j,k}}(\boldsymbol{\omega})|^2 &= \sum_{j=0}^{s-1}\sum_{k=0}^{L} |\widehat{b^{2D}_{2j,k}}(\boldsymbol{\omega})|^2 + \sum_{j=0}^{s-1}\sum_{k=0}^{L} |\widehat{b^{2D}_{2j+1,k}}(\boldsymbol{\omega})|^2 \\ &= \sum_{j=0}^{s-1} |\widehat{b_{2j}}(\omega_1)|^2 \sum_{k=0}^{L} |\widehat{u_k}(\omega_2)|^2 + \sum_{j=0}^{s-2} |\widehat{b_{2j+1}}(\omega_1)|^2 \sum_{k=0}^{L} |\widehat{u_k}(\omega_2 + \pi)|^2 \\ &= \sum_{\ell=0}^{2s-1} |\widehat{b_\ell}(\omega_1)|^2 = 1. \end{aligned}$$

The fact

$$\sum_{j=0}^{2s-1}\sum_{k=0}^{L}\widehat{b_{j,k}^{2D}}(\boldsymbol{\omega})\overline{\widehat{b_{j,k}^{2D}}(\boldsymbol{\omega}+(\pi,\pi))}=0$$

can be proved similarly. Thus $\{b_{j,k}^{2D} : j=0,\dots,2s-1; k=0,\dots,L\}$ is a quincunx tight framelet filter bank. $\square$

Before applying Theorem 4.1 to construct multiple canonical quincunx tight framelets, let us look at the smoothness exponent of the low-pass filter $b_0 \otimes u_0$ in Theorem 4.1.

Theorem 4.2. *Let $1 \le p \le \infty$. The following statements hold.*

(i) *For $a \in l_0(\mathbb{Z})$ with $\widehat{a}(0)=1$, $\operatorname{sr}(a, M_{\sqrt{2}}) = \operatorname{sr}(a, N_{\sqrt{2}}) = \operatorname{sr}(a,2)$; if $\operatorname{sm}_p(a,2) \ge 0$, then $\operatorname{sm}_p(a, M_{\sqrt{2}}) = \operatorname{sm}_p(a,2)$, where a is also regarded as a 2D filter by identifying $\mathbb{Z}$ with $\mathbb{Z}\times\{0\}$ in $\mathbb{Z}^2$.*

(ii) *For $u, v \in l_0(\mathbb{Z}^d)$ with $\widehat{u}(0)=\widehat{v}(0)=1$ and for any $d\times d$ dilation matrix M, $\operatorname{sr}(u*v, M) \ge \operatorname{sr}(u,M)+\operatorname{sr}(v,M)$ and $\operatorname{sm}(u*v,M) \ge \operatorname{sm}_\infty(u*v,M) \ge \operatorname{sm}(u,M)+\operatorname{sm}(v,M)$, where $\widehat{u*v}(\boldsymbol{\omega}) := \widehat{u}(\boldsymbol{\omega})\widehat{v}(\boldsymbol{\omega})$.*

(iii) *For $u, v \in l_0(\mathbb{Z})$ with $\widehat{u}(0)=\widehat{v}(0)=1$, $\operatorname{sr}(u\otimes v, M_{\sqrt{2}}) = \operatorname{sr}(u\otimes v, N_{\sqrt{2}}) \ge \operatorname{sr}(u,2)+\operatorname{sr}(v,2)$; if $\operatorname{sm}(u,2)\ge 0$ and $\operatorname{sm}(v,2)\ge 0$, then $\operatorname{sm}(u\otimes v, M_{\sqrt{2}}) \ge \operatorname{sm}_\infty(u\otimes v, M_{\sqrt{2}}) \ge \operatorname{sm}(u,2)+\operatorname{sm}(v,2)$.*

The proof to Theorem 4.2 is given in Appendix A. For item (i), $\operatorname{sm}(a, N_{\sqrt{2}}) = \operatorname{sm}(a,2)$ often fails. In fact, as Daubechies showed in [8] that $\lim_{n\to\infty}\operatorname{sm}(a_{2n}^I, 2) = \infty$, while numerical calculation in [7] observed that $\lim_{n\to\infty}\operatorname{sm}(a_{2n}^I, N_{\sqrt{2}}) = 0$. Moreover, as we shall see in the proof of Theorem 4.2 in Appendix A, the condition $\operatorname{sm}_p(a,2)\ge 0$ in item (i) cannot be removed to guarantee $\operatorname{sm}_p(a, M_{\sqrt{2}}) = \operatorname{sm}_p(a,2)$.

Let a_n^D with $n \ge 1$ be the Daubechies orthogonal filter with $2n$-nonzero coefficients. Let $b_0 = a_n^D$, $u_0 = a_m^D$ and define b_1 and u_1 by

$$\widehat{b}_1(\omega) = e^{-i\omega}\overline{\widehat{a_n^D}(\omega+\pi)}, \qquad \widehat{u}_1(\omega) = \widehat{a_m^D}(\omega+\pi),\ \omega\in\mathbb{R},$$

then we have double canonical quincunx tight framelet filter banks based on the Daubechies orthogonal filters as summarized in the following corollary.

Corollary 4.1. *Let a_n^D and a_m^D be the Daubechies orthogonal filters. Define*

$$\widehat{b_0^{2D}}(\boldsymbol{\omega}) := \widehat{a_n^D}(\omega_1)\widehat{a_m^D}(\omega_2), \qquad \widehat{b_1^{2D}}(\boldsymbol{\omega}) := e^{-i\omega_1}\overline{\widehat{b_0^{2D}}(\boldsymbol{\omega}+(\pi,\pi))},$$
$$\widehat{b_2^{2D}}(\boldsymbol{\omega}) := \widehat{a_n^D}(\omega_1)\widehat{a_m^D}(\omega_2+\pi), \qquad \widehat{b_3^{2D}}(\boldsymbol{\omega}) := e^{-i\omega_1}\overline{\widehat{b_2^{2D}}(\boldsymbol{\omega}+(\pi,\pi))}.$$

Then $\{b_0^{2D}; b_1^{2D}, b_2^{2D}, b_3^{2D}\}$ is a double canonical quincunx tight framelet filter bank such that

$$\min(\operatorname{vm}(b_1^{2D}), \operatorname{vm}(b_2^{2D}), \operatorname{vm}(b_3^{2D})) \ge \min(m,n)$$

and

$$\operatorname{sm}(b_0^{2D}, M_{\sqrt{2}}) \ge \operatorname{sm}(a_n^D, 2) + \operatorname{sm}(a_m^D, 2) \to \infty, \quad m+n\to\infty.$$

