Matrix Extension with Symmetry and Its Applications

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Abstract In this paper, we are interested in the problems of matrix extension with symmetry, more precisely, the extensions of submatrices of Laurent polynomials satisfying some conditions to square matrices of Laurent polynomials with certain symmetry patterns, which are closely related to the construction of (bi)orthogonal multiwavelets in wavelet analysis and filter banks with the perfect reconstruction property in electronic engineering. We satisfactorily solve the matrix extension problems with respect to both orthogonal and biorthogonal settings. Our results show that the extension matrices do possess certain symmetry patterns and their coefficient supports can be controlled by the given submatrices in certain sense. Moreover, we provide step-by-step algorithms to derive the desired extension matrices. We show that our extension algorithms can be applied not only to the construction of (bi)orthogonal multiwavelets with symmetry, but also to the construction of tight framelets with symmetry and with high order of vanishing moments. Several examples are presented to illustrate the results in this paper.

1 Introduction and Motivation

The matrix extension problems play a fundamental role in many areas such as electronic engineering, system sciences, mathematics, etc. We mention only a few references here on this topic; see [1–3, 5, 8, 10, 12, 19–21, 23–25]. For example, matrix extension is an indispensable tool in the design of filter banks in electronic engineering (see [19,24,25]) and in the construction of multiwavelets in wavelet analysis (see [1–3, 5, 8, 10, 12, 14, 18, 20, 21]). In this section, we shall first introduce the general matrix extension problems and then discuss the connections of the general matrix extension problems to wavelet analysis and filter banks.

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1.1 The Matrix Extension Problems

In order to state the matrix extension problems, let us introduce some notation and definitions first. Let $\mathsf{p}(z) = \sum_{k\in\mathbb{Z}} p_k z^k, z \in \mathbb{C}\backslash\{0\}$ be a Laurent polynomial with complex coefficients $p_k \in \mathbb{C}$ for all $k \in \mathbb{Z}$. We say that p has *symmetry* if its coefficient sequence $\{p_k\}_{k\in\mathbb{Z}}$ has symmetry; more precisely, there exist $\varepsilon \in \{-1,1\}$ and $c \in \mathbb{Z}$ such that

$$p_{c-k} = \varepsilon p_k \qquad \forall\, k \in \mathbb{Z}. \tag{1}$$

If $\varepsilon = 1$, then p is symmetric about the point $c/2$; if $\varepsilon = -1$, then p is antisymmetric about the point $c/2$. Symmetry of a Laurent polynomial can be conveniently expressed using a symmetry operator S defined by

$$\mathsf{Sp}(z) := \frac{\mathsf{p}(z)}{\mathsf{p}(1/z)}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{2}$$

When p is not identically zero, it is evident that (1) holds if and only if $\mathsf{Sp}(z) = \varepsilon z^c$. For the zero polynomial, it is very natural that $\mathsf{S}0$ can be assigned any symmetry pattern; i.e., for every occurrence of $\mathsf{S}0$ appearing in an identity in this paper, $\mathsf{S}0$ is understood to take an appropriate choice of εz^c for some $\varepsilon \in \{-1,1\}$ and some $c \in \mathbb{Z}$ so that the identity holds. If $\mathbb{P}$ is an $r \times s$ matrix of Laurent polynomials with symmetry, then we can apply the operator S to each entry of $\mathbb{P}$, i.e., $\mathsf{S}\mathbb{P}$ is an $r \times s$ matrix such that $[\mathsf{S}\mathbb{P}]_{j,k} := \mathsf{S}([\mathbb{P}]_{j,k})$, where $[\mathbb{P}]_{j,k}$ is the (j,k)-entry of the matrix $\mathbb{P}$.

For two matrices $\mathbb{P}$ and Q of Laurent polynomials with symmetry, even though all the entries in $\mathbb{P}$ and Q have symmetry, their sum $\mathbb{P}+\mathsf{Q}$, difference $\mathbb{P}-\mathsf{Q}$, or product $\mathbb{P}\mathsf{Q}$, if well defined, generally may not have symmetry any more. This is one of the difficulties for matrix extension with symmetry. In order for $\mathbb{P}\pm\mathsf{Q}$ or $\mathbb{P}\mathsf{Q}$ to possess some symmetry, the symmetry patterns of $\mathbb{P}$ and Q should be compatible. For example, if $\mathsf{S}\mathbb{P} = \mathsf{S}\mathsf{Q}$ (i.e., both $\mathbb{P}$ and Q have the same symmetry pattern), then indeed $\mathbb{P}\pm\mathsf{Q}$ has symmetry and $\mathsf{S}(\mathbb{P}\pm\mathsf{Q}) = \mathsf{S}\mathbb{P} = \mathsf{S}\mathsf{Q}$. In the following, we discuss the compatibility of symmetry patterns of matrices of Laurent polynomials.

For an $r \times s$ matrix $\mathbb{P}(z) = \sum_{k\in\mathbb{Z}} P_k z^k$, we denote

$$\mathbb{P}^*(z) := \sum_{k\in\mathbb{Z}} P_k^* z^{-k} \quad \text{with} \quad P_k^* := \overline{P_k}^T, \qquad k \in \mathbb{Z}, \tag{3}$$

where $\overline{P_k}^{\mathrm{T}}$ denotes the transpose of the complex conjugate of the constant matrix P_k in $\mathbb{C}$. We say that *the symmetry of* $\mathbb{P}$ *is compatible* or $\mathbb{P}$ *has compatible symmetry*, if

$$\mathsf{S}\mathbb{P}(z) = (\mathsf{S}\theta_1)^*(z)\mathsf{S}\theta_2(z) \tag{4}$$

for some $1 \times r$ and $1 \times s$ row vectors θ_1 and θ_2 of Laurent polynomials with symmetry. For an $r \times s$ matrix $\mathbb{P}$ and an $s \times t$ matrix Q of Laurent polynomials, we say that $(\mathbb{P},\mathsf{Q})$ *has mutually compatible symmetry* if

$$\mathsf{S}\mathbb{P}(z) = (\mathsf{S}\theta_1)^*(z)\mathsf{S}\theta(z) \quad \text{and} \quad \mathsf{S}\mathsf{Q}(z) = (\mathsf{S}\theta)^*(z)\mathsf{S}\theta_2(z) \tag{5}$$

for some $1 \times r$, $1 \times s$, $1 \times t$ row vectors $\theta_1, \theta, \theta_2$ of Laurent polynomials with symmetry. If $(\mathbb{P}, \mathsf{Q})$ has mutually compatible symmetry as in (5), then their product $\mathbb{P}\mathsf{Q}$ has compatible symmetry and in fact $\mathsf{S}(\mathbb{P}\mathsf{Q}) = (\mathsf{S}\theta_1)^*\mathsf{S}\theta_2$.

For a matrix of Laurent polynomials, another important property is the support of its coefficient sequence. For $\mathbb{P} = \sum_{k\in\mathbb{Z}} P_k z^k$ such that $P_k = 0$ for all $k \in \mathbb{Z}\backslash[m,n]$ with $P_m \neq 0$ and $P_n \neq 0$, we define its coefficient support to be $\operatorname{csupp}(\mathbb{P}) := [m,n]$ and the length of its coefficient support to be $|\operatorname{csupp}(\mathbb{P})| := n - m$. In particular, we define $\operatorname{csupp}(0) := \emptyset$, the empty set, and $|\operatorname{csupp}(0)| := -\infty$. Also, we use $\operatorname{coeff}(\mathbb{P}, k) := P_k$ to denote the coefficient matrix (vector) P_k of z^k in $\mathbb{P}$. In this paper, 0 always denotes a general zero matrix whose size can be determined in the context.

Now, we introduce the general matrix extension problems with symmetry. We shall use r and s to denote two positive integers such that $1 \le r \le s$. I_r denotes the $r \times r$ identity matrix.

Problem 1 (Orthogonal Matrix Extension). Let $\mathbb{F}$ be a subfield of $\mathbb{C}$. Let $\mathbb{P}$ be an $r \times s$ matrix of Laurent polynomials with coefficients in $\mathbb{F}$ such that $\mathbb{P}(z)\mathbb{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ and the symmetry of $\mathbb{P}$ is compatible. Find an $s \times s$ square matrix $\mathbb{P}_e$ of Laurent polynomials with coefficients in $\mathbb{F}$ and with symmetry such that

1. $[I_r, \mathbf{0}]\mathbb{P}_e = \mathbb{P}$ (that is, the submatrix of the first r rows of $\mathbb{P}_e$ is the given matrix $\mathbb{P}$);
2. The symmetry of $\mathbb{P}_e$ is compatible and $\mathbb{P}_e(z)\mathbb{P}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$ (that is, $\mathbb{P}_e$ is *paraunitary*);
3. The length of the coefficient support of $\mathbb{P}_e$ can be controlled by that of $\mathbb{P}$ in some way.

Problem 1 is closely related to the construction of orthonormal multiwavelets in wavelet analysis and the design of filter banks with the perfect reconstruction property in electronic engineering. More generally, Problem 1 can be extended to a more general form with respect to the construction of biorthogonal multiwavelets in wavelet analysis. In a moment, we shall reveal their connections, which also serve as our motivation. The more general form of Problem 1 can be stated as follows.

Problem 2 (Biorthogonal Matrix Extension). Let $\mathbb{F}$ be a subfield of $\mathbb{C}$. Let $(\mathbb{P}, \widetilde{\mathbb{P}})$ be a pair of $r \times s$ matrices of Laurent polynomials with coefficients in $\mathbb{F}$ such that $\mathbb{P}(z)\widetilde{\mathbb{P}}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$, the symmetry of $\mathbb{P}$ or $\widetilde{\mathbb{P}}$ is compatible, and $\mathsf{S}\mathbb{P} = \mathsf{S}\widetilde{\mathbb{P}}$. Find a pair of $s \times s$ square matrices $(\mathbb{P}_e, \widetilde{\mathbb{P}}_e)$ of Laurent polynomials with coefficients in $\mathbb{F}$ and with symmetry such that

1. $[I_r, \mathbf{0}]\mathbb{P}_e = \mathbb{P}$ and $[I_r, \mathbf{0}]\widetilde{\mathbb{P}}_e = \widetilde{\mathbb{P}}$ (that is, the submatrix of the first r rows of $\mathbb{P}_e, \widetilde{\mathbb{P}}_e$ is the given matrix $\mathbb{P}, \widetilde{\mathbb{P}}$, respectively);
2. $(\mathbb{P}_e, \widetilde{\mathbb{P}}_e)$ has mutually compatible symmetry and $\mathbb{P}_e(z)\widetilde{\mathbb{P}}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$ (that is, $(\mathbb{P}_e, \widetilde{\mathbb{P}}_e)$ is *a pair of biorthogonal matrices*);
3. The lengths of the coefficient support of $\mathbb{P}_e$ and $\widetilde{\mathbb{P}}_e$ can be controlled by those of $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ in some way.

1.2 Motivation

The above problems are closely connected to wavelet analysis and filter banks. The key of wavelet construction is the so-called multiresolution analysis (MRA), which contains mainly two parts. One is on the construction of refinable function vectors that satisfies certain desired conditions. Another part is on the derivation of wavelet generators from refinable function vectors obtained in first part, which should be able to inherit certain properties similar to their refinable function vectors. From the point of view of filter banks, the first part corresponds to the design of filters or filter banks with certain desired properties, while the second part can be and is formulated as some matrix extension problems stated previously. In this paper, we shall mainly focus on the second part (with symmetry) of the MRA while assume that the refinable function vectors with certain properties are given in advance (part of Sect. 3 is on the construction of refinable functions satisfying (14)).

We say that d is a *dilation factor* if d is an integer with $|\mathsf{d}| > 1$. Throughout this paper, d denotes a dilation factor. For simplicity of presentation, we further assume that d is positive, while multiwavelets and filter banks with a negative dilation factor can be handled similarly by a slight modification of the statements in this paper.

We say that $\phi = [\phi_1, \dots, \phi_r]^{\mathrm{T}} : \mathbb{R} \to \mathbb{C}^{r\times 1}$ is a d-*refinable function vector* if

$$\phi = \mathsf{d} \sum_{k\in\mathbb{Z}} a_0(k)\phi(\mathsf{d}\cdot - k), \tag{6}$$

where $a_0 : \mathbb{Z} \to \mathbb{C}^{r\times r}$ is a finitely supported sequence of $r\times r$ matrices on $\mathbb{Z}$, called the *low-pass filter (or mask)* for ϕ. The *symbol* of a_0 is denoted by $\mathsf{a}_0(z) := \sum_{k\in\mathbb{Z}} a_0(k)z^k$, which is an $r\times r$ matrix of Laurent polynomials.

In the frequency domain, the refinement equation in (6) can be rewritten as

$$\widehat{\phi}(\mathsf{d}\xi) = \widehat{a_0}(\xi)\widehat{\phi}(\xi), \quad \xi \in \mathbb{R}, \tag{7}$$

where $\widehat{a_0}$ is the *Fourier series* of a_0 given by

$$\widehat{a_0}(\xi) := \sum_{k\in\mathbb{Z}} a_0(k)\mathrm{e}^{-\mathrm{i}k\xi} = \mathsf{a}_0(\mathrm{e}^{-\mathrm{i}\xi}), \quad \xi \in \mathbb{R}. \tag{8}$$

The Fourier transform $\widehat{f}$ of $f \in L_1(\mathbb{R})$ is defined to be $\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)\mathrm{e}^{-\mathrm{i}t\xi}\mathrm{d}t$ and can be extended to square integrable functions and tempered distributions.

We say that a compactly supported d-refinable function vector ϕ in $L_2(\mathbb{R})$ is *orthogonal* if

$$\langle \phi, \phi(\cdot - k)\rangle = \delta(k)I_r, \quad k \in \mathbb{Z}, \tag{9}$$

where δ is the *Dirac sequence* such that $\delta(0) = 1$ and $\delta(k) = 0$ for all $k \neq 0$.

Usually, a wavelet system is generated by some wavelet function vectors $\psi^\ell = [\psi_1^\ell, \dots, \psi_r^\ell]^{\mathrm{T}}$, $\ell = 1, \dots, L$, from a d-refinable function vector ϕ as follows:

$$\widehat{\psi^\ell}(\mathsf{d}\xi) = \widehat{a_\ell}(\xi)\widehat{\phi}(\xi), \quad \ell = 1, \dots, L, \tag{10}$$

where each $a_\ell : \mathbb{Z} \to \mathbb{C}^{r\times r}$ is a finitely supported sequence of $r\times r$ matrices on $\mathbb{Z}$, called the *high-pass filter (or mask)* for ψ^ℓ, $\ell = 1, \dots, L$.

We say that $\{\psi^1,\ldots,\psi^L\}$ generates a d-*multiframe* in $L_2(\mathbb{R})$ if $\{\psi^\ell_{j,k} := \mathsf{d}^{j/2}\psi^\ell(\mathsf{d}^j\cdot - k) : j,k\in\mathbb{Z}, \ell=1,\ldots,L\}$ is a frame in $L_2(\mathbb{R})$, that is, there exist two positive constants C_1, C_2 such that

$$C_1\|f\|^2_{L_2(\mathbb{R})} \le \sum_{\ell=1}^{L}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} |\langle f,\psi^\ell_{j,k}\rangle|^2 \le C_2\|f\|^2_{L_2(\mathbb{R})}, \quad \forall f\in L_2(\mathbb{R}), \tag{11}$$

where $|\langle f,\psi^\ell_{j,k}\rangle|^2 = \langle f,\psi^\ell_{j,k}\rangle\langle\psi^\ell_{j,k},f\rangle$ and $\langle\cdot,\cdot\rangle$ is the inner product defined to be

$$\langle f,g\rangle := \int_{\mathbb{R}} f(t)\overline{g(t)}^{\mathsf{T}}\mathrm{d}t, \quad f\in(L_2(\mathbb{R}))^{s_1\times\ell}, g\in(L_2(\mathbb{R}))^{s_2\times\ell}.$$

If $C_1 = C_2 = 1$ in (11), we say that $\{\psi^1,\ldots,\psi^L\}$ generates a *tight* d-*multiframe* in $L_2(\mathbb{R})$. The wavelet function vectors ψ^ℓ are called *tight multiframelets*. When $r=1$, we usually drop the prefix *multi*.