The Daubechies orthogonal filter-based double canonical quincunx tight framelet filter bank $\{b_0^{2D}; b_1^{2D}, b_2^{2D}, b_3^{2D}\}$ does not have any symmetry property. In this paper we are interested in multiple/double canonical quincunx tight framelet filter banks with symmetry property. We immediately conclude from Theorem 4.1 that all nontrivial real-valued canonical quincunx tight framelet filter banks with symmetry property of the form in (4.3) must have multiplicity at least 6. In fact, if we require both $\{b_0; b_1, \dots, b_{2s-1}\}$ and $\{u_0; u_1, \dots, u_L\}$ in Theorem 4.1 to be of real-valued filters with symmetry property, then $s \ge 2$ and $L \ge 2$ due to the well-known fact that except the Haar-type filter banks, there is no real-valued dyadic orthogonal wavelet filter bank with symmetry property. Consequently, the multiplicity of a nontrivial canonical quincunx tight framelet filter bank with real-valued filters and with symmetry property satisfies $s(L+1) \ge 6$. That is, $\{b_0; b_1, \dots, b_{2s-1}\}$ need to be at least double canonical tight 2-framelet filter bank $\{a, b_1, b_2, b_3\}$ while $\{u_0; u_1, \dots, u_L\}$ need to be at least $\{u_0; u_1, u_2\}$.

We now discuss double canonical tight 2-framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry property satisfying

$$\widehat{b_1}(\omega) = e^{-i\omega}\overline{\widehat{a}(\omega+\pi)}, \qquad \widehat{b_3}(\omega) = e^{-i\omega}\overline{\widehat{b_2}(\omega+\pi)}. \tag{4.4}$$

It follows trivially from the above relations in (4.4) that

$$\overline{\widehat{a}(\omega)}\widehat{a}(\omega+\pi) + \overline{\widehat{b_1}(\omega)}\widehat{b_1}(\omega+\pi) = 0, \qquad \overline{\widehat{b_2}(\omega)}\widehat{b_2}(\omega+\pi) + \overline{\widehat{b_3}(\omega)}\widehat{b_3}(\omega+\pi) = 0.$$

Consequently, every double canonical tight 2-framelet filter bank with symmetry property is a special case of type I tight 2-framelet filter banks $\{a; b_1, b_2, b_3\}$ with symmetry property discussed in [31]. Moreover, Algorithm 1 in [31] can be used to find all possible such type I tight 2-framelet filter banks $\{a; b_1, b_2, b_3\}$ with symmetry property from any given symmetric low-pass filter. For simplicity, we only discuss real-valued filters here. As a special case of [31, Algorithm 1], the following result constructs all possible double canonical tight 2-framelet filter banks with symmetry property.

Theorem 4.3. *Let $a \in l_0(\mathbb{Z})$ be a real-valued low-pass filter which is symmetric and satisfies*

$$\widehat{v}(2\omega) := 1 - |\widehat{a}(\omega)|^2 - |\widehat{a}(\omega+\pi)|^2 \ge 0, \qquad \forall\, \omega \in \mathbb{R}. \tag{4.5}$$

Define a finitely supported real-valued high-pass filter b_2 by either of the following two cases:

(1) *Obtain a real-valued filter $u \in l_0(\mathbb{Z})$ through Fejér-Riesz lemma by $|\widehat{u}(\omega)|^2 = \widehat{v}(\xi)$ and define*

$$\widehat{b_2}(\omega) := (\widehat{u}(2\omega) + \epsilon e^{-i\omega c_b}\overline{\widehat{u}(2\omega)})/2$$

with $\epsilon \in \{-1, 1\}$ and c_b being an odd integer.

(2) *If in addition multiplicity of any zero inside $(0,1)$ of the Laurent polynomial $\sum_{k\in\mathbb{Z}} v(k)z^k$ is even, then one can always construct finitely supported real-valued filters u_1, u_2 with symmetry such that*

$$|\widehat{u_1}(\omega)|^2 + |\widehat{u_2}(\omega)|^2 = \widehat{v}(\xi) \quad \textit{with} \quad \frac{\mathsf{S}\widehat{u_1}(\omega)}{\mathsf{S}\widehat{u_2}(\omega)} = e^{-i\omega},$$

where $\mathsf{S}\widehat{u_1}(\omega) := \frac{\widehat{u_1}(\omega)}{\widehat{u_1}(-\omega)}$ records the symmetry type of the filter u_1. Define

$$\widehat{b_2(\omega)} := (\widehat{u_1}(2\omega) + e^{-i\omega}\widehat{u_2}(\omega))/\sqrt{2}.$$

Define the filters b_1 and b_3 as in (4.4). Then $\{a; b_1, b_2, b_3\}$ is a double canonical tight 2-framelet filter bank such that all the filters have symmetry property (i.e., either symmetric or antisymmetric). Moreover, all finitely supported canonical tight 2-framelet filter banks with symmetry property can be obtained by the above procedure.

The construction of real-value filters $\{u_0; u_1, u_2\}$ satisfying (4.2) and having symmetry property has been completely solved in [29, Theorem 2.7] and [34, Lemma 2.4].

Now we have the main result in this paper on 6-multiple canonical quincunx tight framelet filter banks with symmetry property and vanishing moments.

Theorem 4.4. *Let $a \in l_0(\mathbb{Z})$ be a real-valued low-pass filter satisfying the condition in* (4.5) *such that $\widehat{a}(0) = 1$ and a is symmetric. Then we can always construct by Theorem* 4.3 *a finitely supported real-valued double canonical tight* 2*-framelet filter bank $\{a; b_1, b_2, b_3\}$ with symmetry property and by* [29, Theorem 2.7] *finitely supported real-valued filters u_1 and u_2 with symmetry property such that*

$$|\widehat{a}(\omega)|^2 + |\widehat{u_1}(\omega)|^2 + |\widehat{u_2}(\omega)|^2 = 1. \tag{4.6}$$

Define two-dimensional filters $b^{2D}_{\ell,k}$ as in (4.3) *of Theorem* 4.1 *for $\ell = 0, \ldots, 3$ and $k = 0, 1, 2$ with $b_0 := a$ and $u_0 := a$. Define $a^{2D} := b^{2D}_{0,0}$. Then*

$$\{a^{2D}; b^{2D}_{1,0}, b^{2D}_{2,0}, b^{2D}_{3,0}, b^{2D}_{0,1}, b^{2D}_{1,1}, b^{2D}_{2,1}, b^{2D}_{3,1}, b^{2D}_{0,2}, b^{2D}_{1,2}, b^{2D}_{2,2}, b^{2D}_{3,2}\} \tag{4.7}$$

is a 6*-multiple canonical quincunx tight framelet filter bank such that the real-valued low-pass filter a^{2D} is D_4-symmetric, with*

$$\operatorname{sr}(a^{2D}, M_{\sqrt{2}}) = \operatorname{sr}(a^{2D}, N_{\sqrt{2}}) \ge 2\operatorname{sr}(a, 2),$$
$$\operatorname{sm}(a^{2D}, M_{\sqrt{2}}) = \operatorname{sm}(a^{2D}, N_{\sqrt{2}}) \ge 2\operatorname{sm}(a, 2),$$

and all the eleven high-pass filters are real-valued and have symmetry property with at least order $\min(2\operatorname{sr}(a,2), \operatorname{lpm}(a)/2)$ *vanishing moments. In particular, if we take $a = a^I_{2n}$ with $n \in \mathbb{N}$, then we have a* 6*-multiple canonical quincunx tight framelet filter bank in* (4.7) *such that*

(i) *all the high-pass filters have symmetry property (i.e., either symmetric or antisymmetric) and at least order n vanishing moments;*

(ii) *the low-pass $a^{2D} = a^I_{2n} \otimes a^I_{2n}$ is D_4-symmetric such that*

$$\operatorname{sr}(a^I_{2n} \otimes a^I_{2n}, M_{\sqrt{2}}) = \operatorname{sr}(a^I_{2n} \otimes a^I_{2n}, N_{\sqrt{2}}) \ge 4n$$

and

$$\lim_{n\to\infty} \operatorname{sm}(a^I_{2n} \otimes a^I_{2n}, M_{\sqrt{2}}) = \infty, \quad \lim_{n\to\infty} \operatorname{sm}(a^I_{2n} \otimes a^I_{2n}, N_{\sqrt{2}}) = \infty;$$

(iii) *the tight $M_{\sqrt{2}}$-framelet (or tight $N_{\sqrt{2}}$-framelet) $\{\phi; \psi_1, \ldots, \psi_L\}$ with $L = 11$ in $L_2(\mathbb{R}^2)$ have symmetry property and arbitrarily high orders of vanishing moments and smoothness, where $\phi, \psi_1, \ldots, \psi_L$ is defined in* (1.4).

Proof. Since a is D_4-symmetric, by definition of smoothness exponent, we can directly verify that $\operatorname{sm}_p(a^{2D}, M_{\sqrt{2}}) = \operatorname{sm}_p(a^{2D}, N_{\sqrt{2}})$ for all $1 \le p \le \infty$ (also see Theorem 2.1 and [19, 24]). It is known in [8] that $\lim_{n\to\infty} \operatorname{sm}(a^I_{2n}, 2) = \infty$. By Theorem 4.2, we have $\operatorname{sm}(a^{2D}, M_{\sqrt{2}}) = \operatorname{sm}(a^I_{2n} \otimes a^I_{2n}, M_{\sqrt{2}}) \ge 2\operatorname{sm}(a^I_{2n}, 2)$. Consequently, we have $\lim_{n\to\infty} \operatorname{sm}(a^I_{2n} \otimes a^I_{2n}, M_{\sqrt{2}}) = \infty$. All other claims follow from the results and discussion before Theorem 4.4. ☐

As proved in [25, Theorem 1 and (2.15)], there are finitely supported complex-valued orthogonal 2-wavelet filters with symmetry property and arbitrarily high orders of sum rules. As a consequence, if we relax the constraint on real-valued filters and allow complex-valued filter banks, we can have double canonical quincunx tight framelet filter banks with the symmetry property of form in (4.3).

Corollary 4.2. *For $n \in \mathbb{N}$, let $a_n \in l_0(\mathbb{Z})$ be the finitely supported symmetric complex-valued orthogonal 2-wavelet filter with* $\operatorname{sr}(a_n, 2) = 2n - 1$ *as constructed in* [25, Theorem 1]. *Define*

$$\widehat{a^{2D}}(\omega_1, \omega_2) := \widehat{a_n}(\omega_1)\widehat{a_n}(\omega_2), \qquad \widehat{b_2}(\omega_1, \omega_2) := \widehat{a_n}(\omega_1)\widehat{a_n}(\omega_2 + \pi)$$

and

$$\widehat{b_1}(\omega_1, \omega_2) := e^{-i\omega_1}\overline{\widehat{a^{2D}}(\omega_1 + \pi, \omega_2 + \pi)}, \qquad \widehat{b_3}(\omega_1, \omega_2) := e^{-i\omega_1}\overline{\widehat{b_2}(\omega_1 + \pi, \omega_2 + \pi)}.$$

Then $\{a^{2D}; b_1, b_2, b_3\}$ is a double canonical quincunx tight framelet filter bank such that a^{2D} is D_4-symmetric, with $\operatorname{sr}(a^{2D}, M_{\sqrt{2}}) \geq 2n$ *and*

$$\operatorname{sm}(a^{2D}, M_{\sqrt{2}}) = \operatorname{sm}(a^{2D}, N_{\sqrt{2}}) \geq 2\operatorname{sm}(a_n, 2) \to \infty, \quad \text{as } n \to \infty,$$

and all the high-pass filters b_1, b_2, b_3 have symmetry property and at least order n vanishing moments.

We conclude this section by presenting two examples of 6-multiple canonical quincunx real-valued tight framelet filter banks to illustrate the result in Theorem 4.4.