If ϕ is a compactly supported d-refinable function vector in $L_2(\mathbb{R})$ associated with a low-pass filter a_0, then it is well-known (see [6]) that $\{\psi^1,\ldots,\psi^L\}$ associated with high-pass filters $\{a_1,\ldots,a_L\}$ via (10) generates a tight d-multiframe *if and only if*

$$\sum_{\ell=0}^{L}\widehat{a_\ell}\overline{\widehat{a_\ell}(\cdot+2\pi k/\mathsf{d})}^{\mathsf{T}} = \delta(k)I_r, \quad k=0,\ldots,\mathsf{d}-1. \tag{12}$$

According to various requirements of problems in applications, different desired properties of a wavelet system are needed, which usually can be characterized by conditions on the low-pass filter a_0 for ϕ and the high-pass filters a_ℓ for ψ^ℓ, $\ell=1,\ldots,L$. Among all properties of a wavelet system, high order of vanishing moments, (bi)orthogonality, and symmetry are highly desirable properties in wavelet and filter bank applications. High order of vanishing moments is crucial for the sparsity representation of a wavelet system, which plays an important role in image denoising and compression. (Bi)orthogonality (more general, tightness of a wavelet system) results in simple rules for guaranteeing the perfect reconstruction property. Symmetry usually produces better visual effect and less artifact in signal/image processing; not to mention the double reduction of the computational cost for a symmetric system.

A framelet ψ has *vanishing moments of order n* if

$$\int_{\mathbb{R}} t^k\psi(t)\mathrm{d}t = 0 \qquad k=0,\ldots,n-1, \tag{13}$$

which is equivalent to saying that $\frac{\mathrm{d}^k}{\mathrm{d}t^k}\widehat{\psi}(0)=0$ for all $k=0,\ldots,n-1$. If (12) holds and the low-pass filter a_0 satisfies

$$1-|\widehat{a_0}(\xi)|^2 = O(|\xi|^{2n}), \quad \xi\to 0, \tag{14}$$

which means $1-|\widehat{a_0}(\xi)|^2$ has *zero of order* $2n$ near the origin, then the framelet system generated by $\{\psi^1,\ldots,\psi^L\}$ has vanishing moments of order n (see [6]). We shall see in Sect. 3 on the connection of tight frames to the orthogonal matrix extension problem and on the construction of symmetric complex tight framelets with high order of vanishing moments via the technique of matrix extension with symmetry.

Next, let us review the construction of tight d-multiframes in the point of view of filters and filter banks. Let $\mathbb{F}$ be a subfield of $\mathbb{C}$. Let $a_0 : \mathbb{Z} \to \mathbb{F}^{r\times r}$ be a low-pass filter with *multiplicity* r for a d-refinable function vector $\phi = [\phi_1, \dots, \phi_r]^{\mathrm{T}}$. The d-*band subsymbols (polyphase components)* of a_0 are defined to be

$$\mathsf{a}_{0;\gamma}(z) := \sqrt{\mathsf{d}} \sum_{k\in\mathbb{Z}} a_0(\gamma + \mathsf{d}k) z^k, \quad \gamma \in \mathbb{Z}. \tag{15}$$

Let $a_1, \dots, a_L : \mathbb{Z} \to \mathbb{F}^{r\times r}$ be high-pass filters for function vectors $\psi^1, \dots, \psi^L$, respectively. The *polyphase matrix* for the filter bank $\{a_0, a_1, \dots, a_L\}$ (or $\{\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_L\}$) is defined to be

$$\mathbf{P}(z) = \begin{bmatrix} \mathsf{a}_{0;0}(z) & \cdots & \mathsf{a}_{0;\mathsf{d}-1}(z) \\ \mathsf{a}_{1;0}(z) & \cdots & \mathsf{a}_{1;\mathsf{d}-1}(z) \\ \vdots & \vdots & \vdots \\ \mathsf{a}_{L;0}(z) & \cdots & \mathsf{a}_{L;\mathsf{d}-1}(z) \end{bmatrix}, \tag{16}$$

where $\mathsf{a}_{\ell;\gamma}$ are subsymbols of a_ℓ similarly defined as in (15) for $\gamma = 0, \dots, \mathsf{d}-1$ and $\ell = 1, \dots, L$.

If ϕ is a compactly supported d-refinable function vectors in $L_2(\mathbb{R})$, then it is well-known (see [6]) that $\{\psi^1, \dots, \psi^L\}$ associated with $\{a_1, \dots, a_L\}$ via (10) generates a tight d-multiframe, i.e., (12) holds, *if and only if*,

$$\mathbf{P}^*(z)\mathbf{P}(z) = I_{\mathsf{d}r}, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{17}$$

Note that the polyphase matrix $\mathbf{P}$ is not necessarily a square matrix (only if $L = \mathsf{d}-1$). When the d-refinable function vector ϕ associated with a low-pass filter a_0 is orthogonal, the multiframlet system generated by $\{\psi^1, \dots, \psi^{\mathsf{d}-1}\}$ via (10) becomes an *orthonormal multiwavelet basis* for $L_2(\mathbb{R})$. In this case, the polyphase matrix $\mathbf{P}$ associated with the filter bank $\{\mathsf{a}_0, \dots, \mathsf{a}_{\mathsf{d}-1}\}$ is indeed a square matrix. Moreover, the low-pass filter a_0 for ϕ is a d-*band orthogonal filter*:

$$\sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{0;\gamma}(z)\mathsf{a}^*_{0;\gamma}(z) = I_r, \qquad z \in \mathbb{C}\backslash\{0\}. \tag{18}$$

Now, one can show that the derivation of high-pass filters $\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}$ from a_0 so that the filter bank $\{\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}\}$ has the *perfect reconstruction property* as in (17) is simply a special case of Problem 1 (orthogonal matrix extension). More generally, for $L = \mathsf{d}-1$, one can consider the construction of *biorthogonal multiwavelets* (see Sect. 4), which corresponds to Problem 2. Our main focus of this paper is on matrix extension with symmetry with respect to Problems 1 and 2. We shall study in Sects. 2 and 4 on the orthogonal matrix extension problem and the biorthogonal matrix extension problem, respectively.

1.3 Prior Work and Our Contributions

Without considering symmetry issue, it is known in the engineering literature that Problem 1 or 2 can be solved by representing the given matrices in *cascade structures*; see [19, 24]. In the context of wavelet analysis, orthogonal matrix extension without symmetry was discussed by Lawton, Lee, and Shen in their paper [20]. In electronic engineering, an algorithm using the cascade structure for orthogonal matrix extension without symmetry was given in [24] for filter banks with perfect reconstruction property. The algorithms in [20,24] mainly deal with the special case that $\mathbb{P}$ is a row vector (that is, $r = 1$ in our case) without symmetry, and the coefficient support of the derived matrix $\mathbb{P}_e$ indeed can be controlled by that of $\mathbb{P}$. The algorithms in [20,24] for the special case $r = 1$ can be employed to handle a general $r \times s$ matrix $\mathbb{P}$ without symmetry; see [20, 24] for detail. However, for the general case $r > 1$, it is no longer clear whether the coefficient support of the derived matrix $\mathbb{P}_e$ obtained by the algorithms in [20, 24] can still be controlled by that of $\mathbb{P}$. For $r = 1$, Goh et al. in [9] considered the biorthogonal matrix extension problem without symmetry. They provided a step-by-step algorithm for deriving the extension matrices, yet they did not concern about the support control of the extension matrices nor the symmetry patterns of the extension matrices. For $r > 1$, there are only a few results in the literature [1,4] and most of them only consider about some very special cases. The difficulty comes from the flexibility of the biorthogonality relation between the given pair $(\mathbb{P}, \widetilde{\mathbb{P}})$ of biorthogonal matrices.

Several special cases of matrix extension with symmetry were considered in the literature. For $\mathbb{F} = \mathbb{R}$ and $r = 1$, orthogonal matrix extension with symmetry was considered in [21]. For $r = 1$, orthogonal matrix extension with symmetry was studied in [12] and a simple algorithm is given there. In the context of wavelet analysis, several particular cases of matrix extension with symmetry related to the construction of (bi)orthogonal multiwavelets were investigated in [1, 3, 10, 12, 19, 21]. However, for the general case of an $r \times s$ matrix, the approaches on orthogonal matrix extension with symmetry in [12,21] for the particular case $r = 1$ cannot be employed to handle the general case. The algorithms in [12, 21] are very difficult to be generalized to the general case $r > 1$, partially due to the complicated relations of the symmetry patterns between different rows of $\mathbb{P}$. For the general case of matrix extension with symmetry, it becomes much harder to control the coefficient support of the derived matrix $\mathbb{P}_e$, comparing with the special case $r = 1$. Extra effort is needed in any algorithm of deriving $\mathbb{P}_e$ so that its coefficient support can be controlled by that of $\mathbb{P}$.

The contributions of this paper lie in the following aspects. First, we satisfactorily solve the matrix extension problems with symmetry for any r, s such that $1 \le r \le s$. More importantly, we obtain a complete representation for any $r \times s$ paraunitary matrix $\mathbb{P}$ or pairs of biorthogonal matrices $(\mathbb{P}, \widetilde{\mathbb{P}})$ having compatible symmetry with $1 \le r \le s$. This representation leads to step-by-step algorithms for deriving a desired matrix $\mathbb{P}_e$ or the pair of extension matrices $(\mathbb{P}_e, \widetilde{\mathbb{P}}_e)$ from a given matrix $\mathbb{P}$ or a pair $(\mathbb{P}, \widetilde{\mathbb{P}})$. Second, we obtain an optimal result in the sense of (21) on controlling the coefficient support of the desired matrix $\mathbb{P}_e$ derived from a given matrix $\mathbb{P}$ by our algorithm for orthogonal matrix extension with symmetry. This is of importance in both theory and application, since short support of a filter or a multiwavelet is a highly desirable property and short support usually means a fast algorithm and simple implementation in practice. Third, we introduce the notion of compatibility of symmetry, which plays a critical role in the study of the general matrix extension problems with symmetry ($r \geq 1$). Fourth, we provide a complete analysis and a systematic construction algorithm for symmetric filter banks with the perfect reconstruction property and symmetric (bi)orthogonal multiwavelets. Finally, most of the literature on the matrix extension problem only consider Laurent polynomials with coefficients in the special field $\mathbb{C}$ (see [20]) or $\mathbb{R}$ (see [2,21]). In this paper, our setting is under a general field $\mathbb{F}$, which can be any subfield of $\mathbb{C}$ satisfies certain conditions (see (19) for the case of orthogonal matrix extension).

1.4 Outline

Here is the structure of this paper. In Sect. 2, we shall study the orthogonal matrix extension with symmetry and present a step-by-step algorithm for this problem. We shall also apply our algorithm in this section to the design of symmetric filter banks in electronic engineering and to the construction of symmetric orthonormal multiwavelets in wavelet analysis. In Sect. 3, we shall discuss the construction of symmetric complex tight framelets with high order of vanishing moments and with symmetry via our algorithm for orthogonal matrix extension with symmetry. In Sect. 4, we shall study the biorthogonal matrix extension problem corresponding to the construction of symmetric biorthogonal multiwavelets. We also provide a step-by-step algorithm for the construction of the desired pair of biorthogonal extension matrices. Examples will be provided to illustrate our algorithms and results.

2 Orthogonal Matrix Extension with Symmetry

In this section, we shall study the orthogonal matrix extension problem with symmetry. The Laurent polynomials that we shall consider in this section have their coefficients in a subfield $\mathbb{F}$ of the complex field $\mathbb{C}$ such that $\mathbb{F}$ is closed under the operations of complex conjugate of $\mathbb{F}$ and square roots of positive numbers in $\mathbb{F}$. In other words, the subfield $\mathbb{F}$ of $\mathbb{C}$ satisfies the following properties:

$$\bar{x} \in \mathbb{F} \quad \text{and} \quad \sqrt{y} \in \mathbb{F} \qquad \forall\, x, y \in \mathbb{F} \quad \text{with} \quad y > 0. \tag{19}$$

Two particular examples of such subfields $\mathbb{F}$ are $\mathbb{F} = \mathbb{R}$ (the field of real numbers) and $\mathbb{F} = \mathbb{C}$ (the field of complex numbers). A nontrivial example is the field of all algebraic number, i.e., the algebraic closure $\overline{\mathbb{Q}}$ of the rational number $\mathbb{Q}$. A subfield of $\mathbb{R}$ given by $\overline{\mathbb{Q}} \cap \mathbb{R}$ also satisfies (19).

Problem 1 is completely solved by the following theorem.

Theorem 1. *Let $\mathbb{F}$ be a subfield of $\mathbb{C}$ such that (19) holds. Let $\mathbb{P}$ be an $r \times s$ matrix of Laurent polynomials with coefficients in the subfield $\mathbb{F}$ such that the symmetry of $\mathbb{P}$ is compatible, i.e., $\mathsf{S}\mathbb{P} = (\mathsf{S}\theta_1)^*\mathsf{S}\theta_2$ for some $1 \times r$, $1 \times s$ vectors θ_1, θ_2 of Laurent polynomials with symmetry. Then $\mathbb{P}(z)\mathbb{P}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$ (that is, $\mathbb{P}$ is paraunitary), if and only if, there exists an $s \times s$ square matrix $\mathbb{P}_e$ of Laurent polynomials with coefficients in $\mathbb{F}$ such that*

(1) $[I_r, \mathbf{0}]\mathbb{P}_e = \mathbb{P}$*; that is, the submatrix of the first r rows of $\mathbb{P}_e$ is $\mathbb{P}$;*
(2) $\mathbb{P}_e$ *is paraunitary:* $\mathbb{P}_e(z)\mathbb{P}_e^*(z) = I_s$ *for all* $z \in \mathbb{C}\backslash\{0\}$*;*
(3) *The symmetry of $\mathbb{P}_e$ is compatible: $\mathsf{S}\mathbb{P}_e = (\mathsf{S}\theta)^*\mathsf{S}\theta_2$ for some $1 \times s$ vector θ of Laurent polynomials with symmetry;*
(4) *$\mathbb{P}_e$ can be represented as products of some $s \times s$ matrices $\mathbb{P}_0, \mathbb{P}_1, \ldots, \mathbb{P}_{J+1}$ of Laurent polynoimals with coefficient in $\mathbb{F}$:*

$$\mathbb{P}_e(z) = \mathbb{P}_{J+1}(z)\mathbb{P}_J(z)\cdots\mathbb{P}_1(z)\mathbb{P}_0(z); \tag{20}$$

(5) $\mathbb{P}_j, 1 \le j \le J$ *are elementary:* $\mathbb{P}_j(z)\mathbb{P}_j^*(z) = I_s$ *and* $\mathrm{csupp}(\mathbb{P}_j) \subseteq [-1, 1]$*;*
(6) $(\mathbb{P}_{j+1}, \mathbb{P}_j)$ *has mutually compatible symmetry for all* $0 \le j \le J$*;*
(7) $\mathbb{P}_0 = \mathsf{U}_{\mathsf{S}\theta_2}^*$ *and* $\mathbb{P}_{J+1} = diag(\mathsf{U}_{\mathsf{S}\theta_1}, I_{s-r})$*, where* $\mathsf{U}_{\mathsf{S}\theta_1}$*,* $\mathsf{U}_{\mathsf{S}\theta_2}$ *are products of a permutation matrix with a diagonal matrix of monomials, as defined in (23);*
(8) *The coefficient support of $\mathbb{P}_e$ is controlled by that of $\mathbb{P}$ in the following sense:*

$$|\mathrm{csupp}([\mathbb{P}_e]_{j,k})| \le \max_{1 \le n \le r} |\mathrm{csupp}([\mathbb{P}]_{n,k})|, \qquad 1 \le j, k \le s. \tag{21}$$

The representation in (20) is often called the *cascade structure* in the literature of engineering, see [19, 24]. The key of Theorem 1 is to construct the elementary paraunitary matrices $\mathbb{P}_1, \ldots, \mathbb{P}_J$ step by step such that $\mathbb{P}_j$'s have the properties stated as in Items (4)–(7) of the theorem. We shall provide such a step-by-step algorithm next, which not only provides a detailed construction of such $\mathbb{P}_j$'s, but also leads to a constructive proof of Theorem 1. For a complete and detailed proof of Theorem 1 using our algorithm, one may refer to [16, Sect. 4].

2.1 An Algorithm for the Orthogonal Matrix Extension with Symmetry

Now we present a step-by-step algorithm on orthogonal matrix extension with symmetry to derive the desired matrix $\mathbb{P}_e$ in Theorem 1 from a given matrix $\mathbb{P}$. Our algorithm has three steps: initialization, support reduction, and finalization. The step of initialization reduces the symmetry pattern of $\mathbb{P}$ to a standard form. The step of support reduction is the main body of the algorithm, producing a sequence of elementary matrices $\mathsf{A}_1, \ldots, \mathsf{A}_J$ that reduce the length of the coefficient support of $\mathbb{P}$ to 0. The step of finalization generates the desired matrix $\mathbb{P}_e$ as in Theorem 1. More precisely, see Algorithm 1 for our algorithm written in the form of *pseudo-code*.

Algorithm 1 Orthogonal matrix extension with symmetry

(a) **Input**: $\mathbb{P}$ as in Theorem 1 with $\mathsf{S}\mathbb{P} = (\mathsf{S}\theta_1)^*\mathsf{S}\theta_2$ for some $1\times r$ and $1\times s$ row vectors θ_1 and θ_2 of Laurent polynomials with symmetry.