Example 4.1. Consider $a = a_2^I = \{-\frac{1}{32}, 0, \frac{9}{32}, \boxed{\frac{1}{2}}, \frac{9}{32}, 0, -\frac{1}{32}\}_{[-3,3]}$ with $\operatorname{sr}(a, 2) = 4$ and $\operatorname{lpm}(a) = 4$. Then

$$1 - |\widehat{a}(\omega)|^2 - |\widehat{a}(\omega + \pi)|^2 = -\frac{1}{64}(\cos^3(2x) - 9\cos^2(2x) + 15\cos(2x) - 7) \geq 0.$$

By the Fejér-Riesz lemma, we can obtain $u \in l_0(\mathbb{Z})$ such that $|\widehat{u}(2\omega)|^2 = 1 - |\widehat{a}(\omega)|^2 - |\widehat{a}(\omega + \pi)|^2$ as follows:

$$\widehat{u}(\omega) = \frac{\sqrt{2}}{32}e^{i\omega}(t_0 + t_1 e^{-i\omega} + t_2 e^{-i2\omega} + t_3 e^{-i3\omega}),$$

where $t_0 = 2 - \sqrt{3}$, $t_1 = -6 + \sqrt{3}$, $t_2 = 6 + \sqrt{3}$, $t_3 = -2 - \sqrt{3}$. Define b_1, b_2, b_3 by

$$\begin{cases} \widehat{b_1}(\omega) = e^{-i\omega}\overline{\widehat{a}(\omega + \pi)}, \\ \widehat{b_2}(\omega) = (\widehat{u}(2\omega) + e^{-i\omega}\overline{\widehat{u}(2\omega)})/2, \\ \widehat{b_3}(\omega) = e^{-i\omega}\overline{\widehat{b_2}(\omega + \pi)}. \end{cases} \tag{4.8}$$

Then,

$$\widehat{b_1}(\omega) = e^{-i\omega}\left(\frac{1}{2} - \frac{9}{32}(e^{i\omega} + e^{-i\omega}) + \frac{1}{32}(e^{i3\omega} + e^{-i3\omega})\right),$$

$$\widehat{b_2}(\omega) = \frac{\sqrt{2}}{64}\left(t_3(e^{i3\omega} + e^{-i4\omega}) + t_0(e^{i2\omega} + e^{-i3\omega}) + t_2(e^{i\omega} + e^{-i2\omega}) + t_1(1 + e^{-i\omega})\right),$$

$$\widehat{b_3}(\omega) = -\frac{\sqrt{2}}{64}\left(t_3(e^{i3\omega} - e^{-i4\omega}) - t_0(e^{i2\omega} - e^{-i3\omega}) + t_2(e^{i\omega} - e^{-i2\omega}) - t_1(1 - e^{-i\omega})\right).$$

Note that b_1 is symmetric about 1 and supported on $[-2, 4]$, b_2 is symmetric about $1/2$ and supported on $[-3, 4]$, b_3 is antisymmetric about $1/2$ and supported on $[-3, 4]$. The filter bank $\{a; b_1, b_2, b_3\}$ forms a double canonical tight 2-framelet filter bank. See [31, Examples 4 and 9] for other tight 2-framelet filter banks $\{a; b_1, b_2, b_3\}$ with symmetry property derived from the interpolatory low-pass filter $a = a_2^I$.

Next, define $\widehat{v}(\omega) := 1 - |\widehat{a}(\omega)|^2$. Then,

$$\widehat{v}(\xi) = \Big|\frac{1 - e^{-i\omega}}{2}\Big|^4 \Big|\frac{e^{-i\omega} + 2 - \sqrt{3}}{\sqrt{4 - 2\sqrt{3}}}\Big|^2 \Big(\frac{6 + 3\cos(x) - \cos^3(x)}{4}\Big).$$

By the Fejér-Riesz lemma, we can obtain u_0 such that $|\widehat{u_0}(\omega)|^2 = v(\omega)$ as follows:

$$\widehat{u_0}(\omega) = e^{i3\omega}\Big(\frac{1-e^{-i\omega}}{2}\Big)^2\Big(\frac{e^{-i\omega}+2-\sqrt{3}}{\sqrt{4-2\sqrt{3}}}\Big)\Big(\frac{e^{-i\omega}-r_1}{2\sqrt{2r_1}}\Big)\frac{(e^{-2i\omega}+(r_2+\overline{r_2})e^{-i\omega}+|r_2|^2)}{2|r_2|},$$

where

$$r_1 := c_0 - \sqrt{c_0^2-1},\ r_2 := c_1 - \sqrt{c_1^2-1},$$

with

$$t = (3+2\sqrt{2})^{1/3},\ c_0 := t + \frac{1}{t},\ c_1 := \frac{c_0}{2} - \frac{\sqrt{3}}{2}i(t-1/t).$$

Define u_1, u_2 by

$$\widehat{u_1}(\omega) = (\widehat{u_0}(\omega) + e^{-i\omega}\overline{\widehat{u_0}(\omega)})/2, \quad \widehat{u_2}(\omega) = (\widehat{u_0}(\omega) - e^{-i\omega}\overline{\widehat{u_0}(\omega)})/2.$$

Then, u_1 is symmetric about $1/2$ with support $[-3,4]\cap\mathbb{Z}$, u_2 is antisymmetric about $1/2$ with support $[-3,4]$, and $|\widehat{a}(\omega)|^2 + |\widehat{u_1}(\omega)|^2 + |\widehat{u_2}(\omega)|^2 = 1$.

Finally, we can define

$$\{a^{2D}; b^{2D}_{1,0}, b^{2D}_{2,0}, b^{2D}_{3,0}, b^{2D}_{0,1}, b^{2D}_{1,1}, b^{2D}_{2,1}, b^{2D}_{3,1}, b^{2D}_{0,2}, b^{2D}_{1,2}, b^{2D}_{2,2}, b^{2D}_{3,2}\}$$

as in Theorem 4.4, which gives a 6-multiple canonical quincunx tight framelet filter bank. a^{2D} has at least order 4 sum rules and is D_4-symmetric about the origin. All the eleven high-pass filters are real-valued and have some symmetry properties with at least order 2 vanishing moments.

Example 4.2. Consider $a = \{-\frac{3}{64}, \frac{5}{64}, \boxed{\frac{15}{32}}, \frac{15}{32}, \frac{5}{64}, -\frac{3}{64}\}_{[-2,3]}$ with $\mathrm{sr}(a,2) = 3$ and $\mathrm{lpm}(a) = 4$. Then

$$1 - |\widehat{a}(\omega)|^2 - |\widehat{a}(\omega+\pi)|^2 = -\frac{15}{256}(1-\cos(2x))^2 \ge 0.$$

Then $\widehat{u}(\omega) := \frac{\sqrt{15}}{32}(2 - e^{-i\omega} - e^{i\omega})$ satisfies $1 - |\widehat{a}(\omega)|^2 - |\widehat{a}(\omega+\pi)|^2 = |\widehat{u}(2\omega)|^2$. Define b_1, b_2, b_3 as in (4.8). Then,

$$\begin{aligned}
\widehat{b_1}(\omega) &= \frac{15}{32}(-1+e^{-i\omega}) + \frac{5}{64}(e^{i\omega} - e^{-i2\omega}) + \frac{3}{64}(e^{i2\omega} - e^{-i3\omega}),\\
\widehat{b_2}(\omega) &= \frac{\sqrt{15}}{64}\left(2(1+e^{-i\omega}) - (e^{i\omega}+e^{-i2\omega}) - (e^{i2\omega}+e^{-i3\omega})\right),\\
\widehat{b_3}(\omega) &= \frac{\sqrt{15}}{64}\left(2(-1+e^{-i\omega}) - (e^{i\omega}-e^{-i2\omega}) + (e^{i2\omega}-e^{-i3\omega})\right).
\end{aligned}$$

Note that high-pass filter b_1 is antisymmetric about $1/2$ and supported on $[-2,3]$, the high-pass filter b_2 is symmetric about $1/2$ and supported on $[-2,3]$, and the high-pass filter b_3 is antisymmetric about $1/2$ and supported on $[-2,3]$. The filter bank $\{a; b_1, b_2, b_3\}$ forms a double canonical tight 2-framelet filter bank with $\mathrm{vm}(b_1) = 3$, $\mathrm{vm}(b_2) = 2$, and $\mathrm{vm}(b_3) = 3$. See [29, Example 3] for a tight 2-framelet filter bank $\{a; b_1, b_2\}$ with symmetry property derived from the low-pass filter a.

Next, define $\widehat{v}(\omega) := 1 - |\widehat{a}(\omega)|^2 = -\frac{1}{128}(\cos(x)-1)^2(9\cos^3(x) + 3\cos^2(x) - 53\cos(x) - 79)$. Then,

$$\widehat{v}(\xi) = \Big|\frac{1-e^{-i\omega}}{2}\Big|^4 \frac{9(\cos(x)-c_0)(\cos(x)-c_1)(\cos(x)-\overline{c_1})}{-32},$$

where

$$c_0 = t_1+t_2-1/9,\ c_1 = -(t_1+t_2+2/9-\sqrt{3}i(t_1-t_2))/2,\ t_1 = 8\sqrt[3]{10}/9,\ t_2 = 2\sqrt[3]{100}/9.$$

Consequently, we can obtain u_0 such that $|\widehat{u_0}(\omega)|^2 = v(\omega)$ as follows:

$$\widehat{u_0}(\omega) = 3e^{i2\omega}\Big(\frac{1-e^{-i\omega}}{2}\Big)^2\Big(\frac{e^{-i\omega}-r_0}{2\sqrt{r_0}}\Big)\frac{e^{-2i\omega}-(r_1+\overline{r_1})e^{-i\omega}+|r_1|^2}{8|r_1|},$$

where $r_0 = c_0 - \sqrt{c_0^2-1}$ and $r_1 = c_1 - \sqrt{c_1^2-1}$. Define u_1, u_2 by

$$\widehat{u_1}(\omega) = (\widehat{u_0}(\omega) + e^{-i\omega}\overline{\widehat{u_0}(\omega)})/2, \quad \widehat{u_2}(\omega) = (\widehat{u_0}(\omega) - e^{-i\omega}\overline{\widehat{u_0}(\omega)})/2.$$

Then, u_1 is symmetric about $1/2$ with support $[-2,3]\cap\mathbb{Z}$, u_2 is antisymmetric about $1/2$ with support $[-2,3]\cap\mathbb{Z}$, and $|\widehat{a}(\omega)|^2 + |\widehat{u_1}(\omega)|^2 + |\widehat{u_2}(\omega)|^2 = 1$. We also have $\mathrm{vm}(u_1) = 2$ and $\mathrm{vm}(u_2) = 3$.

Finally, we can define

$$\{a^{2D}; b^{2D}_{1,0}, b^{2D}_{2,0}, b^{2D}_{3,0}, b^{2D}_{0,1}, b^{2D}_{1,1}, b^{2D}_{2,1}, b^{2D}_{3,1}, b^{2D}_{0,2}, b^{2D}_{1,2}, b^{2D}_{2,2}, b^{2D}_{3,2}\}$$

as in Theorem 4.4, which gives a 6-multiple canonical quincunx tight frame filter bank. a^{2D} has at least order 6 sum rule and is D_4-symmetric about the origin. All the eleven high-pass filters are real-valued and have certain symmetry properties with at least 2 vanishing moments.

We remark that other 6-multiple canonical quincunx tight framelet filter banks with high orders of vanishing moments can be obtained by considering other low-pass filters and following the above procedure.

Appendix A. Proofs of Theorems 2.1 and 4.2

Proof of Theorem 2.1. The existence of such a filter $a^{2D}_{2n,2n}$ has been proved in Proposition 3.1. Let a be such a filter $a^{2D}_{2n,2n}$. We now prove the uniqueness of such a filter a satisfying all the properties in Theorem 2.1.

The filter a having orders $2n$ sum rules with respect to $M = M_{\sqrt{2}}$ is equivalent to

$$\text{(A.1)} \qquad \sum_{k\in\mathbb{Z}^2} a(Mk+(1,0))(Mk+(1,0))^\mu = \sum_{k\in\mathbb{Z}^2} a(Mk)(Mk)^\mu, \quad \forall |\mu| < 2n,$$

and a having order $2n$ linear-phase moments with phase $\mathbf{c} = (1/2, 1/2)$ is equivalent to

$$\text{(A.2)} \qquad \sum_{k\in\mathbb{Z}^2} a(k)k^\mu = \mathbf{c}^\mu \quad \forall\, |\mu| < 2n,$$

where $\mu = (\mu_1, \mu_2) \in \mathbb{N}_0^2$. It is easily seen that (A.1) and (A.2) are equivalent to

$$\text{(A.3)} \qquad \begin{cases} \sum_{k\in\mathbb{Z}^2} a(Mk+(1,0))(Mk+(1,0))^\mu &= \frac{1}{2}\mathbf{c}^\mu, \\ & \qquad\qquad |\mu| < 2n. \\ \sum_{k\in\mathbb{Z}^2} a(Mk)(Mk)^\mu &= \frac{1}{2}\mathbf{c}^\mu, \end{cases}$$