(b) **Initialization**: Let $\mathsf{Q} := \mathsf{U}^*_{\mathsf{S}\theta_1}\mathbb{P}\mathsf{U}_{\mathsf{S}\theta_2}$. Then the symmetry pattern of Q is

$$\mathsf{S}\mathsf{Q} = [1_{r_1}, -1_{r_2}, z1_{r_3}, -z1_{r_4}]^{\mathrm{T}}[1_{s_1}, -1_{s_2}, z^{-1}1_{s_3}, -z^{-1}1_{s_4}], \tag{22}$$

where all nonnegative integers $r_1,\dots,r_4,s_1,\dots,s_4$ are uniquely determined by $\mathsf{S}\mathbb{P}$.

(c) **Support Reduction**: Let $\mathbb{P}_0 := \mathsf{U}^*_{\mathsf{S}\theta_2}$ and $J := 1$.

while $(|\mathrm{csupp}(\mathsf{Q})| > 0)$ **do**
  Let $\mathsf{Q}_0 := \mathsf{Q}$, $[k_1,k_2] := \mathrm{csupp}(\mathsf{Q})$, and $\mathsf{A}_J := I_s$.
  if $k_2 = -k_1$ **then**
    for $j = 1$ to r **do**
      Let $\mathsf{q} := [\mathsf{Q}_0]_{j,:}$ and $\mathsf{p} := [\mathsf{Q}]_{j,:}$ be the jth rows of Q_0 and Q, respectively. Let $[\ell_1,\ell_2] := \mathrm{csupp}(\mathsf{q})$, $\ell := \ell_2 - \ell_1$, and $\mathsf{B}_j := I_s$.
      if $\mathrm{csupp}(\mathsf{q}) = \mathrm{csupp}(\mathsf{p})$ and $\ell \ge 2$ and $(\ell_1 = k_1$ or $\ell_2 = k_2)$ **then**
        $\mathsf{B}_j := \mathsf{B}_{\mathsf{q}}$. $\mathsf{A}_J := \mathsf{A}_J\mathsf{B}_j$. $\mathsf{Q}_0 := \mathsf{Q}_0\mathsf{B}_j$.
      end if
    end for
    Q_0 takes the form in (31). Let $\mathsf{B}_{(-k_2,k_2)} := I_s$, $\mathsf{Q}_1 := \mathsf{Q}_0$, $j_1 := 1$ and $j_2 := r_3 + r_4 + 1$.
    while $j_1 \le r_1 + r_2$ and $j_2 \le r$ **do**
      Let $\mathsf{q}_1 := [\mathsf{Q}_1]_{j_1,:}$ and $\mathsf{q}_2 := [\mathsf{Q}_1]_{j_2,:}$.
      if $\mathrm{coeff}(\mathsf{q}_1,k_1) = 0$ **then** $j_1 := j_1 + 1$. **end if**
      if $\mathrm{coeff}(\mathsf{q}_2,k_2) = 0$ **then** $j_2 := j_2 + 1$. **end if**
      if $\mathrm{coeff}(\mathsf{q}_1,k_1) \ne \mathbf{0}$ and $\mathrm{coeff}(\mathsf{q}_2,k_2) \ne \mathbf{0}$ **then**
        $\mathsf{B}_{(-k_2,k_2)} := \mathsf{B}_{(-k_2,k_2)}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$. $\mathsf{Q}_1 := \mathsf{Q}_1\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$. $\mathsf{A}_J := \mathsf{A}_J\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$. $j_1 := j_1 + 1$. $j_2 := j_2 + 1$.
      end if
    end while *// end inner while loop*
  end if
  Q_1 takes the form in (31) with either $\mathrm{coeff}(\mathsf{Q}_1,-k) = 0$ or $\mathrm{coeff}(\mathsf{Q}_1,k) = 0$. Let $\mathsf{A}_J := \mathsf{A}_J\mathsf{B}_{\mathsf{Q}_1}$ and $\mathsf{Q} := \mathsf{Q}\mathsf{A}_J$. Then

$$\mathsf{S}\mathsf{Q} = [1_{r_1}, -1_{r_2}, z1_{r_3}, -z1_{r_4}]^{\mathrm{T}}[1_{s'_1}, -1_{s'_2}, z^{-1}1_{s'_3}, -z^{-1}1_{s'_4}].$$

  Replace $s_1,\dots,s_4$ by $s'_1,\dots,s'_4$, respectively. Let $\mathbb{P}_J := \mathsf{A}^*_J$ and $J := J+1$.
end while *// end outer while loop*

(d) **Finalization**: $\mathsf{Q} = \mathrm{diag}(F_1,F_2,F_3,F_4)$ for some $r_j \times s_j$ constant matrices F_j in $\mathbb{F}$, $j = 1,\dots,4$. Let $U := \mathrm{diag}(U_{F_1},U_{F_2},U_{F_3},U_{F_4})$ so that $\mathsf{Q}U = [I_r, 0]$. Define $\mathbb{P}_J := U^*$ and $\mathbb{P}_{J+1} := \mathrm{diag}(\mathsf{U}_{\mathsf{S}\theta_1}, I_{s-r})$.

(e) **Output**: A desired matrix $\mathbb{P}_e$ satisfying all the properties in Theorem 1

In the following subsections, we present detailed constructions of the matrices $\mathsf{U}_{\mathsf{S}\theta}$, B_{q}, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, $\mathsf{B}_{\mathsf{Q}_1}$, and U_F appearing in Algorithm 1.

2.1.1 Initialization

Let θ be a $1\times n$ row vector of Laurent polynomials with symmetry such that $\mathsf{S}\theta = [\varepsilon_1 z^{c_1},\dots,\varepsilon_n z^{c_n}]$ for some $\varepsilon_1,\dots,\varepsilon_n \in \{-1,1\}$ and $c_1,\dots,c_n \in \mathbb{Z}$. Then, the symmetry of any entry in the vector $\theta\mathrm{diag}(z^{-\lceil c_1/2\rceil},\dots,z^{-\lceil c_n/2\rceil})$ belongs to $\{\pm 1,\pm z^{-1}\}$. Thus, there is a permutation matrix E_θ to regroup these four types of symmetries together so that

$$\mathsf{S}(\theta \mathsf{U}_{\mathsf{S}\theta}) = [\mathbf{1}_{n_1}, -\mathbf{1}_{n_2}, z^{-1}\mathbf{1}_{n_3}, -z^{-1}\mathbf{1}_{n_4}], \tag{23}$$

where $\mathsf{U}_{\mathsf{S}\theta} := \mathrm{diag}(z^{-\lceil c_1/2\rceil},\dots,z^{-\lceil c_n/2\rceil})E_\theta$, $\mathbf{1}_m$ denotes the $1\times m$ row vector $[1,\dots,1]$, and $n_1,\dots,n_4$ are nonnegative integers uniquely determined by $\mathsf{S}\theta$. Since $\mathbb{P}$ satisfies (4), $\mathsf{Q} := \mathsf{U}^*_{\mathsf{S}\theta_1}\mathbb{P}\mathsf{U}_{\mathsf{S}\theta_2}$ has the symmetry pattern as in (22). Note that $\mathsf{U}_{\mathsf{S}\theta_1}$ and $\mathsf{U}_{\mathsf{S}\theta_2}$ do not increase the length of the coefficient support of $\mathbb{P}$.

2.1.2 Support Reduction

For a $1\times n$ row vector $\mathtt{f}$ in $\mathbb{F}$ such that $\|\mathtt{f}\|\ne 0$, we define $n_{\mathtt{f}}$ to be the number of nonzero entries in $\mathtt{f}$ and $\varepsilon_j := [0,\dots,0,1,0,\dots,0]$ to be the jth unit coordinate row vector in $\mathbb{R}^n$. Let $E_{\mathtt{f}}$ be a permutation matrix such that $\mathtt{f}E_{\mathtt{f}} = [f_1,\dots,f_{n_{\mathtt{f}}},0,\dots,0]$ with $f_j\ne 0$ for $j=1,\dots,n_{\mathtt{f}}$. We define

$$V_{\mathtt{f}} := \begin{cases} \frac{\bar f_1}{|f_1|}, & \text{if } n_{\mathtt{f}} = 1;\\ \frac{\bar f_1}{|f_1|}\left(I_n - \frac{2}{\|v_{\mathtt{f}}\|^2}v_{\mathtt{f}}^* v_{\mathtt{f}}\right), & \text{if } n_{\mathtt{f}} > 1, \end{cases} \tag{24}$$

where $v_{\mathtt{f}} := \mathtt{f} - \frac{f_1}{|f_1|}\|\mathtt{f}\|\varepsilon_1$. Observing that $\|v_{\mathtt{f}}\|^2 = 2\|\mathtt{f}\|(\|\mathtt{f}\|-|f_1|)$, we can verify that $V_{\mathtt{f}}V_{\mathtt{f}}^* = I_n$ and $\mathtt{f}E_{\mathtt{f}}V_{\mathtt{f}} = \|\mathtt{f}\|\varepsilon_1$. Let $U_{\mathtt{f}} := E_{\mathtt{f}}V_{\mathtt{f}}$. Then $U_{\mathtt{f}}$ is unitary and satisfies $U_{\mathtt{f}} = [\frac{\mathtt{f}^*}{\|\mathtt{f}\|}, F^*]$ for some $(n-1)\times n$ matrix F in $\mathbb{F}$ such that $\mathtt{f}U_{\mathtt{f}} = [\|\mathtt{f}\|,0,\dots,0]$. We also define $U_{\mathtt{f}} := I_n$ if $\mathtt{f}=0$ and $U_{\mathtt{f}} := \emptyset$ if $\mathtt{f}=\emptyset$. Here, $U_{\mathtt{f}}$ plays the role of reducing the number of nonzero entries in $\mathtt{f}$. More generally, for an $r\times n$ nonzero matrix G of rank m in $\mathbb{F}$, employing the above procedure to each row of G, we can obtain an $n\times n$ unitary matrix U_G such that $GU_G = [R,0]$ for some $r\times m$ lower triangular matrix R of rank m. If $G_1G_1^* = G_2G_2^*$, then the above procedure produces two matrices U_{G_1}, U_{G_2} such that $G_1U_{G_1} = [R,0]$ and $G_2U_{G_2} = [R,0]$ for some lower triangular matrix R of full rank. It is important to notice that the constructions of $U_{\mathtt{f}}$ and U_G only involve the nonzero entries of $\mathtt{f}$ and nonzero columns of G, respectively. In other words, up to a permutation, we have

$$\begin{aligned} [U_{\mathtt{f}}]_{j,:} &= ([U_{\mathtt{f}}]_{:,j})^{\mathrm{T}} = \varepsilon_j, \text{ if } [\mathtt{f}]_j = 0,\\ [U_G]_{j,:} &= ([U_G]_{:,j})^{\mathrm{T}} = \varepsilon_j, \text{ if } [G]_{:,j} = 0. \end{aligned} \tag{25}$$

Denote $\mathsf{Q} := \mathsf{U}^*_{\mathsf{S}\theta_1}\mathbb{P}\mathsf{U}_{\mathsf{S}\theta_2}$ as in Algorithm 1. The outer *while* loop produces a sequence of elementary paraunitary matrices $\mathsf{A}_1,\dots,\mathsf{A}_J$ that reduce the length of the coefficient support of Q gradually to 0. The construction of each A_j has three parts: $\{\mathsf{B}_1,\dots,\mathsf{B}_r\}$, $\mathsf{B}_{(-k,k)}$, and $\mathsf{B}_{\mathsf{Q}_1}$. The first part $\{\mathsf{B}_1,\dots,\mathsf{B}_r\}$ (see the *for* loop) is constructed recursively for each of the r rows of Q so that $\mathsf{Q}_0 := \mathsf{Q}\mathsf{B}_1\cdots\mathsf{B}_r$ has a special form as in (31). If both $\mathrm{coeff}(\mathsf{Q}_0,-k)\ne 0$ and $\mathrm{coeff}(\mathsf{Q}_0,k)\ne 0$, then the second part

$\mathsf{B}_{(-k,k)}$ (see the inner *while* loop) is further constructed so that $\mathsf{Q}_1 := \mathsf{Q}_0\mathsf{B}_{(-k,k)}$ takes the form in (31) with at least one of $\mathrm{coeff}(\mathsf{Q}_1,-k)$ and $\mathrm{coeff}(\mathsf{Q}_1,k)$ being 0. $\mathsf{B}_{\mathsf{Q}_1}$ is constructed to handle the case that $\mathrm{csupp}(\mathsf{Q}_1) = [-k,k-1]$ or $\mathrm{csupp}(\mathsf{Q}_1) = [-k+1,k]$ so that $\mathrm{csupp}(\mathsf{Q}_1\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-k+1,k-1]$.

Let q denote an arbitrary row of Q with $|\mathrm{csupp}(\mathsf{q})| \geq 2$. We first explain how to construct B_q for a given row q such that B_q reduces the length of the coefficient support of q by 2 and keeps its symmetry pattern. Note that in the *for* loop, B_j is simply B_q with q being the current jth row of $\mathsf{Q}\mathsf{B}_0\cdots\mathsf{B}_{j-1}$, where $\mathsf{B}_0 := I_s$.

By (22), we have $\mathsf{S}\mathsf{q} = \varepsilon z^c[\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$ for some $\varepsilon \in \{-1,1\}$ and $c \in \{0,1\}$. For $\varepsilon = -1$, there is a permutation matrix E_ε such that $\mathsf{S}(\mathsf{q}E_\varepsilon) = z^c[\mathbf{1}_{s_2}, -\mathbf{1}_{s_1}, z^{-1}\mathbf{1}_{s_4}, -z^{-1}\mathbf{1}_{s_3}]$. For $\varepsilon = 1$, we let $E_\varepsilon := I_s$. Then, $\mathsf{q}E_\varepsilon$ must take the form in either (26) or (27) with $\mathtt{f}_1 \neq 0$ as follows:

$$
\begin{aligned}
\mathsf{q}E_\varepsilon =& [\mathtt{f}_1, -\mathtt{f}_2, \mathtt{g}_1, -\mathtt{g}_2]z^{\ell_1} + [\mathtt{f}_3, -\mathtt{f}_4, \mathtt{g}_3, -\mathtt{g}_4]z^{\ell_1+1} + \sum_{\ell=\ell_1+2}^{\ell_2-2} \mathrm{coeff}(\mathsf{q}E_\varepsilon,\ell)z^\ell \\
&+ [\mathtt{f}_3, \mathtt{f}_4, \mathtt{g}_1, \mathtt{g}_2]z^{\ell_2-1} + [\mathtt{f}_1, \mathtt{f}_2, \mathbf{0}, 0]z^{\ell_2};
\end{aligned}
\tag{26}
$$

$$
\begin{aligned}
\mathsf{q}E_\varepsilon =& [0, 0, \mathtt{f}_1, -\mathtt{f}_2]z^{\ell_1} + [\mathtt{g}_1, -\mathtt{g}_2, \mathtt{f}_3, -\mathtt{f}_4]z^{\ell_1+1} + \sum_{\ell=\ell_1+2}^{\ell_2-2} \mathrm{coeff}(\mathsf{q}E_\varepsilon,\ell)z^\ell \\
&+ [\mathtt{g}_3, \mathtt{g}_4, \mathtt{f}_3, \mathtt{f}_4]z^{\ell_2-1} + [\mathtt{g}_1, \mathtt{g}_2, \mathtt{f}_1, \mathtt{f}_2]z^{\ell_2}.
\end{aligned}
\tag{27}
$$

If $\mathsf{q}E_\varepsilon$ takes the form in (27), we further construct a permutation matrix E_q such that $[\mathtt{g}_1, \mathtt{g}_2, \mathtt{f}_1, \mathtt{f}_2]E_\mathsf{q} = [\mathtt{f}_1, \mathtt{f}_2, \mathtt{g}_1, \mathtt{g}_2]$ and define $\mathsf{U}_{\mathsf{q},\varepsilon} := E_\varepsilon E_\mathsf{q}\,\mathrm{diag}(I_{s-s_\mathtt{g}}, z^{-1}I_{s_\mathtt{g}})$, where $s_\mathtt{g}$ is the size of the row vector $[\mathtt{g}_1, \mathtt{g}_2]$. Then, $\mathsf{q}\mathsf{U}_{\mathsf{q},\varepsilon}$ takes the form in (26). For $\mathsf{q}E_\varepsilon$ of form (26), we simply let $\mathsf{U}_{\mathsf{q},\varepsilon} := E_\varepsilon$. In this way, $\mathsf{q}_0 := \mathsf{q}\mathsf{U}_{\mathsf{q},\varepsilon}$ always takes the form in (26) with $\mathtt{f}_1 \neq 0$.