Define

$$\Lambda_0 := \{k = (k_1,k_2) \in \mathbb{Z}^2 : k_1 + k_2 \text{ even}, k \in [-n+1, n]^2\},$$
$$\Lambda_1 := \{k = (k_1,k_2) \in \mathbb{Z}^2 : k_1 - k_2 \text{ odd}, k \in [-n+1, n]^2\}.$$

Then, $\#\Lambda_0 = \#\Lambda_1 = 2n^2$, $\Lambda_0 \cap \Lambda_1 = \emptyset$, and $\Lambda_0 \cup \Lambda_1 = [-n+1,n]^2 \cap \mathbb{Z}^2$. Moreover, $\Lambda_0 = M\mathbb{Z}^2 \cap [-n+1,n]^2$ and $\Lambda_1 = (M\mathbb{Z}^2 + (1,0)) \cap [-n+1,n]^2$. On the other hand, consider the index set

$$\Gamma_n := \{\mu \in \mathbb{N}_0^2 : |\mu| < 2n, \mu_2 < 2n-1\} \setminus \{(0, 2j-1) : j = 1, \dots, n-1\}.$$

Then, it is easy to show that $\#\Lambda_0 = \#\Lambda_1 = \#\Gamma_n = 2n^2$. Using this notation and noting that Γ_n is a subset of $\{\mu \in \mathbb{N}_0^2 : |\mu| < 2n\}$, (A.3) implies that a must also satisfy the following conditions:

$$\sum_{k\in\Lambda_\epsilon} a(k)k^\mu = \frac{1}{2}\mathbf{c}^\mu, \quad \mu \in \Gamma_n, \epsilon \in \{0,1\}. \tag{A.4}$$

Note that

$$\#\big(\Lambda_0 \cap \{\boldsymbol{x} = (x_1,x_2) \in \mathbb{R}^2 : x_1 + x_2 = 2j\}\big) = 4 - |2j-1|, j = -n+1,\dots,n$$

and

$$\#\big(\Lambda_1 \cap \{\boldsymbol{x} = (x_1,x_2) \in \mathbb{R}^2 : x_1 - x_2 = 2j+1\}\big) = 4 - |2j+1|, j = -n,\dots,n-1.$$

By [33, Lemma 3.1], The matrices $(k^\mu)_{k\in\Lambda_\epsilon,\, \mu\in\Gamma_n}, \epsilon = 0,1$ are nonsingular. Consequently, a must be unique.

Item (i) follows from Proposition 3.1. For item (ii), notice that $\widehat{a^{2D}_{2n,2n}}(\boldsymbol{\omega}) = \hat{\mathring{a}}(\boldsymbol{\omega})e^{-i\mathbf{c}\cdot\boldsymbol{\omega}}$, where

$$\hat{\mathring{a}}(\boldsymbol{\omega}) := \widehat{a^I_{2n}}\left(\frac{\omega_1+\omega_2}{2}\right) + \widehat{a^I_{2n}}\left(\frac{\omega_1-\omega_2}{2}\right) - 1. \tag{A.5}$$

One can easily show that $\hat{\mathring{a}}$ satisfies $\hat{\mathring{a}}(E^\top\cdot) = \hat{\mathring{a}}$ for all $E \in D_4$ due to the fact that a^I_{2n} satisfies $\widehat{a^I_{2n}}(-\omega) = \widehat{a^I_{2n}}(\omega)$ for $\omega \in \mathbb{R}$. Consequently,

$$\widehat{a^{2D}_{2n,2n}}(E^\top\boldsymbol{\omega}) = \hat{\mathring{a}}(E^\top\boldsymbol{\omega})e^{-i\mathbf{c}\cdot E^\top\boldsymbol{\omega}} = \widehat{a^{2D}_{2n,2n}}(\boldsymbol{\omega})e^{i\mathbf{c}\cdot(I_2-E^\top)\boldsymbol{\omega}},\ \boldsymbol{\omega} \in \mathbb{R}^2,$$

which is equivalent to (1.8), i.e., $a^{2D}_{2n,2n}$ is D_4-symmetric about $\mathbf{c} = (1/2,1/2)$.

Item (iii) is a direct consequence of [24, Proposition 2.1] (also see [19, Theorem 2.3]). In fact, by $N_{\sqrt{2}} = EM_{\sqrt{2}}$ with $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $N_{\sqrt{2}}$ is D_4-equivalent to $M_{\sqrt{2}}$. Thus, by [24, Proposition 2.1], $\phi^{N_{\sqrt{2}}} = \phi^{M_{\sqrt{2}}}(\cdot + \mathring{\mathbf{c}})$, where $\mathring{\mathbf{c}} := (M_{\sqrt{2}}-I_2)^{-1}\mathbf{c}-(N_{\sqrt{2}}-I_2)^{-1}\mathbf{c} = (1,1)$. (2.12) follows from [24, Proposition 2.1]. $\square$

Proof of Theorem 4.2. By the definition of sum rules and $M_{\sqrt{2}}\mathbb{Z}^2 = N_{\sqrt{2}}\mathbb{Z}^2$, it is straightforward to check that $\mathrm{sr}(a, M_{\sqrt{2}}) = \mathrm{sr}(a, N_{\sqrt{2}}) = \mathrm{sr}(a,2)$. We now prove $\mathrm{sm}_p(a, M_{\sqrt{2}}) = \mathrm{sm}_p(a,2)$. Let $M = M_{\sqrt{2}}$. By the definition of the subdivision operator in (2.7), we have

$$\widehat{\mathcal{S}^n_{a,M}v}(\boldsymbol{\omega}) = |\det(M)|^n\hat{v}((M^\top)^n\boldsymbol{\omega})\hat{a}(\xi)\cdots\hat{a}((M^\top)^{n-1}\boldsymbol{\omega}). \tag{A.6}$$