Note that $\mathsf{U}_{\mathsf{q},\varepsilon}\mathsf{U}_{\mathsf{q},\varepsilon}^* = I_s$ and $\|\mathtt{f}_1\| = \|\mathtt{f}_2\|$ if $\mathsf{q}_0\mathsf{q}_0^* = 1$, where $\|\mathtt{f}\| := \sqrt{\mathtt{f}\mathtt{f}^*}$. Now we construct an $s \times s$ paraunitary matrix $\mathsf{B}_{\mathsf{q}_0}$ to reduce the coefficient support of q_0 as in (26) from $[\ell_1, \ell_2]$ to $[\ell_1+1, \ell_2-1]$ as follows:

$$
\mathsf{B}_{\mathsf{q}_0}^* := \frac{1}{c}\left[\begin{array}{c|c|c|c}
\mathtt{f}_1(z+\frac{c_0}{c_{\mathtt{f}_1}}+\frac{1}{z}) & \mathtt{f}_2(z-\frac{1}{z}) & \mathtt{g}_1(1+\frac{1}{z}) & \mathtt{g}_2(1-\frac{1}{z}) \\
cF_1 & 0 & 0 & 0 \\
\hline
-\mathtt{f}_1(z-\frac{1}{z}) & -\mathtt{f}_2(z-\frac{c_0}{c_{\mathtt{f}_1}}+\frac{1}{z}) & -\mathtt{g}_1(1-\frac{1}{z}) & -\mathtt{g}_2(1+\frac{1}{z}) \\
0 & cF_2 & 0 & 0 \\
\hline
\frac{c_{\mathtt{g}_1}}{c_{\mathtt{f}_1}}\mathtt{f}_1(1+z) & -\frac{c_{\mathtt{g}_1}}{c_{\mathtt{f}_1}}\mathtt{f}_2(1-z) & c_{\mathtt{g}_1'}\mathtt{g}_1' & 0 \\
0 & 0 & cG_1 & 0 \\
\hline
\frac{c_{\mathtt{g}_2}}{c_{\mathtt{f}_1}}\mathtt{f}_1(1-z) & -\frac{c_{\mathtt{g}_2}}{c_{\mathtt{f}_1}}\mathtt{f}_2(1+z) & 0 & c_{\mathtt{g}_2'}\mathtt{g}_2' \\
0 & 0 & 0 & cG_2
\end{array}\right], \tag{28}
$$

where $c_{\mathtt{f}_1} := \|\mathtt{f}_1\|, c_{\mathtt{g}_1} := \|\mathtt{g}_1\|, c_{\mathtt{g}_2} := \|\mathtt{g}_2\|, c_0 := \frac{1}{c_{\mathtt{f}_1}}\mathrm{coeff}(\mathsf{q}_0, \ell_1+1)\mathrm{coeff}(\mathsf{q}_0^*, -\ell_2)$,

$$\begin{aligned}
&c_{\mathtt{g}_1'} := \begin{cases} \frac{-2c_{\mathtt{f}_1} - \overline{c_0}}{c_{\mathtt{g}_1}} & \text{if } \mathtt{g}_1 \neq 0; \\ c & \text{otherwise,} \end{cases} \qquad c_{\mathtt{g}_2'} := \begin{cases} \frac{2c_{\mathtt{f}_1} - \overline{c_0}}{c_{\mathtt{g}_2}} & \text{if } \mathtt{g}_2 \neq 0; \\ c & \text{otherwise,} \end{cases} \\
&c := (4c_{\mathtt{f}_1}^2 + 2c_{\mathtt{g}_1}^2 + 2c_{\mathtt{g}_2}^2 + |c_0|^2)^{1/2},
\end{aligned} \tag{29}$$

and $[\frac{\mathtt{f}_j^*}{\|\mathtt{f}_j\|}, F_j^*] = U_{\mathtt{f}_j}$, $[\mathtt{g}_j'^*, G_j^*] = U_{\mathtt{g}_j}$ for $j = 1, 2$ are unitary constant extension matrices in $\mathbb{F}$ for vectors $\mathtt{f}_j, \mathtt{g}_j$ in $\mathbb{F}$, respectively. Here, the role of a unitary constant matrix $U_{\mathtt{f}}$ in $\mathbb{F}$ is to reduce the number of nonzero entries in $\mathtt{f}$ such that $\mathtt{f}U_{\mathtt{f}} = [\|\mathtt{f}\|, 0, \ldots, 0]$. The operations for the emptyset $\emptyset$ are defined by $\|\emptyset\| = \emptyset$, $\emptyset + A = A$ and $\emptyset \cdot A = \emptyset$ for any object A.

Define $\mathsf{B}_\mathsf{q} := \mathsf{U}_{\mathsf{q},\varepsilon} \mathsf{B}_{\mathsf{q}_0} \mathsf{U}_{\mathsf{q},\varepsilon}^*$. Then, B_q is paraunitary. Due to the particular form of $\mathsf{B}_{\mathsf{q}_0}$ as in (28), direct computations yield the following very important properties of the paraunitary matrix B_q:

(P1) $\mathsf{SB}_\mathsf{q} = [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z\mathbf{1}_{s_3}, -z\mathbf{1}_{s_4}]^{\mathrm{T}} [\mathbf{1}_{s_1}, -\mathbf{1}_{s_2}, z^{-1}\mathbf{1}_{s_3}, -z^{-1}\mathbf{1}_{s_4}]$, $\mathrm{csupp}(\mathsf{B}_\mathsf{q}) = [-1, 1]$, and $\mathrm{csupp}(\mathsf{qB}_\mathsf{q}) = [\ell_1+1, \ell_2-1]$. That is, B_q has compatible symmetry with coefficient support on $[-1, 1]$ and B_q reduces the length of the coefficient support of q exactly by 2. Moreover, $\mathsf{S}(\mathsf{qB}_\mathsf{q}) = \mathsf{Sq}$.

(P2) If $(\mathsf{p}, \mathsf{q}^*)$ has mutually compatible symmetry and $\mathsf{pq}^* = 0$, then $\mathsf{S}(\mathsf{pB}_\mathsf{q}) = \mathsf{S}(\mathsf{p})$ and $\mathrm{csupp}(\mathsf{pB}_\mathsf{q}) \subseteq \mathrm{csupp}(\mathsf{p})$. That is, B_q keeps the symmetry pattern of p and does not increase the length of the coefficient support of p.

Next, let us explain the construction of $\mathsf{B}_{(-k,k)}$. For $\mathrm{csupp}(\mathsf{Q}) = [-k, k]$ with $k \geq 1$, Q is of the form as follows:

$$\begin{aligned}
\mathsf{Q} = &\left[\begin{array}{cccc} F_{11} & -F_{21} & G_{31} & -G_{41} \\ -F_{12} & F_{22} & -G_{32} & G_{42} \\ \hline 0 & 0 & F_{31} & -F_{41} \\ 0 & 0 & -F_{32} & F_{42} \end{array}\right] z^{-k} + \left[\begin{array}{cccc} F_{51} & -F_{61} & G_{71} & -G_{81} \\ -F_{52} & F_{61} & -G_{72} & G_{82} \\ \hline G_{11} & -G_{21} & F_{71} & -F_{81} \\ -G_{12} & G_{22} & -F_{72} & F_{82} \end{array}\right] z^{-k+1} \\
&+ \sum_{n=2-k}^{k-2} \mathrm{coeff}(\mathsf{Q}, n) z^n + \left[\begin{array}{cccc} F_{51} & F_{61} & G_{31} & G_{41} \\ F_{52} & F_{61} & G_{32} & G_{42} \\ \hline G_{51} & G_{61} & F_{71} & F_{81} \\ G_{52} & G_{62} & F_{72} & F_{82} \end{array}\right] z^{k-1} + \left[\begin{array}{cccc} F_{11} & F_{21} & 0 & 0 \\ F_{12} & F_{22} & 0 & 0 \\ \hline G_{11} & G_{21} & F_{31} & F_{41} \\ G_{12} & G_{22} & F_{32} & F_{42} \end{array}\right] z^k
\end{aligned} \tag{30}$$

with all F_{jk}'s and G_{jk}'s being constant matrices in $\mathbb{F}$ and $F_{11}, F_{22}, F_{31}, F_{42}$ being of size $r_1 \times s_1$, $r_2 \times s_2$, $r_3 \times s_3$, $r_4 \times s_4$, respectively. Due to Properties (P1) and (P2) of B_q, the *for* loop in Algorithm 1 reduces Q in (30) to $\mathsf{Q}_0 := \mathsf{QB}_1 \cdots \mathsf{B}_r$ as follows:

$$\left[\begin{array}{cccc} 0 & 0 & \widetilde{G}_{31} & -\widetilde{G}_{41} \\ 0 & 0 & -\widetilde{G}_{32} & \widetilde{G}_{42} \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] z^{-k} + \cdots + \left[\begin{array}{cccc} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline \widetilde{G}_{11} & \widetilde{G}_{21} & 0 & 0 \\ \widetilde{G}_{12} & \widetilde{G}_{22} & 0 & 0 \end{array}\right] z^k. \tag{31}$$

If either $\operatorname{coeff}(\mathsf{Q}_0,-k)=0$ or $\operatorname{coeff}(\mathsf{Q}_0,k)=0$, then the inner *while* loop does nothing and $\mathsf{B}_{(-k,k)}=I_s$. If both $\operatorname{coeff}(\mathsf{Q}_0,-k)\neq 0$ and $\operatorname{coeff}(\mathsf{Q}_0,k)\neq 0$, then $\mathsf{B}_{(-k,k)}$ is constructed recursively from pairs $(\mathsf{q}_1,\mathsf{q}_2)$ with $\mathsf{q}_1,\mathsf{q}_2$ being two rows of Q_0 satisfying $\operatorname{coeff}(\mathsf{q}_1,-k)\neq 0$ and $\operatorname{coeff}(\mathsf{q}_2,k)\neq 0$. The construction of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ with respect to such a pair $(\mathsf{q}_1,\mathsf{q}_2)$ in the inner *while* loop is as follows.

Similar to the discussion before (26), there is a permutation matrix $E_{(\mathsf{q}_1,\mathsf{q}_2)}$ such that $\mathsf{q}_1E_{(\mathsf{q}_1,\mathsf{q}_2)}$ and $\mathsf{q}_2E_{(\mathsf{q}_1,\mathsf{q}_2)}$ take the following form:

$$\begin{aligned}\begin{bmatrix}\widetilde{\mathsf{q}}_1\\ \widetilde{\mathsf{q}}_2\end{bmatrix} := \begin{bmatrix}\mathsf{q}_1\\ \mathsf{q}_2\end{bmatrix}E_{(\mathsf{q}_1,\mathsf{q}_2)} &= \left[\begin{array}{cccc} 0 & 0 & \widetilde{g}_3 & -\widetilde{g}_4\\ \hline 0 & 0 & 0 & 0\end{array}\right] z^{-k} + \left[\begin{array}{cccc} \widetilde{f}_5 & -\widetilde{f}_6 & \widetilde{g}_7 & -\widetilde{g}_8\\ \hline \widetilde{g}_1 & -\widetilde{g}_2 & \widetilde{f}_7 & -\widetilde{f}_8\end{array}\right] z^{-k+1}\\ &\quad + \sum_{n=2-k}^{k-2}\operatorname{coeff}\left(\begin{bmatrix}\widetilde{\mathsf{q}}_1\\ \widetilde{\mathsf{q}}_2\end{bmatrix},n\right)z^n + \left[\begin{array}{cccc} \widetilde{f}_5 & \widetilde{f}_6 & \widetilde{g}_3 & \widetilde{g}_4\\ \hline \widetilde{g}_5 & \widetilde{g}_6 & \widetilde{f}_7 & \widetilde{f}_8\end{array}\right] z^{k-1} + \left[\begin{array}{cccc} 0 & 0 & 0 & 0\\ \hline \widetilde{g}_1 & \widetilde{g}_2 & 0 & 0\end{array}\right] z^{k},\end{aligned} \tag{32}$$

where $\widetilde{\mathsf{g}}_1,\widetilde{\mathsf{g}}_2,\widetilde{\mathsf{g}}_3,\widetilde{\mathsf{g}}_4$ are all nonzero row vectors. Note that $\|\widetilde{\mathsf{g}}_1\|=\|\widetilde{\mathsf{g}}_2\|=:c_{\widetilde{\mathsf{g}}_1}$ and $\|\widetilde{\mathsf{g}}_3\|=\|\widetilde{\mathsf{g}}_4\|=:c_{\widetilde{\mathsf{g}}_3}$. Construct an $s\times s$ paraunitary matrix $\mathsf{B}_{(\widetilde{\mathsf{q}}_1,\widetilde{\mathsf{q}}_2)}$ as follows:

$$\mathsf{B}^*_{(\widetilde{\mathsf{q}}_1,\widetilde{\mathsf{q}}_2)} := \frac{1}{c}\left[\begin{array}{c|c|c|c}
\frac{c_0}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_1 & 0 & \widetilde{\mathsf{g}}_3(1+\frac{1}{z}) & \widetilde{\mathsf{g}}_4(1-\frac{1}{z})\\
c\widetilde{G}_1 & 0 & 0 & 0\\ \hline
0 & \frac{c_0}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_2 & -\widetilde{\mathsf{g}}_3(1-\frac{1}{z}) & -\widetilde{\mathsf{g}}_4(1+\frac{1}{z})\\
0 & c\widetilde{G}_2 & 0 & 0\\ \hline
\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_1(1+z) & -\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_2(1-z) & -\frac{\overline{c_0}}{c_{\widetilde{\mathsf{g}}_3}}\widetilde{\mathsf{g}}_3 & 0\\
0 & 0 & c\widetilde{G}_3 & 0\\ \hline
\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_1(1-z) & -\frac{c_{\widetilde{\mathsf{g}}_3}}{c_{\widetilde{\mathsf{g}}_1}}\widetilde{\mathsf{g}}_2(1+z) & 0 & -\frac{\overline{c_0}}{c_{\widetilde{\mathsf{g}}_3}}\widetilde{\mathsf{g}}_4\\
0 & 0 & 0 & c\widetilde{G}_4
\end{array}\right], \tag{33}$$

where $c_0:=\frac{1}{c_{\widetilde{\mathsf{g}}_1}}\operatorname{coeff}(\widetilde{\mathsf{q}}_1,-k+1)\operatorname{coeff}(\widetilde{\mathsf{q}}_2^*,-k)$, $c:=(|c_0|^2+4c_{\widetilde{\mathsf{g}}_3}^2)^{\frac{1}{2}}$, and $[\frac{\widetilde{\mathsf{g}}_j^*}{\|\widetilde{\mathsf{g}}_j\|},\widetilde{G}_j^*]=U_{\widetilde{\mathsf{g}}_j}$ are unitary constant extension matrices in $\mathbb{F}$ for vectors $\widetilde{\mathsf{g}}_j$ in $\mathbb{F}$, $j=1,\dots,4$, respectively. Let $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}:=E_{(\mathsf{q}_1,\mathsf{q}_2)}\mathsf{B}_{(\widetilde{\mathsf{q}}_1,\widetilde{\mathsf{q}}_2)}E^{\mathrm{T}}_{(\mathsf{q}_1,\mathsf{q}_2)}$. Similar to Properties (P1) and (P2) of B_{q}, we have the following very important properties of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$:

(P3) $\mathsf{S}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}=[\mathbf{1}_{s_1},-\mathbf{1}_{s_2},z\mathbf{1}_{s_3},-z\mathbf{1}_{s_4}]^{\mathrm{T}}[\mathbf{1}_{s_1},-\mathbf{1}_{s_2},z^{-1}\mathbf{1}_{s_3},-z^{-1}\mathbf{1}_{s_4}]$, $\operatorname{csupp}(\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)})=[-1,1]$, $\operatorname{csupp}(\mathsf{q}_1\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)})\subseteq[-k+1,k-1]$ and $\operatorname{csupp}(\mathsf{q}_2\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)})\subseteq[-k+1,k-1]$. That is, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ has compatible symmetry with coefficient support on

$[-1,1]$ and $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ reduces the length of both the coefficient supports of q_1 and q_2 by 2. Moreover, $\mathsf{S}(\mathsf{q}_1\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathsf{S}\mathsf{q}_1$ and $\mathsf{S}(\mathsf{q}_2\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathsf{S}\mathsf{q}_2$.

(P4) If both $(\mathsf{p},\mathsf{q}_1^*)$ and $(\mathsf{p},\mathsf{q}_2^*)$ have mutually compatible symmetry and $\mathsf{p}\mathsf{q}_1^* = \mathsf{p}\mathsf{q}_2^* = 0$, then $\mathsf{S}(\mathsf{p}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) = \mathsf{S}\mathsf{p}$ and $\mathrm{csupp}(\mathsf{p}\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}) \subseteq \mathrm{csupp}(\mathsf{p})$. That is, $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$ keeps the symmetry pattern of p and does not increase the length of the coefficient support of p.

Now, due to Properties (P3) and (P4) of $\mathsf{B}_{(\mathsf{q}_1,\mathsf{q}_2)}$, $\mathsf{B}_{(-k,k)}$ constructed in the inner *while* loop reduces Q_0 of the form in (31) with both $\mathrm{coeff}(\mathsf{Q}_0,-k) \neq 0$ and $\mathrm{coeff}(\mathsf{Q}_0,k) \neq 0$, to $\mathsf{Q}_1 := \mathsf{Q}_0\mathsf{B}_{(-k,k)}$ of the form in (31) with either $\mathrm{coeff}(\mathsf{Q}_1,-k) = \mathrm{coeff}(\mathsf{Q}_1,k) = 0$ (for this case, simply let $\mathsf{B}_{\mathsf{Q}_1} := I_s$) or one of $\mathrm{coeff}(\mathsf{Q}_1,-k)$ and $\mathrm{coeff}(\mathsf{Q}_1,k)$ is nonzero. For the latter case, $\mathsf{B}_{\mathsf{Q}_1} := \mathrm{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})E$ with U_1,W_1 constructed with respect to $\mathrm{coeff}(\mathsf{Q}_1,k) \neq 0$ or $\mathsf{B}_{\mathsf{Q}_1} := \mathrm{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)E$ with U_3,W_3 constructed with respect to $\mathrm{coeff}(\mathsf{Q}_1,-k) \neq 0$, where E is a permutation matrix. $\mathsf{B}_{\mathsf{Q}_1}$ is constructed so that $\mathrm{csupp}(\mathsf{Q}_1\mathsf{B}_{\mathsf{Q}_1}) \subseteq [-k+1,k-1]$. Let Q_1 take form in (31). The matrices U_1,W_1 or U_3,W_3, and E are constructed as follows.

Let $U_1 := \mathrm{diag}(U_{\widetilde{G}_1}, U_{\widetilde{G}_2})$ and $U_3 := \mathrm{diag}(U_{\widetilde{G}_3}, U_{\widetilde{G}_4})$ with

$$\widetilde{G}_1 := \begin{bmatrix} \widetilde{G}_{11} \\ \widetilde{G}_{12} \end{bmatrix}, \widetilde{G}_2 := \begin{bmatrix} \widetilde{G}_{21} \\ \widetilde{G}_{22} \end{bmatrix}, \widetilde{G}_3 := \begin{bmatrix} \widetilde{G}_{31} \\ \widetilde{G}_{32} \end{bmatrix}, \widetilde{G}_4 := \begin{bmatrix} \widetilde{G}_{41} \\ \widetilde{G}_{42} \end{bmatrix}. \tag{34}$$

Here, for a nonzero matrix G with rank m, U_G is a unitary matrix such that $GU_G = [R,0]$ for some matrix R of rank m. For $G = 0, U_G := I$ and for $G = \emptyset, U_G := \emptyset$. When $G_1G_1^* = G_2G_2^*$, U_{G_1} and U_{G_2} can be constructed such that $G_1U_{G_1} = [R,0]$ and $G_2U_{G_2} = [R,0]$.

Let m_1, m_3 be the ranks of $\widetilde{G}_1$, $\widetilde{G}_3$, respectively ($m_1 = 0$ when $\mathrm{coeff}(\mathsf{Q}_1,k) = 0$ and $m_3 = 0$ when $\mathrm{coeff}(\mathsf{Q}_1,-k) = 0$). Note that $\widetilde{G}_1\widetilde{G}_1^* = \widetilde{G}_2\widetilde{G}_2^*$ or $\widetilde{G}_3\widetilde{G}_3^* = \widetilde{G}_4\widetilde{G}_4^*$ due to $\mathsf{Q}_1\mathsf{Q}_1^* = I_r$. The matrices $\mathsf{W}_1,\mathsf{W}_3$ are then constructed by

$$\mathsf{W}_1 := \left[\begin{array}{c|c|c|c} \mathsf{U}_1 & & \mathsf{U}_2 & \\ \hline & I_{s_1-m_1} & & \\ \hline \mathsf{U}_2 & & \mathsf{U}_1 & \\ \hline & & & I_{s_2-m_1} \end{array}\right], \mathsf{W}_3 := \left[\begin{array}{c|c|c|c} \mathsf{U}_3 & & \mathsf{U}_4 & \\ \hline & I_{s_3-m_3} & & \\ \hline \mathsf{U}_4 & & \mathsf{U}_3 & \\ \hline & & & I_{s_4-m_3} \end{array}\right], \tag{35}$$

where $\mathsf{U}_1(z) = -\mathsf{U}_2(-z) := \frac{1+z^{-1}}{2}I_{m_1}$ and $\mathsf{U}_3(z) = \mathsf{U}_4(-z) := \frac{1+z}{2}I_{m_3}$.

Let $\mathsf{W}_{\mathsf{Q}_1} := \mathrm{diag}(U_1\mathsf{W}_1, I_{s_3+s_4})$ for the case that $\mathrm{coeff}(\mathsf{Q}_1,k) \neq 0$ or $\mathsf{W}_{\mathsf{Q}_1} := \mathrm{diag}(I_{s_1+s_2}, U_3\mathsf{W}_3)$ for the case that $\mathrm{coeff}(\mathsf{Q}_1,-k) \neq 0$. Then $\mathsf{W}_{\mathsf{Q}_1}$ is paraunitary. By the symmetry pattern and orthogonality of Q_1, $\mathsf{W}_{\mathsf{Q}_1}$ reduces the coefficient support of Q_1 to $[-k+1,k-1]$, i.e., $\mathrm{csupp}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1}) = [-k+1,k-1]$. Moreover, $\mathsf{W}_{\mathsf{Q}_1}$ changes the symmetry pattern of Q_1 such that

$$\mathsf{S}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1}) = [1_{r_1}, -1_{r_2}, z1_{r_3}, -z1_{r_4}]^{\mathrm{T}}\mathsf{S}\theta_1,$$

with

$$\mathsf{S}\theta_1 = [z^{-1}1_{m_1}, 1_{s_1-m_1}, -z^{-1}1_{m_1}, -1_{s_2-m_1}, 1_{m_3}, z^{-1}1_{s_3-m_3}, -1_{m_3}, -z^{-1}1_{s_4-m_3}].$$

E is then the permutation matrix such that

$$\mathsf{S}(\mathsf{Q}_1\mathsf{W}_{\mathsf{Q}_1})E = [1_{r_1}, -1_{r_2}, z1_{r_3}, -z1_{r_4},]^{\mathsf{T}}\mathsf{S}\theta,$$

with $\mathsf{S}\theta = [1_{s_1-m_1+m_3}, , -1_{s_2-m_1+m_3}, z^{-1}1_{s_3-m_3+m_1}, -z^{-1}1_{s_4-m_3+m_1}] = (\mathsf{S}\theta_1)E.$

2.2 Application to Filter Banks and Orthonormal Multiwavelets with Symmetry

In this subsection, we shall discuss the application of our results on orthogonal matrix extension with symmetry to d-band symmetric paraunitary filter banks in electronic engineering and to orthonormal multiwavelets with symmetry in wavelet analysis.

Symmetry of the filters in a filter bank is a very much desirable property in many applications. We say that the low-pass filter a_0 with multiplicity r has symmetry if

$$\mathsf{a}_0(z) = \mathrm{diag}(\varepsilon_1 z^{\mathsf{d}c_1}, \dots, \varepsilon_r z^{\mathsf{d}c_r})\mathsf{a}_0(1/z)\mathrm{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}) \tag{36}$$

for some $\varepsilon_1, \dots, \varepsilon_r \in \{-1, 1\}$ and $c_1, \dots, c_r \in \mathbb{R}$ such that $\mathsf{d}c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$. If a_0 has symmetry as in (36) and if 1 is a simple eigenvalue of $\mathsf{a}_0(1)$, then it is well known that the d-refinable function vector ϕ in (6) associated with the low-pass filter a_0 has the following symmetry:

$$\phi_1(c_1 - \cdot) = \varepsilon_1\phi_1, \quad \phi_2(c_2 - \cdot) = \varepsilon_2\phi_2, \quad \dots, \quad \phi_r(c_r - \cdot) = \varepsilon_r\phi_r. \tag{37}$$

Under the symmetry condition in (36), to apply Theorem 1, we first show that there exists a suitable paraunitary matrix U acting on $\mathbb{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]$ so that $\mathbb{P}_{\mathsf{a}_0}\mathsf{U}$ has compatible symmetry. Note that $\mathbb{P}_{\mathsf{a}_0}$ itself may not have any symmetry.

Lemma 1. *Let $\mathbb{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]$, where $\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}$ are d-band subsymbols of a d-band orthogonal filter a_0 satisfying (36). Then there exists a $\mathsf{d}r \times \mathsf{d}r$ paraunitary matrix U such that $\mathbb{P}_{\mathsf{a}_0}\mathsf{U}$ has compatible symmetry.*

Proof. From (36), we deduce that

$$[\mathsf{a}_{0;\gamma}(z)]_{\ell,j} = \varepsilon_\ell\varepsilon_j z^{R^\gamma_{\ell,j}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}(z^{-1})]_{\ell,j}, \; \gamma = 0, \dots, \mathsf{d}-1; \ell, j = 1, \dots, r, \tag{38}$$

where $\gamma, Q^\gamma_{\ell,j} \in \Gamma := \{0, \dots, \mathsf{d}-1\}$ and $R^\gamma_{\ell,j}$, $Q^\gamma_{\ell,j}$ are uniquely determined by

$$\mathsf{d}c_\ell - c_j - \gamma = \mathsf{d}R^\gamma_{\ell,j} + Q^\gamma_{\ell,j} \quad Q^\gamma_{\ell,j} \in \Gamma.\text{with} \quad R^\gamma_{\ell,j} \in \mathbb{Z}, \tag{39}$$

Since $\mathsf{d}c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$, we have $c_\ell - c_j \in \mathbb{Z}$ for all $\ell, j = 1, \dots, r$ and therefore, $Q^\gamma_{\ell,j}$ is independent of ℓ. Consequently, by (38), for every $1 \le j \le r$, the jth column of the matrix $\mathsf{a}_{0;\gamma}$ is a flipped version of the jth column of the matrix

$\mathsf{a}_{0;Q^\gamma_{\ell,j}}$. Let $\kappa_{j,\gamma} \in \mathbb{Z}$ be an integer such that $|\mathrm{csupp}([\mathsf{a}_{0;\gamma}]_{:,j} + z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j})|$ is as small as possible. Define $\mathbb{P} := [\mathsf{b}_{0;0}, \dots, \mathsf{b}_{0;\mathsf{d}-1}]$ as follows:

$$[\mathsf{b}_{0;\gamma}]_{:,j} := \begin{cases} [\mathsf{a}_{0;\gamma}]_{:,j}, & \gamma = Q^\gamma_{\ell,j}; \\ \frac{1}{\sqrt{2}}([\mathsf{a}_{0;\gamma}]_{:,j} + z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j}), & \gamma < Q^\gamma_{\ell,j}; \\ \frac{1}{\sqrt{2}}([\mathsf{a}_{0;\gamma}]_{:,j} - z^{\kappa_{j,\gamma}}[\mathsf{a}_{0;Q^\gamma_{\ell,j}}]_{:,j}), & \gamma > Q^\gamma_{\ell,j}, \end{cases} \tag{40}$$

where $[\mathsf{a}_{0;\gamma}]_{:,j}$ denotes the jth column of $\mathsf{a}_{0;\gamma}$. Let U denote the unique transform matrix corresponding to (40) such that $\mathbb{P} := [\mathsf{b}_{0;0}, \dots, \mathsf{b}_{0;\mathsf{d}-1}] = [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]\mathsf{U}$. It is evident that U is paraunitary and $\mathbb{P} = \mathbb{P}_{\mathsf{a}_0}\mathsf{U}$. We now show that $\mathbb{P}$ has compatible symmetry. Indeed, by (38) and (40),

$$[\mathsf{S}\mathsf{b}_{0;\gamma}]_{\ell,j} = \mathrm{sgn}(Q^\gamma_{\ell,j} - \gamma)\varepsilon_\ell \varepsilon_j z^{R^\gamma_{\ell,j} + \kappa_{j,\gamma}}, \tag{41}$$

where $\mathrm{sgn}(x) = 1$ for $x \ge 0$ and $\mathrm{sgn}(x) = -1$ for $x < 0$. By (39) and noting that $Q^\gamma_{\ell,j}$ is independent of ℓ, we have

$$\frac{[\mathsf{S}\mathsf{b}_{0;\gamma}]_{\ell,j}}{[\mathsf{S}\mathsf{b}_{0;\gamma}]_{n,j}} = \varepsilon_\ell \varepsilon_n z^{R^\gamma_{\ell,j} - R^\gamma_{n,j}} = \varepsilon_\ell \varepsilon_n z^{c_\ell - c_n}, \quad \ell, n = 1, \dots, r,$$

which is equivalent to saying that $\mathbb{P}$ has compatible symmetry. □ □

Now, for a d-band orthogonal low-pass filter a_0 satisfying (36), we have an algorithm to construct high-pass filters $\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}$ such that they form a symmetric paraunitary filter bank with the perfect reconstruction property. See Algorithm 2.

Algorithm 2 Construction of orthonormal multiwavelets with symmetry

(a) **Input**: An orthogonal d-band filter a_0 with symmetry in (36).
(b) **Initialization**: Construct U with respect to (40) such that $\mathbb{P} := \mathbb{P}_{\mathsf{a}_0}\mathsf{U}$ has compatible symmetry: $\mathsf{S}\mathbb{P} = [\varepsilon_1 z^{k_1}, \dots, \varepsilon_r z^{k_r}]^{\mathsf{T}}\mathsf{S}\theta$ for some $k_1, \dots, k_r \in \mathbb{Z}$ and some $1 \times \mathsf{d}r$ row vector θ of Laurent polynomials with symmetry.
(c) **Extension**: Derive $\mathbb{P}_e$ with all the properties as in Theorem 1 from $\mathbb{P}$ by Algorithm 1.
(d) **High-pass Filters**: Let $\mathbf{P} := \mathbb{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0 \le m,\gamma \le \mathsf{d}-1}$ as in (16). Define high-pass filters

$$\mathsf{a}_m(z) := \frac{1}{\sqrt{\mathsf{d}}} \sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{m;\gamma}(z^{\mathsf{d}}) z^\gamma, \qquad m = 1, \dots, \mathsf{d}-1. \tag{42}$$

(f) **Output**: A symmetric filter bank $\{\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}\}$ with the perfect reconstruction property, i.e., $\mathbf{P}$ in (16) is paraunitary and all filters a_m, $m = 1, \dots, \mathsf{d}-1$, have symmetry:

$$\mathsf{a}_m(z) = \mathrm{diag}(\varepsilon^m_1 z^{\mathsf{d}c^m_1}, \dots, \varepsilon^m_r z^{\mathsf{d}c^m_r})\mathsf{a}_m(1/z)\,\mathrm{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}), \tag{43}$$

where $c^m_\ell := (k^m_\ell - k_\ell) + c_\ell \in \mathbb{R}$ and all $\varepsilon^m_\ell \in \{-1, 1\}$, $k^m_\ell \in \mathbb{Z}$, for $\ell, j = 1, \dots, r$ and $m = 1, \dots, \mathsf{d}-1$, are determined by the symmetry pattern of $\mathbb{P}_e$ as follows:

$$[\varepsilon_1 z^{k_1}, \dots, \varepsilon_r z^{k_r}, \varepsilon^1_1 z^{k^1_1}, \dots, \varepsilon^1_r z^{k^1_r}, \dots, z^{k^{\mathsf{d}-1}_1}, \dots, \varepsilon^{\mathsf{d}-1}_r z^{k^{\mathsf{d}-1}_r}]^{\mathsf{T}}\mathsf{S}\theta := \mathsf{S}\mathbb{P}_e. \tag{44}$$

Proof (of Algorithm 2). Rewrite $\mathbb{P}_e = (\mathsf{b}_{m;\gamma})_{0\le m,\gamma\le \mathsf{d}-1}$ as a $\mathsf{d}\times\mathsf{d}$ block matrix with $r\times r$ blocks $\mathsf{b}_{m;\gamma}$. Since $\mathbb{P}_e$ has compatible symmetry as in (44), we have $[\mathsf{S}\mathsf{b}_{m;\gamma}]_{\ell,:} = \varepsilon_\ell^m \varepsilon_\ell z^{k_\ell^m - k_\ell}[\mathsf{S}\mathsf{b}_{0;\gamma}]_{\ell,:}$ for $\ell = 1,\dots,r$ and $m = 1,\dots,\mathsf{d}-1$. By (41), we have

$$[\mathsf{S}\mathsf{b}_{m;\gamma}]_{\ell,j} = \operatorname{sgn}(Q_{\ell,j}^{\gamma} - \gamma)\varepsilon_\ell^m \varepsilon_j z^{R_{\ell,j}^{\gamma} + k_{j,\gamma} + k_\ell^m - k_\ell}, \qquad \ell, j = 1,\dots,r. \tag{45}$$

By (45) and the definition of U^* in (40), we deduce that

$$[\mathsf{a}_{m;\gamma}]_{\ell,j} = \varepsilon_\ell^m \varepsilon_j z^{R_{\ell,j}^{\gamma} + k_\ell^m - k_\ell}[\mathsf{a}_{m;Q_{\ell,j}^{\gamma}}(z^{-1})]_{\ell,j}. \tag{46}$$

This implies that $[\mathsf{S}\mathsf{a}_m]_{\ell,j} = \varepsilon_\ell^m \varepsilon_j z^{\mathsf{d}(k_\ell^m - k_\ell + c_\ell) - c_j}$, which is equivalent to (43) with $c_\ell^m := k_\ell^m - k_\ell + c_\ell$ for $m = 1,\dots,\mathsf{d}-1$ and $\ell = 1,\dots,r$. $\square$ $\square$

Since the high-pass filters $\mathsf{a}_1,\dots,\mathsf{a}_{\mathsf{d}-1}$ satisfy (43), it is easy to verify that each $\psi^m = [\psi_1^m,\dots,\psi_r^m]^{\mathrm{T}}$ defined in (10) also has the following symmetry:

$$\psi_1^m(c_1^m - \cdot) = \varepsilon_1^m \psi_1^m, \quad \psi_2^m(c_2^m - \cdot) = \varepsilon_2^m \psi_2^m, \quad \dots, \quad \psi_r^m(c_r^m - \cdot) = \varepsilon_r^m \psi_r^m. \tag{47}$$

In the following, let us present an example to demonstrate our results and illustrate our algorithms (for more examples, see [16]).