In particular, noting that $M^2 = 2I_2$, we have

$$\widehat{\mathcal{S}^n_{a,M}\delta}(\boldsymbol{\omega}) = 2^n\hat{a}(\boldsymbol{\omega})\cdots\hat{a}((M^\top)^{n-1}\boldsymbol{\omega}) = \widehat{\mathcal{S}^{n_1}_{a,2}\delta}(\omega_1)\widehat{\mathcal{S}^{n_2}_{a,2}\delta}(\omega_1+\omega_2),$$

where

$$n_1 := \lfloor\frac{n+1}{2}\rfloor, \qquad n_2 := n - n_1. \tag{A.7}$$

Therefore, for $\mu_1,\mu_2 \in \mathbb{N}_0 := \mathbb{N}\cup\{0\}$, we deduce from the above identity that

$$[\nabla^{\mu_1}_{e_1}\nabla^{\mu_2}_{e_1+e_2}\mathcal{S}^n_{a,M}\delta](j,k) = [\nabla^{\mu_1}\mathcal{S}^{n_1}_{a,2}\delta](j-k)[\nabla^{\mu_2}\mathcal{S}^{n_2}_{a,2}\delta](k), \qquad j,k \in \mathbb{Z},$$

from which we have

$$\|\nabla^{\mu_1}_{e_1}\nabla^{\mu_2}_{e_1+e_2}\mathcal{S}^n_{a,M}\delta\|_{l_p(\mathbb{Z}^2)} = \|\nabla^{\mu_1}\mathcal{S}^{n_1}_{a,2}\delta\|_{l_p(\mathbb{Z})}\|\nabla^{\mu_2}\mathcal{S}^{n_2}_{a,2}\delta\|_{l_p(\mathbb{Z})},\ \mu_1,\mu_2,n \in \mathbb{N}_0, \tag{A.8}$$

where n_1 and n_2 are defined in (A.7). Let $m := \mathrm{sr}(a,2)$. By $\widehat{a}(0) = 1$, it is known in [21] and [20, Theorem 3.1] that $\rho_j(a,2)_p \ge 2^{1/p-j}$ for all $j \in \mathbb{N}_0$ and

$$\rho_j(a,2)_p = \max(2^{1/p-j}, \rho_m(a,2)_p), \qquad j = 0,\dots,m. \tag{A.9}$$

Taking $\mu_1 = m$ and $\mu_2 = 0$ in (A.8), by $\rho_0(a,2)_p \ge 2^{1/p} > 0$ and $\lim_{n\to\infty} n_1/n = 1/2 = \lim_{n\to\infty} n_2/n$, we have

$$\begin{aligned}\sqrt{\rho_m(a,2)_p}\sqrt{\rho_0(a,2)_p} &= \lim_{n\to\infty} \|\nabla^m \mathcal{S}_{a,2}^{n_1}\delta\|_{l_p(\mathbb{Z})}^{1/n} \lim_{n\to\infty} \|\mathcal{S}_{a,2}^{n_2}\delta\|_{l_p(\mathbb{Z})}^{1/n}\\ &= \lim_{n\to\infty} \|\nabla_{e_1}^m \mathcal{S}_{a,M}^{n}\delta\|_{l_p(\mathbb{Z}^2)}^{1/n}\\ &\le \rho_m(a,M)_p.\end{aligned}$$

Since $\rho_0(a,2)_p \ge 2^{1/p}$, we conclude from the above inequality that $\rho_m(a,2)_p \le 2^{-1/p}(\rho_m(a,M)_p)^2$. Consequently, by $|\det(M)| = 2$ and $\mathrm{sr}(a,M) = m$, we have

$$\begin{aligned}\mathrm{sm}_p(a,2) &= \tfrac{1}{p} - \log_2 \rho_m(a,2)_p \ge \tfrac{1}{p} - \log_2[2^{-1/p}(\rho_m(a,M)_p)^2]\\ &= \tfrac{2}{p} - 2\log_2 \rho_m(a,M)_p\\ &= \mathrm{sm}_p(a,M).\end{aligned}$$

This proves $\mathrm{sm}_p(a,2) \ge \mathrm{sm}_p(a,M)$. Conversely, taking $\mu_1 = j$ and $\mu_2 = m-j$ in (A.8) with $0 \le j \le m$, we have

$$\begin{aligned}&\lim_{n\to\infty} \|\nabla_{e_1}^j \nabla_{e_1+e_2}^{m-j} \mathcal{S}_{a,M}^{n}\delta\|_{l_p(\mathbb{Z}^2)}^{2/n}\\ =&\lim_{n\to\infty} \|\nabla^j \mathcal{S}_{a,2}^{n_1}\delta\|_{l_p(\mathbb{Z})}^{2/n} \lim_{n\to\infty} \|\nabla^{m-j} \mathcal{S}_{a,2}^{n_2}\delta\|_{l_p(\mathbb{Z})}^{2/n}\\ \le&\rho_j(a,2)_p\rho_{m-j}(a,2)_p.\end{aligned} \tag{A.10}$$

By $\nabla_{e_2}\delta = \nabla_{e_1+e_2}\delta - [\nabla_{e_1}\delta](\cdot - e_2)$, we see that all $\nabla_{e_1}^{\mu_1}\nabla_{e_2}^{m-\mu_1}\delta$ with $\mu_1 = 0,\dots,m$ are finitely linear combinations of $[\nabla_{e_1}^j \nabla_{e_1+e_2}^{m-j}\delta](\cdot - k)$, $j = 0,\dots,m$ and $k \in \mathbb{Z}^2$. If we can prove

$$\rho_j(a,2)_p\rho_{m-j}(a,2)_p \le 2^{1/p}\rho_m(a,2)_p, \qquad \forall\, j = 0,\dots,m, \tag{A.11}$$

then it follows from (A.10) that $(\rho_m(a,M)_p)^2 \le 2^{1/p}\rho_m(a,2)_p$. Since $m = \mathrm{sr}(a,M)$ and $|\det(M)| = 2$,

$$\begin{aligned}\mathrm{sm}_p(a,M) &= \tfrac{2}{p} - 2\log_2 \rho_m(a,M)_p\\ &\ge \tfrac{2}{p} - 2\log_2 \sqrt{2^{1/p}\rho_m(a,2)_p}\\ &= \tfrac{1}{p} - \log_2 \rho_m(a,2)_p = \mathrm{sm}_p(a,2).\end{aligned}$$

Hence, $\mathrm{sm}_p(a,M) \ge \mathrm{sm}_p(a,2)$ and this completes the proof of item (i).