Example 1. Let $\mathsf{d} = 3$ and $r = 2$. Let a_0 be the 3-band orthogonal low-pass filter with multiplicity 2 obtained in [15, Example 4]. Then

$$\mathsf{a}_0(z) = \frac{1}{540}\begin{bmatrix} a_{11}(z) + a_{11}(z^{-1}) & a_{12}(z) + z^{-1}a_{12}(z^{-1}) \\ a_{21}(z) + z^3 a_{21}(z^{-1}) & a_{22}(z) + z^2 a_{22}(z^{-1}) \end{bmatrix},$$

where

$$\begin{aligned}
a_{11}(z) &= 90 + (55 - 5\sqrt{41})z - (8 + 2\sqrt{41})z^2 + (7\sqrt{41} - 47)z^4,\\
a_{12}(z) &= 145 + 5\sqrt{41} + (1 - \sqrt{41})z^2 + (34 - 4\sqrt{41})z^3,\\
a_{21}(z) &= (111 + 9\sqrt{41})z^2 + (69 - 9\sqrt{41})z^4,\\
a_{22}(z) &= 90z + (63 - 3\sqrt{41})z^2 + (3\sqrt{41} - 63)z^3.
\end{aligned}$$

The low-pass filter a_0 satisfies (36) with $c_1 = 0, c_2 = 1$ and $\varepsilon_1 = \varepsilon_2 = 1$. From $\mathbb{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \mathsf{a}_{0;1}, \mathsf{a}_{0;2}]$, the matrix U constructed by Lemma 1 is given by

$$\mathsf{U} := \frac{1}{\sqrt{2}}\begin{bmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & z & 0 & -z & 0 \\ 0 & z & 0 & 0 & 0 & -z \end{bmatrix}.$$

Let

$$c_0 = 11-\sqrt{41}, \quad t_{12} = 5(7-\sqrt{41}), \quad c_{12} = 10(29+\sqrt{41}), \quad t_{13} = -5c_0,$$
$$t_{16} = 3c_0, \quad t_{15} = 3(3\sqrt{41}-13), \quad t_{25} = 6(7+3\sqrt{41}), \quad t_{26} = 6(21-\sqrt{41}),$$
$$t_{53} = 400\sqrt{6}/c_0, \quad t_{55} = 12\sqrt{6}(\sqrt{41}-1), \quad t_{56} = 6\sqrt{6}(4+\sqrt{41}), \quad c_{66} = 3\sqrt{6}(3+7\sqrt{41}).$$

Then, $\mathbb{P} := \mathbb{P}_{\mathsf{a}_0}\mathsf{U}$ satisfies $\mathsf{S}\mathbb{P} = [1,z]^{\mathrm{T}}[1,1,1,z^{-1},-1,-1]$ and is given by

$$\mathbb{P} = \frac{\sqrt{6}}{1080}\begin{bmatrix} 180\sqrt{2} & b_{12}(z) & b_{13}(z) & 0 & t_{15}(z-z^{-1}) & t_{16}(z-z^{-1}) \\ 0 & 0 & 180(1+z) & 180\sqrt{2} & t_{25}(1-z) & t_{26}(1-z) \end{bmatrix},$$

where $b_{12}(z) = t_{12}(z+z^{-1}) + c_{12}$ and $b_{13}(z) = t_{13}(z-2+z^{-1})$. Applying Algorithm 1, we obtain a desired paraunitary matrix $\mathbb{P}_e$ as follows:

$$\mathbb{P}_e = \frac{\sqrt{6}}{1080}\left[\begin{array}{cccccc} 180\sqrt{2} & b_{12}(z) & b_{13}(z) & 0 & t_{15}(z-\frac{1}{z}) & t_{16}(z-\frac{1}{z}) \\ 0 & 0 & 180(1+z) & 180\sqrt{2} & t_{25}(1-z) & t_{26}(1-z) \\ \hline 360 & -\frac{b_{12}(z)}{\sqrt{2}} & -\frac{b_{13}(z)}{\sqrt{2}} & 0 & \frac{t_{15}}{\sqrt{2}}(\frac{1}{z}-z) & \frac{t_{16}}{\sqrt{2}}(\frac{1}{z}-z) \\ 0 & 0 & 90\sqrt{2}(1+z) & -360 & \frac{t_{25}}{\sqrt{2}}(1-z) & \frac{t_{26}}{\sqrt{2}}(1-z) \\ \hline 0 & \sqrt{6}t_{13}(1-z) & t_{53}(1-z) & 0 & t_{55}(1+z) & t_{56}(1+z) \\ 0 & \frac{\sqrt{6}t_{12}}{2}(\frac{1}{z}-z) & \frac{\sqrt{6}t_{13}}{2}(\frac{1}{z}-z) & 0 & b_{65}(z) & b_{66}(z) \end{array}\right],$$

where $b_{65}(z) = -\sqrt{6}(5t_{15}(z+z^{-1}) + 3c_{12})/10$ and $b_{66}(z) = -\sqrt{6}t_{16}(z+z^{-1})/2 + c_{66}$. Note that $\mathsf{S}\mathbb{P}_e = [1,z,1,z,-z,-1]^{\mathrm{T}}[1,1,1,z^{-1},-1,-1]$ and the coefficient support of $\mathbb{P}_e$ satisfies $\mathrm{csupp}([\mathbb{P}_e]_{:,j}) \subseteq \mathrm{csupp}([\mathbb{P}]_{:,j})$ for all $1 \le j \le 6$. From the polyphase matrix $\mathbf{P} := \mathbb{P}_e\mathsf{U}^* =: (\mathsf{a}_{m;\gamma})_{0\le m,\gamma\le 2}$, we derive two high-pass filters $\mathsf{a}_1, \mathsf{a}_2$ as follows:

$$\mathsf{a}_1(z) = \frac{\sqrt{2}}{1080}\begin{bmatrix} a^1_{11}(z) + a^1_{11}(z^{-1}) & a^1_{12}(z) + z^{-1}a^1_{12}(z^{-1}) \\ a^1_{21}(z) + z^3 a^1_{21}(z^{-1}) & a^1_{22}(z) + z^2 a^1_{22}(z^{-1}) \end{bmatrix},$$

$$\mathsf{a}_2(z) = \frac{\sqrt{6}}{1080}\begin{bmatrix} a^2_{11}(z) - z^3 a^2_{11}(z^{-1}) & a^2_{12}(z) - z^2 a^2_{12}(z^{-1}) \\ a^2_{21}(z) - a^2_{21}(z^{-1}) & a^2_{22}(z) - z^{-1}a^2_{22}(z^{-1}) \end{bmatrix},$$

where

$$a^1_{11}(z) = (47-7\sqrt{41})z^4 + 2(4+\sqrt{41})z^2 + 5(\sqrt{41}-11)z + 180,$$
$$a^1_{12}(z) = 2(2\sqrt{41}-17)z^3 + (\sqrt{41}-1)z^2 - 5(29+\sqrt{41}),$$
$$a^1_{21}(z) = 3(37+3\sqrt{41})z + 3(23-3\sqrt{41})z^{-1},$$
$$a^1_{22}(z) = -180z + 3(21-\sqrt{41}) - 3(21-\sqrt{41})z^{-1},$$
$$a^2_{11}(z) = (43+17\sqrt{41})z + (67-7\sqrt{41})z^{-1},$$
$$a^2_{12}(z) = 11\sqrt{41} - 31 - (79+\sqrt{41})z^{-1},$$
$$a^2_{21}(z) = (47-7\sqrt{41})z^4 + 2(4+\sqrt{41})z^2 - 3(29+\sqrt{41})z,$$
$$a^2_{22}(z) = 2(2\sqrt{41}-17)z^3 + (\sqrt{41}-1)z^2 + 3(3+7\sqrt{41}).$$

Then the high-pass filters a_1, a_2 satisfy (43) with $c_1^1 = 0$, $c_2^1 = 1$, $\varepsilon_1^1 = \varepsilon_2^1 = 1$ and $c_1^2 = 1$, $c_2^2 = 0$, $\varepsilon_1^2 = \varepsilon_2^2 = -1$. See Fig. 1 for graphs of the 3-refinable function vector ϕ associated with the low-pass filter a_0 and the multiwavelet function vectors ψ^1, ψ^2 associated with the high-pass filters $\mathsf{a}_1, \mathsf{a}_2$, respectively.

3 Construction of Symmetric Complex Tight Framelets

Redundant wavelet systems ($L \geq \mathsf{d}$ in (17)) have been proved to be quit useful in many applications, for examples, signal denoising, image processing, and numerical algorithm. As a redundant system, it can possess many desirable properties such as symmetry, short support, high vanishing moments, and so on, simultaneously (see [6, 7, 12, 22]). In this section, we are interested in the construction of tight framelets with such desirable properties. Due to [6], the whole picture of constructing tight framelets with high order of vanishing moments is more or less clear. Yet, when comes to symmetry, there is no general way of deriving tight framelet systems with symmetry. Especially when one requires the number of framelet generators is as less as possible. In this section, we first provide a general result on the construction of d-refinable functions with symmetry such that (14) holds. Once such a d-refinable function is obtained, we then show that using our results on orthogonal matrix extension with symmetry studied in Sect. 2, we can construct a symmetric tight framelet system with only d or $\mathsf{d}+1$ framelet generators.

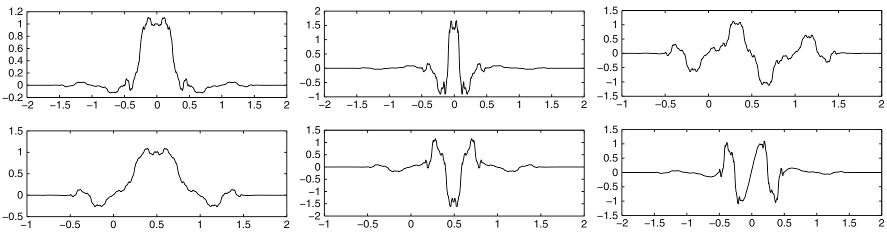

Fig. 1: Graphs of the 3-refinable function vector $\phi = [\phi_1, \phi_2]^{\mathrm{T}}$ associated with a_0 (*left column*), multiwavelet function vector $\psi^1 = [\psi_1^1, \psi_2^1]^{\mathrm{T}}$ associated with a_1 (*middle column*), and multiwavelet function vector $\psi^2 = [\psi_1^2, \psi_2^2]^{\mathrm{T}}$ associated with a_2 (*right column*) in Example 1

3.1 *Symmetric Complex* d*-Refinable Functions*

Let ϕ be a d-refinable functions associated with a low-pass filters a_0. To have high order of vanishing moments for a tight framelet system, we need to design a_0 such that (14) holds for some $n \in \mathbb{N}$. To guarantee that the d-refinable function ϕ associated with a_0 has certain regularity and polynomial reproducibility, usually the low-pass filter a_0 satisfies the *sum rules of order m* for some $m \in \mathbb{N}$. More precisely, $\widehat{a_0}$ is of the form:

$$\widehat{a_0}(\xi) = \left(\frac{1+e^{-i\xi}+\cdots+e^{-i(\mathsf{d}-1)\xi}}{\mathsf{d}}\right)^m \widehat{\mathscr{L}}(\xi), \quad \xi \in \mathbb{R} \tag{48}$$

for some 2π-periodic trigonometric polynomial $\widehat{\mathscr{L}}(\xi)$ with $\widehat{\mathscr{L}}(0) = 1$. For $\widehat{\mathscr{L}}(\xi) \equiv 1$. a_0 is the low-pass filter for B-spline of order m: $\widehat{B_m}(\xi) = (1-e^{-i\xi})^m/(i\xi)^m$.

Define a function h by

$$h(y) := \prod_{k=1}^{\mathsf{d}-1}\left(1-\frac{y}{\sin^2(k\pi/\mathsf{d})}\right), \quad y \in \mathbb{R}. \tag{49}$$

One can show that

$$h(\sin^2(\xi/2)) = \frac{|1+\cdots+e^{-i(\mathsf{d}-1)\xi}|^2}{\mathsf{d}^2} = \frac{\sin^2(\mathsf{d}\xi/2)}{\mathsf{d}^2\sin^2(\xi/2)} \tag{50}$$

and

$$h(y)^{-m} = \left[\prod_{k=1}^{\mathsf{d}-1}\left(\sum_{j_k=0}^{\infty}\frac{y^{j_k}}{\sin^{2j_k}(k\pi/\mathsf{d})}\right)\right]^{-m} = \sum_{j=0}^{\infty} c_{m,j}y^j, \quad |y| < \sin^2(\pi/\mathsf{d}), \tag{51}$$

where

$$c_{m,j} = \sum_{j_1+\cdots+j_{\mathsf{d}-1}=j}\prod_{k=1}^{\mathsf{d}-1}\binom{m-1+j_k}{j_k}\sin(k\pi/\mathsf{d})^{-2j_k}, \quad j \in \mathbb{N}. \tag{52}$$

Define $P_{m,n}(y)$ a polynomial of degree $n-1$ as follows:

$$P_{m,n}(y) = \sum_{j=0}^{n-1}\left[\sum_{j_1+\cdots+j_{\mathsf{d}-1}=j}\prod_{k=1}^{\mathsf{d}-1}\binom{m-1+j_k}{j_k}\sin(k\pi/\mathsf{d})^{-2j_k}\right]y^j. \tag{53}$$

By convention, $\binom{m}{j} = 0$ if $j < 0$. Note that $P_{m,n}(y) = \sum_{j=0}^{n-1} c_{m,j}y^j$. Then, it is easy to show the following result by Taylor expansion.

Lemma 2. *Let $m,n \in \mathbb{N}$ be such that $n \le m$; let $P_{m,n}$ and h be polynomials defined as in (53) and (49), respectively. Then $P_{m,n}(\sin^2(\xi/2))$ is the unique positive trigonometric polynomial of minimal degree such that*

$$1-h(\sin^2(\xi/2))^m P_{m,n}(\sin^2(\xi/2)) = O(|\xi|^{2n}), \quad \xi \to 0. \tag{54}$$

For $m,n\in\mathbb{N}$ such that $1\le n\le m$, let $\widehat{{}_{II}a_0}(\xi):=h(\sin^2(\xi/2))^m P_{m,n}(\sin^2(\xi/2))$. Then the d-refinable function ${}_{II}\phi$ associated with ${}_{II}a_0$ by (6) is called the d-*refinable pseudo spline of type II with order* (m,n). By Lemma 2, using Riesz Lemma, one can derive a low-pass filter ${}_{I}a_0$ from ${}_{II}a_0$ such that $|\widehat{{}_{I}a_0}(\xi)|^2=\widehat{{}_{II}a_0}(\xi)$. The d-refinable function ${}_{I}\phi$ associated with such ${}_{I}a_0$ by (6) is referred as *real* d-*refinable pseudo spline of type I with order* (m,n). Interesting readers can refer to [6, 7, 22] for more details on this subject for the special case $\mathsf{d}=2$.

Note that ${}_{I}a_0$ satisfies (14). One can construct high-pass filters $a_1,\dots,a_L$ from $a_0:={}_{I}a_0$ such that (12) holds. Then $\psi^1,\dots,\psi^L$ defined by (10) are real-valued functions. $\{\psi^1,\dots,\psi^L\}$ has vanishing moment of order n and generates a tight d-frame. However, $\{\psi^1,\dots,\psi^L\}$ does not necessarily have symmetry since the low-pass filter ${}_{I}a_0$ from ${}_{II}a_0$ via Riesz lemma might not possess any symmetry pattern. In the following, we shall show that we can achieve symmetry for any odd integer $n\in\mathbb{N}$ if considering complex-valued wavelet generators.

For $1\le n\le m$, we have the following lemma regarding the positiveness of $P_{m,n}(y)$, which generalizes [12, Theorem 5] and [22, Theorem 2.4]. See [26, Theorem 2] for its technical proof.

Lemma 3. *Let $m,n\in\mathbb{N}$ be such that $n\le m$. Then $P_{m,n}(y)>0$ for all $y\in\mathbb{R}$ if and only if n is an odd number.*

Now, by $P_{m,2n-1}(y)>0$ for all $y\in\mathbb{R}$ and $2n-1\le m$, $P_{m,2n-1}(y)$ can only have complex roots. Hence, we must have

$$P_{m,2n-1}(y)=c_0\prod_{j=1}^{n-1}(y-z_j)(y-\overline{z_j}),\quad z_1,\overline{z_1},\dots,z_{n-1},\overline{z_{n-1}}\in\mathbb{C}\setminus\mathbb{R}.$$

In view of Lemmas 2 and 3, we have the following result.

Theorem 2. *Let $\mathsf{d}>1$ be a dilation factor. Let $m,n\in\mathbb{N}$ be positive integers such that $2n-1\le m$. Let $P_{m,n}(y)$ be the polynomial defined in (53). Then,*

$$P_{m,2n-1}(y)=|Q_{m,n}(y)|^2,\tag{55}$$

where $Q_{m,n}(y)=c(y-z_1)\cdots(y-z_{n-1})$ with $c=(-1)^{n-1}(z_1\cdots z_{n-1})^{-1}$ and $z_1,\overline{z_1}$, $\dots$, $z_{n-1},\overline{z_{n-1}}\in\mathbb{C}\setminus\mathbb{R}$ are all the complex roots of $P_{m,2n-1}(y)$. Define a low-pass filter a_0 by

$$\widehat{a_0}(\xi):=e^{i\lfloor\frac{m(\mathsf{d}-1)}{2}\rfloor\xi}\left(\frac{1+e^{-i\xi}+\cdots+e^{-i(\mathsf{d}-1)\xi}}{\mathsf{d}}\right)^m Q_{m,n}(\sin^2(\xi/2)),\tag{56}$$

where $\lfloor\cdot\rfloor$ is the floor operation. Then,

$$\widehat{a_0}(-\xi)=e^{i\varepsilon\xi}\widehat{a_0}(\xi)\quad\textit{with}\quad\varepsilon=m(\mathsf{d}-1)-2\lfloor\frac{m(\mathsf{d}-1)}{2}\rfloor\tag{57}$$

and

$$\operatorname{csupp}(\mathsf{a}_0) = \left[-\lfloor\frac{m(\mathsf{d}-1)}{2}\rfloor - n + 1, \lfloor\frac{m(\mathsf{d}-1)}{2}\rfloor + n - 1 + \varepsilon\right].$$

Let ϕ be the standard d*-refinable function associated with the low-pass filter a_0, that is, $\widehat{\phi}(\xi) := \prod_{j=1}^{\infty} \widehat{a_0}(\mathsf{d}^{-j}\xi)$. Then, ϕ is a compactly supported* d*-refinable function in $L_2(\mathbb{R})$ with symmetry satisfying $\phi(\frac{\varepsilon}{\mathsf{d}-1} - \cdot) = \phi$.*

For $m, n \in \mathbb{N}$ such that $2n-1 \le m$, we shall refer the d-refinable function ϕ associated with the low-pass filter a_0 defined in Theorem 2 as *complex* d*-refinable pseudo spline of type I with order* $(m, 2n-1)$.

Now, we have the following result which shall play an important role in our construction of tight framelet systems in this section.

Corollary 1. *Let* $\mathsf{d} > 1$ *be a dilation factor. Let $m, n \in \mathbb{N}$ be such that $2n-1 \le m$ and a_0 be the low-pass filter for the complex* d*-refinable pseudo spline of type I with order $(m, 2n-1)$. Then*

$$1 - \sum_{j=0}^{\mathsf{d}-1} |\widehat{a_0}(\xi + 2\pi j/\mathsf{d})|^2 = |\widehat{b}(\mathsf{d}\xi)|^2, \tag{58}$$

for some 2π-periodic trigonometric function $\widehat{b}(\xi)$ with real coefficients. In particular,

$$|\widehat{b}(\xi)|^2 = \begin{cases} 0 & m = 2n-1; \\ c_{2n,2n-1}[\sin^2(\xi/2)/\mathsf{d}^2]^{2n-1} & m = 2n, \end{cases}$$

where $c_{2n,2n-1}$ is the coefficient given in (52).

Proof. We first show that $1 - \sum_{j=0}^{\mathsf{d}-1} |\widehat{a_0}(\xi + 2\pi j/\mathsf{d})|^2 \ge 0$ for all $\xi \in \mathbb{R}$. Let $y_j := \sin^2(\xi/2 + \pi j/\mathsf{d})$ for $j = 0, \ldots, \mathsf{d}-1$. Noting that $|\widehat{a_0}(\xi)|^2 = h(y_0)^m P_{m,2n-1}(y_0)$, we have

$$\begin{aligned} 1 - \sum_{j=0}^{\mathsf{d}-1} |\widehat{a_0}(\xi + 2\pi j/\mathsf{d})|^2 &= 1 - \sum_{j=0}^{\mathsf{d}-1} h(y_j)^m P_{m,2n-1}(y_j) \\ &= 1 - \sum_{j=0}^{\mathsf{d}-1} h(y_j)^m P_{m,m}(y_j) + \sum_{j=0}^{\mathsf{d}-1} h(y_j)^m \sum_{k=2n-1}^{m-1} c_{m,k} y_j^k \\ &= \sum_{j=0}^{\mathsf{d}-1} h(y_j)^m \sum_{k=2n-1}^{m-1} c_{m,k} y_j^k \\ &\ge 0. \end{aligned}$$

The last equality follows from the fact that the low-pass filter a_0, which is defined by factorizing $h(y_0)^m P_{m,m}(y_0)$ such that $|\widehat{a_0}(\xi)|^2 := h(y_0)^m P_{m,m}(y_0)$, is an orthogonal low-pass filter (see [17]). Now, by that $1 - \sum_{j=0}^{\mathsf{d}-1} |\widehat{a_0}(\xi + 2\pi j/\mathsf{d})|^2$ is of period $2\pi/\mathsf{d}$, (58) follows from Riesz Lemma.

Obviously, $\widehat{b}(\xi) \equiv 0$ when $m = 2n-1$ since a_0 is then an orthogonal low-pass filter. For $m = 2n$, noting that $h(y_j)y_j = \sin^2(\mathsf{d}\xi/2)/\mathsf{d}^2$ for $j = 0, \ldots, \mathsf{d}-1$, we have

$$
\begin{aligned}
|\widehat{b}(\mathsf{d}\xi)|^2 &= c_{2n,2n-1} \sum_{j=0}^{\mathsf{d}-1} h(y_j)^{2n} y_j^{2n-1} = c_{2n,2n-1} \sum_{j=0}^{\mathsf{d}-1} [h(y_j)y_j]^{2n-1} h(y_j) \\
&= c_{2n,2n-1} [\sin^2(\mathsf{d}\xi/2)/\mathsf{d}^2]^{2n-1} \sum_{j=0}^{\mathsf{d}-1} h(y_j) P_{1,1}(y_j) \\
&= c_{2n,2n-1} [\sin^2(\mathsf{d}\xi/2)/\mathsf{d}^2]^{2n-1},
\end{aligned}
$$

which completes our proof. □

3.2 Tight Framelets via Matrix Extension

Fixed $m, n \in \mathbb{N}$ such that $1 \le 2n-1 \le m$, we next show that we can construct a vector of Laurent polynomial with symmetry from a low-pass filter a_0 for the complex d-refinable pseudo spline of type I with order $(m, 2n-1)$ to which Algorithm 1 is applicable. Indeed, by (40), we have a $1 \times \mathsf{d}$ vector of Laurent polynomial $\mathsf{p}(z) := [\mathsf{b}_{0;0}(z), \ldots, \mathsf{b}_{0;\mathsf{d}-1}(z)]$ from a_0. Note $\mathsf{p}\mathsf{p}^* = 1$ when $m = 2n-1$ while $\mathsf{p}\mathsf{p}^* \neq 1$ when $2n-1 < m$. To apply our matrix extension algorithm, we need to append extra entries to p when $\mathsf{p}\mathsf{p}^* < 1$. It is easy to show that

$$
1 - \sum_{j=0}^{\mathsf{d}-1} |\widehat{a}_0(\xi + 2j\pi/\mathsf{d})|^2 = 1 - \sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{0;\gamma}(z^{\mathsf{d}}) \mathsf{a}_{0;\gamma}^*(z^{\mathsf{d}}), \quad z = \mathrm{e}^{-\mathrm{i}\xi},
$$

where $\mathsf{a}_{0;\gamma}, \gamma = 0, \ldots, \mathsf{d}-1$ are the subsymbols of a_0. By Corollary 1, we have

$$
1 - \sum_{j=0}^{\mathsf{d}-1} |\widehat{a}_0(\xi + 2j\pi/\mathsf{d})|^2 = |\widehat{b}(\mathsf{d}\xi)|^2.
$$

for some 2π-periodic trigonometric function $\widehat{b}$ with real coefficients. Hence, we can construct a Laurent polynomial $\mathsf{a}_{0;\mathsf{d}}(z)$ from $\widehat{b}$ such that $\mathsf{a}_{0;\mathsf{d}}(\mathrm{e}^{-\mathrm{i}\xi}) = \widehat{b}(\xi)$. Then, the vector of Laurent polynomials $\mathsf{q}(z) = [\mathsf{a}_{0;0}(z), \ldots, \mathsf{a}_{0;\mathsf{d}-1}(z), \mathsf{a}_{0;\mathsf{d}}(z)]$ satisfies $\mathsf{q}\mathsf{q}^* = 1$.

For $m = 2n$, by Corollary 1, $|\widehat{b}(\xi)|^2 = c_{2n,2n-1}[\sin^2(\xi/2)/\mathsf{d}^2]^{2n-1}$. In this case, $\mathsf{a}_{0;\mathsf{d}}(z)$ can be constructed explicitly as follows:

$$
\mathsf{a}_{0;\mathsf{d}}(z) = \sqrt{c_{2n,2n-1}} \left(\frac{2 - z - 1/z}{4\mathsf{d}^2} \right)^{n-1} \frac{1-z}{2\mathsf{d}}. \tag{59}
$$

$\mathsf{a}_{0;\mathsf{d}}(z)$ has symmetry $\mathsf{S}\mathsf{a}_{0;\mathsf{d}} = -z$. Let $\mathsf{b}_{0;\mathsf{d}}(z) := \mathsf{a}_{0;\mathsf{d}}(z)$. Then $\mathsf{p} := [\mathsf{b}_{0;0}, \ldots, \mathsf{b}_{0;\mathsf{d}}]$ is a $1 \times (\mathsf{d}+1)$ vector of Laurent polynomials with symmetry satisfying $\mathsf{p}\mathsf{p}^* = 1$.

For $m \neq 2n$, $\mathsf{a}_{0;\mathsf{d}}(z)$ does not necessary have symmetry. We can further let $\mathsf{b}_{0;\mathsf{d}}(z) := (\mathsf{a}_{0;\mathsf{d}}(z) + \mathsf{a}_{0;\mathsf{d}}(1/z))/2$ and $\mathsf{b}_{0;\mathsf{d}+1}(z) := (\mathsf{a}_{0;\mathsf{d}}(z) - \mathsf{a}_{0;\mathsf{d}}(1/z))/2$. In this way, $\mathsf{p} := [\mathsf{b}_{0;0}, \dots, \mathsf{b}_{0;\mathsf{d}}, \mathsf{b}_{0;\mathsf{d}+1}]$ is a $1 \times (\mathsf{d}+2)$ vector of Laurent polynomials with symmetry satisfying $\mathsf{p}\mathsf{p}^* = 1$.

Consequently, we can summerize the above discussion as follows:

Theorem 3. *Let $m,n \in \mathbb{N}$ be such that $1 \leq 2n-1 < m$. Let a_0 (with symbol a_0) be the low-pass filter for the complex d-refinable pseudo spline of type I with order $(m,2n-1)$ defined in (56). Then one can derive Laurent polynomials $\mathsf{a}_{0;\mathsf{d}}, \dots, \mathsf{a}_{0;L}, L \in \{\mathsf{d}, \mathsf{d}+1\}$ such that $\mathsf{p}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}, \dots, \mathsf{a}_{0;L}]$ satisfies $\mathsf{p}_{\mathsf{a}_0}\mathsf{p}_{\mathsf{a}_0}^* = 1$, where $\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}$ are subsymbols of a_0. Moreover, one can construct an $(L+1) \times (L+1)$ paraunitary matrix U such that $\mathsf{p}_{\mathsf{a}_0}\mathsf{U}$ is a vector of Laurent polynomials with symmetry. In particular, if $m = 2n$, then $L = \mathsf{d}$ and $\mathsf{a}_{0;\mathsf{d}}$ is given by (59).*

Now, applying Theorem 3 and Algorithm 1, we have the following algorithm to construct high-pass filters $\mathsf{a}_1, \dots, \mathsf{a}_L$ from a low-pass filter a_0 for a complex d-refinable pseudo spline of type I with order $(m,2n-1)$ so that $\psi^1, \dots, \psi^L$ defined by (10) generates a tight framelet system.

Algorithm 3 Construction of symmetric complex tight framelets

(a) **Input**: A low-pass filter a_0 for a complex d-refinable pseudo spline of type I with order $(m,2n-1)$, $1 \leq 2n-1 < m$. Note that a_0 satisfies (36) for $r = 1$.

(b) **Initialization**: Construct $\mathsf{p}_{\mathsf{a}_0}(z)$ and U as in Theorem 3 such that $\mathsf{p} := \mathsf{p}_{\mathsf{a}_0}\mathsf{U}$ is a $1 \times (L+1)$ row vector of Laurent polynomials with symmetry ($L = \mathsf{d}$ when $m = 2n$ while $L = \mathsf{d}+1$ when $m \neq 2n$).

(c) **Extension**: Derive $\mathbb{P}_e$ from p by Algorithm 1 with all the properties as in Theorem 1 for the case $r = 1$.

(d) **High-pass Filters**: Let $\mathbf{P} := [\mathbb{P}_e\mathsf{U}^*]_{0:L,0:\mathsf{d}-1} =: (\mathsf{a}_{m;\gamma})_{0 \leq m \leq L, 0 \leq \gamma \leq \mathsf{d}-1}$ as in (16). Define high-pass filters

$$\mathsf{a}_m(z) := \frac{1}{\sqrt{\mathsf{d}}} \sum_{\gamma=0}^{\mathsf{d}-1} \mathsf{a}_{m;\gamma}(z^{\mathsf{d}}) z^{\gamma}, \qquad m = 1, \dots, L. \tag{60}$$

Note that we only need the first d columns of $\mathbb{P}_e\mathsf{U}^*$.

(e) **Output**: A symmetric filter bank $\{\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_L\}$ with the perfect reconstruction property, i.e. $\mathbf{P}^*(z)\mathbf{P}(z) = I_{\mathsf{d}}$ for all $z \in \mathbb{C}\backslash\{0\}$. All filters a_m, $m = 1, \dots, L$, have symmetry:

$$\mathsf{a}_m(z) = \varepsilon_m z^{\mathsf{d}c_m - c_0} \mathsf{a}_m(1/z), \tag{61}$$

where $c_m := k_m + c_0 \in \mathbb{R}$ and all $\varepsilon_m \in \{-1,1\}$, $k_m \in \mathbb{Z}$ for $m = 1, \dots, L$ are determined by the symmetry pattern of $\mathbb{P}_e$ as follows:

$$[1, \varepsilon_1 z^{k_1}, \dots, \varepsilon_L z^{k_L}]^{\mathrm{T}} \mathsf{S}\mathsf{p} := \mathsf{S}\mathbb{P}_e. \tag{62}$$

Since the high-pass filters $\mathsf{a}_1, \dots, \mathsf{a}_L$ satisfy (43), it is easy to verify that $\psi^1, \dots, \psi^L$ defined in (10) also has the following symmetry:

$$\psi^1(c_1 - \cdot) = \varepsilon_1 \psi^1, \quad \psi^2(c_2 - \cdot) = \varepsilon_2 \psi^2, \quad \dots, \quad \psi^L(c_L - \cdot) = \varepsilon_L \psi^1. \tag{63}$$

In the following, let us present an example to demonstrate our results and illustrate our algorithms. More examples can be obtained in the same way.