We now prove (A.11). According to (A.9), we have four cases to consider. If $\rho_j(a,2)_p = 2^{1/p-j}$ and $\rho_{m-j}(a,2)_p = 2^{1/p-(m-j)}$, then (A.11) holds, since

$$\rho_j(a,2)_p\rho_{m-j}(a,2)_p = 2^{1/p-j}2^{1/p-(m-j)} = 2^{2/p-m} = 2^{1/p}2^{1/p-m} \le 2^{1/p}\rho_m(a,2)_p,$$

where we used the fact that $\rho_m(a,2)_p \ge 2^{1/p-m}$. If $\rho_j(a,2)_p = \rho_m(a,2)_p$ and $\rho_{m-j}(a,2)_p = 2^{1/p-(m-j)}$, then (A.11) holds, since

$$\rho_j(a,2)_p\rho_{m-j}(a,2)_p = 2^{1/p-(m-j)}\rho_m(a,2)_p \le 2^{1/p}\rho_m(a,2)_p.$$

The case $\rho_j(a,2)_p = 2^{1/p-j}$ and $\rho_{m-j}(a,2)_p = \rho_m(a,2)_p$ is similar.

If $\rho_j(a,2)_p = \rho_m(a,2)_p$ and $\rho_{m-j}(a,2)_p = \rho_m(a,2)_p$, then

$$\rho_j(a,2)_p \rho_{m-j}(a,2)_p = \rho_m(a,2)_p \rho_m(a,2)_p \le 2^{1/p} \rho_m(a,2)_p,$$

where we used the inequality $\rho_m(a,2)_p \le 2^{1/p}$ which is guaranteed by our assumption $\operatorname{sm}(a,2)_p \ge 0$. Therefore, (A.11) is verified and this completes the proof of item (i).

We now prove item (ii). The claim $\operatorname{sr}(u*v,M) \ge \operatorname{sr}(u,M) + \operatorname{sr}(v,M)$ can be directly verified by using the definition of sum rules. By (A.6) and $\widehat{u*v}(\boldsymbol{\omega}) = \widehat{u}(\boldsymbol{\omega})\widehat{v}(\boldsymbol{\omega})$, for $\mu,\nu \in \mathbb{N}_0^d$, we have

$$\nabla^{\mu+\nu}\mathcal{S}^n_{u*v,M}\delta = |\det(M)|^{-n}[\nabla^\mu \mathcal{S}^n_{u,M}\delta] * [\nabla^\nu \mathcal{S}^n_{v,M}\delta].$$

Consequently, by Cauchy-Schwarz inequality, we have

$$\|\nabla^{\mu+\nu}\mathcal{S}^n_{u*v,M}\delta\|_{l_\infty(\mathbb{Z}^d)} \le |\det(M)|^{-n}\|\nabla^\mu \mathcal{S}^n_{u,M}\delta\|_{l_2(\mathbb{Z}^d)}\|\nabla^\nu \mathcal{S}^n_{v,M}\delta\|_{l_2(\mathbb{Z}^d)}$$

Let $m_1 := \operatorname{sr}(u,M)$ and $m_2 := \operatorname{sr}(v,M)$. Taking $\mu,\nu \in \mathbb{N}_0^d$ with $|\mu| = m_1$ and $|\nu| = m_2$ in the above inequality, we have

$$\begin{aligned}\lim_{n\to\infty} \|\nabla^{\mu+\nu}\mathcal{S}^n_{u*v,M}\delta\|^{1/n}_{l_\infty(\mathbb{Z}^d)} &\le |\det(M)|^{-1} \lim_{n\to\infty}\|\nabla^\mu \mathcal{S}^n_{u,M}\delta\|^{1/n}_{l_2(\mathbb{Z}^d)} \lim_{n\to\infty}\|\nabla^\nu \mathcal{S}^n_{v,M}\delta\|^{1/n}_{l_2(\mathbb{Z}^d)}\\ &\le |\det(M)|^{-1}\rho_{m_1}(u,M)_2\rho_{m_2}(v,M)_2.\end{aligned}$$

Note that any element $\eta \in \mathbb{N}_0^d$ with $|\eta| = m_1 + m_2$ can be written as $\eta = \mu + \nu$ with $|\mu| = m_1$ and $|\nu| = m_2$ for some $\mu,\nu \in \mathbb{N}_0^d$. Thus, we deduce from the above inequality that $\rho_{m_1+m_2}(u*v,M)_\infty \le |\det(M)|^{-1}\rho_{m_1}(u,M)_2\rho_{m_2}(v,M)_2$. Let $m := \operatorname{sr}(u*v,M)$. By $m \ge m_1 + m_2$, we have

$$\rho_m(u*v,M)_\infty \le \rho_{m_1+m_2}(u*v,M)_\infty \le |\det(M)|^{-1}\rho_{m_1}(u,M)_2\rho_{m_2}(v,M)_2,$$

from which we have

$$\begin{aligned}\operatorname{sm}_\infty(u*v,M) &= -d\log_{|\det(M)|}\rho_m(u*v,M)_\infty\\ &\ge -d\log_{|\det(M)|}[|\det(M)|^{-1}\rho_{m_1}(u,M)_2\rho_{m_2}(v,M)_2]\\ &= \tfrac{d}{2} - d\log_{|\det(M)|}\rho_{m_1}(u,M)_2 + \tfrac{d}{2} - d\log_{|\det(M)|}\rho_{m_2}(v,M)_2\\ &= \operatorname{sm}_2(u,M) + \operatorname{sm}_2(v,M).\end{aligned}$$

The proof of item (ii) is completed by noting that $\operatorname{sm}_\infty(u*v,M) \le \operatorname{sm}_2(u*v,M)$ always holds.

To prove item (iii), we define $\tilde{u}(k,j) := u(k)\delta(j)$ and $\tilde{v}(j,k) := v(k)\delta(j)$ for all $j,k \in \mathbb{Z}$. That is, $\tilde{u}$ is the 2D filter by identifying u on $\mathbb{Z}$ with $\mathbb{Z}\times\{0\}$, while $\tilde{v}$ is the 2D filter by identifying v on $\mathbb{Z}$ with $\{0\}\times\mathbb{Z}$. Since $\operatorname{sm}(u,2) \ge 0$ and $\operatorname{sm}(v,2) \ge 0$, by item (i), we have $\operatorname{sr}(\tilde{u},M_{\sqrt{2}}) = \operatorname{sr}(u,2)$, $\operatorname{sr}(\tilde{v},M_{\sqrt{2}}) = \operatorname{sr}(v,2)$ and $\operatorname{sm}(\tilde{u},M_{\sqrt{2}}) = \operatorname{sm}(u,2)$, $\operatorname{sm}(\tilde{v},M_{\sqrt{2}}) = \operatorname{sm}(v,2)$. Note that $u\otimes v = \tilde{u}*\tilde{v}$. Now the claim in item (iii) follows directly from item (ii). $\square$

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