Example 2. Consider dilation factor $\mathsf{d}=3$. Let $m=4$ and $n=2$. Then $P_{4,3}(y)=1+\frac{32}{3}y+64y^2)$. The low-pass filter a_0 with its symbol a_0 for the complex 3-refinable pseudo spline of order $(4,3)$ is given by

$$\mathsf{a}_0(z)=\left(\frac{\frac{1}{z}+1+z}{3}\right)^4\left[-\left(\frac{4}{3}+\frac{2\sqrt{5}}{3}\mathrm{i}\right)\frac{1}{z}+\left(\frac{11}{3}+\frac{4\sqrt{5}}{3}\mathrm{i}\right)-\left(\frac{4}{3}+\frac{2\sqrt{5}}{3}\mathrm{i}\right)z\right].$$

Note that $\mathrm{csupp}(\mathsf{a}_0)=[-5,5]$ and $\mathsf{a}(z)=\mathsf{a}(z^{-1})$. In this case, $m=2n$. By Theorem 3, we can obtain $\mathsf{p}_{\mathsf{a}_0}=[\mathsf{a}_{0;0}(z),\mathsf{a}_{0;1}(z),\mathsf{a}_{0;2}(z),\mathsf{a}_{0;3}(z)]$ as follows:

$$\begin{aligned}
\mathsf{a}_{0;0}(z)&=-\frac{\sqrt{15}\mathrm{i}}{405}\left(10z+27\sqrt{5}\mathrm{i}-20+10z^{-1}\right);\\
\mathsf{a}_{0;1}(z)&=\frac{\sqrt{3}}{243}(-(4+2\sqrt{5}\mathrm{i})z^{-2}+30z^{-1}+60+6\sqrt{5}\mathrm{i}-(5+4\sqrt{5}\mathrm{i})z);\\
\mathsf{a}_{0;2}(z)&=\frac{\sqrt{3}}{243}(-(5+4\sqrt{5}\mathrm{i})z^{-2}+(60+6\sqrt{5}\mathrm{i})z^{-1}+30-(4+2\sqrt{5}\mathrm{i})z)\\
\mathsf{a}_{0;3}(z)&=-\frac{2\sqrt{10}}{81}(z-2+z^{-1})(1-z).
\end{aligned}$$

We have $\mathsf{a}_{0;1}(z)=z^{-1}\mathsf{a}_{0;2}(z^{-1})$. Let $\mathsf{p}=\mathsf{p}_{\mathsf{a}_0}\mathsf{U}$ with U being the paraunitary matrix given by

$$\mathsf{U}:=\mathrm{diag}(1,U_0,z^{-1})\quad\text{with}\quad U_0=\begin{bmatrix}\frac{1}{\sqrt{2}}&\frac{1}{\sqrt{2}}\\\frac{1}{\sqrt{2}}&-\frac{1}{\sqrt{2}}\end{bmatrix}.$$

Then p is a 1×4 vector of Laurent polynomials with symmetry pattern satisfying $\mathsf{Sp}=[1,z^{-1},-z^{-1},-z^{-1}]$. Applying Algorithm 3, we can obtain a 4×4 extension matrix $\mathbb{P}_e^*=[\mathsf{p}_{\mathsf{a}_0}^*,\mathsf{p}_{\mathsf{a}_1}^*,\mathsf{p}_{\mathsf{a}_2}^*,\mathsf{p}_{\mathsf{a}_3}^*]$ with $\mathsf{p}_{\mathsf{a}_1}:=[\mathsf{a}_{1;0},\mathsf{a}_{1;1},\mathsf{a}_{1;2},\mathsf{a}_{1;3}]$, $\mathsf{p}_{\mathsf{a}_2}:=[\mathsf{a}_{2;0},\mathsf{a}_{2;1},\mathsf{a}_{2;2},\mathsf{a}_{2;3}]$, and $\mathsf{p}_{\mathsf{a}_3}:=[\mathsf{a}_{3;0},\mathsf{a}_{3;1},\mathsf{a}_{3;2},\mathsf{a}_{3;3}]$. The coefficient support of $\mathbb{P}_e$ satisfies $\mathrm{csupp}([\mathbb{P}_e]_{:,j})\subseteq\mathrm{csupp}([\mathsf{p}_{\mathsf{a}_0}]_j)$ for $j=1,2,3,4$. The high-pass filters a_1, $\mathsf{a}_2,\mathsf{a}_3$ constructed from $\mathsf{p}_{\mathsf{a}_1}$, $\mathsf{p}_{\mathsf{a}_2}$, and $\mathsf{p}_{\mathsf{a}_3}$ via (60) are then given by

$$\mathsf{a}_1(z)=c_1(b_1(z)+b_1(z^{-1}));\ \mathsf{a}_2(z)=c_2(b_2(z)-b_2(z^{-1}));\ \mathsf{a}_3(z)=c_3(b_3(z)-z^3b_3(z^{-1})).$$

where $c_1=\frac{\sqrt{19178}}{4660254}$, $c_2=\frac{\sqrt{218094}}{17665614}$, $c_3=\frac{2\sqrt{1338}}{54189}$, and

$$\begin{aligned}
b_1(z)&=\left(-172-86i\sqrt{5}\right)z^5+\left(-215-172i\sqrt{5}\right)z^4-258i\sqrt{5}z^3\\
&\quad+\left(1470+1224i\sqrt{5}\right)z^2+\left(1860+2328i\sqrt{5}\right)z-3036i\sqrt{5}-2943\\
b_2(z)&=\left(-652-326i\sqrt{5}\right)z^5+\left(-815-652i\sqrt{5}\right)z^4-978i\sqrt{5}z^3\\
&\quad+\left(1832i\sqrt{5}+1750\right)z^2+\left(3508i\sqrt{5}+3020\right)z\\
b_3(z)&=\left(4\sqrt{5}+10i\right)z^5+\left(5\sqrt{5}+20i\right)z^4+30iz^3+\left(-53\sqrt{5}-260i\right)z^2.
\end{aligned}$$

We have $\mathsf{a}_1(z) = \mathsf{a}_1(z^{-1})$, $\mathsf{a}_2(z) = -\mathsf{a}_2(z^{-1})$, and $\mathsf{a}_3(z) = -z^3\mathsf{a}_3(z^{-1})$. Let ϕ be the 3-refinable function associated with the low-pass filter a_0. Let ψ^1, ψ^2, ψ^3 be the wavelet functions associated with the high-pass filters $\mathsf{a}_1, \mathsf{a}_2, \mathsf{a}_3$ by (10), respectively. Then $\phi(-\cdot) = \phi$, $\psi^1(-\cdot) = \psi^1$, $\psi^2(-\cdot) = -\psi^2$, and $\psi^3(1-\cdot) = -\psi^3$. See Fig. 2 for the graphs of ϕ, ψ^1, ψ^2, and ψ^3.

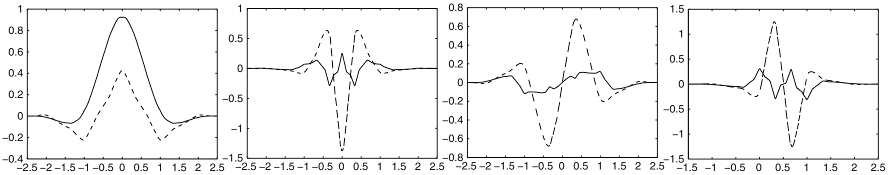

Fig. 2: The graphs of ϕ, ψ^1, ψ^2, and ψ^3 (*left to right*) in Example 2. Real part: *solid line*. Imaginary part: *dashed line*

4 Biorthogonal Matrix Extension with Symmetry

In this section, we shall discuss the construction of biorthogonal multiwavelets with symmetry, which corresponds to Problem 2. Due to the flexibility of biorthogonality relation $\mathbb{P}\widetilde{\mathbb{P}}^* = I_r$, the biorthogonal matrix extension problem becomes far more complicated than that for the orthogonal matrix extension problem we considered in Sect. 2. The difficulty here is not the symmetry patterns of the extension matrices, but the support control of the extension matrices. Without considering any issue on support control, almost all results of Theorem 1 can be transferred to the biorthogonal case without much difficulty. In Theorem 1, the length of the coefficient support of the extension matrix can never exceed the length of the coefficient support of the given matrix. Yet, for the extension matrices in the biorthogonal extension case, we can no longer expect such nice result, that is, in this case, the length of the coefficient supports of the extension matrices might not be controlled by one of the given matrices. Nevertheless, we have the following result.

Theorem 4. *Let $\mathbb{F}$ be any subfield of $\mathbb{C}$. Let $(\mathbb{P}, \widetilde{\mathbb{P}})$ be a pair of $r \times s$ matrices of Laurent polynomials with coefficients in $\mathbb{F}$ such that $\mathsf{S}\mathbb{P} = \mathsf{S}\widetilde{\mathbb{P}} = (\mathsf{S}\theta_1)^*\mathsf{S}\theta_2$ for some $1 \times r$, $1 \times s$ vectors θ_1, θ_2 of Laurent polynomials with symmetry. Moreover, $\mathbb{P}(z)\widetilde{\mathbb{P}}^*(z) = I_r$ for all $z \in \mathbb{C}\backslash\{0\}$. Then there exists a pair of $s \times s$ square matrices $(\mathbb{P}_e, \widetilde{\mathbb{P}}_e)$ of Laurent polynomials with coefficients in $\mathbb{F}$ such that*

(1) $[I_r, 0]\mathbb{P}_e = \mathbb{P}$ *and* $[I_r, 0]\widetilde{\mathbb{P}}_e = \widetilde{\mathbb{P}}$*; that is, the submatrices of the first r rows of $\mathbb{P}_e, \widetilde{\mathbb{P}}_e$ are $\mathbb{P}, \widetilde{\mathbb{P}}$, respectively;*
(2) $(\mathbb{P}_e, \widetilde{\mathbb{P}}_e)$ *is a pair of biorthogonal matrices: $\mathbb{P}_e(z)\widetilde{\mathbb{P}}_e^*(z) = I_s$ for all $z \in \mathbb{C}\backslash\{0\}$;*

(3) *the symmetry of each* $\mathbb{P}_e, \widetilde{\mathbb{P}}_e$ *is compatible:* $\mathsf{S}\mathbb{P}_e = \mathsf{S}\widetilde{\mathbb{P}}_e = (\mathsf{S}\theta)^*\mathsf{S}\theta_2$ *for some* $1 \times s$ *vector* θ *of Laurent polynomials with symmetry;*

(4) $\mathbb{P}_e, \widetilde{\mathbb{P}}_e$ *can be represented as:*

$$\mathbb{P}_e(z) = \mathbb{P}_J(z) \cdots \mathbb{P}_1(z), \quad \widetilde{\mathbb{P}}_e(z) = \widetilde{\mathbb{P}}_J(z) \cdots \widetilde{\mathbb{P}}_1(z), \tag{64}$$

where $(\mathbb{P}_j, \widetilde{\mathbb{P}}_j), 1 \le j \le J$ *are pairs of* $s \times s$ *biorthogonal matrices of Laurent polynomials with symmetry. Moreover, each pair of* $(\mathbb{P}_{j+1}, \mathbb{P}_j)$ *and* $(\widetilde{\mathbb{P}}_{j+1}, \widetilde{\mathbb{P}}_j)$ *has mutually compatible symmetry for all* $j = 1, \ldots, J-1$.

(5) *if* $r = 1$, *then the coefficient supports of* $\mathbb{P}_e, \widetilde{\mathbb{P}}_e$ *are controlled by those of* $\mathbb{P}, \widetilde{\mathbb{P}}$ *in the following sense:*

$$\max_{1\le j,k\le s} \{|\mathrm{csupp}([\mathbb{P}_e]_{j,k})|, |\mathrm{csupp}([\widetilde{\mathbb{P}}_e]_{j,k})|\} \le \max_{1\le \ell\le s} |\mathrm{csupp}([\mathbb{P}]_\ell)| + \max_{1\le \ell\le s} |\mathrm{csupp}([\widetilde{\mathbb{P}}]_\ell)|. \tag{65}$$

4.1 Proof of Theorem 4 and an Algorithm

In this section, we shall prove Theorem 4. Based on the proof, we shall provide a step-by-step extension algorithm for deriving the desired pair of biorthogonal extension matrices in Theorem 4.

In this section, $\mathbb{F}$ denote any subfield of $\mathbb{C}$. The next lemma shows that for a pair of constant vectors $(\mathtt{f}, \widetilde{\mathtt{f}})$ in $\mathbb{F}$, we can find a pair of constant biorthogonal matrices $(U_{(\mathtt{f},\widetilde{\mathtt{f}})}, \widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})})$ in $\mathbb{F}$ such that up to a constant multiplication, it normalizes $(\mathtt{f}, \widetilde{\mathtt{f}})$ to a pair of unit vectors.

Lemma 4. *Let* $(\mathtt{f}, \widetilde{\mathtt{f}})$ *be a pair of nonzero* $1 \times n$ *vectors in* $\mathbb{F}$. *Then,*

(1) *if* $\mathtt{f}\widetilde{\mathtt{f}}^* \ne 0$, *then there exists a pair of* $n \times n$ *matrices* $(U_{(\mathtt{f},\widetilde{\mathtt{f}})}, \widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})})$ *in* $\mathbb{F}$ *such that* $U_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{\widetilde{f}}{\widetilde{c}})^*, F]$, $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{f}{c})^*, \widetilde{F}]$, *and* $U_{(\mathtt{f},\widetilde{\mathtt{f}})}\widetilde{U}^*_{(\mathtt{f},\widetilde{\mathtt{f}})} = I_n$, *where* $F, \widetilde{F}$ *are* $n \times (n-1)$ *constant matrices in* $\mathbb{F}$ *and* $c, \widetilde{c}$ *are two nonzero numbers in* $\mathbb{F}$ *such that* $\mathtt{f}\widetilde{\mathtt{f}}^* = c\overline{\widetilde{c}}$. *In this case,* $\mathtt{f}U_{(\mathtt{f},\widetilde{\mathtt{f}})} = c\varepsilon_1$ *and* $\widetilde{\mathtt{f}}\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} = \widetilde{c}\varepsilon_1$;

(2) *if* $\mathtt{f}\widetilde{\mathtt{f}}^* = 0$, *then there exists a pair of* $n \times n$ *matrices* $(U_{(\mathtt{f},\widetilde{\mathtt{f}})}, \widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})})$ *in* $\mathbb{F}$ *such that* $U_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{f}{\widetilde{c}_1})^*, (\frac{\widetilde{f}}{c_2})^*, F]$, $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{f}{c_1})^*, (\frac{\widetilde{f}}{\widetilde{c}_2})^*, \widetilde{F}]$, *and* $U_{(\mathtt{f},\widetilde{\mathtt{f}})}\widetilde{U}^*_{(\mathtt{f},\widetilde{\mathtt{f}})} = I_n$, *where* $F, \widetilde{F}$ *are* $n \times (n-2)$ *constant matrices in* $\mathbb{F}$ *and* $c_1, c_2, \widetilde{c}_1, \widetilde{c}_2$ *are nonzero numbers in* $\mathbb{F}$ *such that* $\|\mathtt{f}\|^2 = c_1\overline{\widetilde{c}_1}, \|\widetilde{\mathtt{f}}\|^2 = c_2\overline{\widetilde{c}_2}$. *In this case,* $\mathtt{f}U_{(\mathtt{f},\widetilde{\mathtt{f}})} = c_1\varepsilon_1$ *and* $\widetilde{\mathtt{f}}\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} = c_2\varepsilon_2$.

Proof. If $\mathtt{f}\widetilde{\mathtt{f}}^* \ne 0$, there exists $\{\mathtt{f}_2, \ldots, \mathtt{f}_n\}$ being a basis of the orthogonal compliment of the linear span of $\{\mathtt{f}\}$ in $\mathbb{F}^n$. Let $F := [\mathtt{f}_2^*, \ldots, \mathtt{f}_n^*]$ and $U_{(\mathtt{f},\widetilde{\mathtt{f}})} := [(\frac{\widetilde{\mathtt{f}}}{\widetilde{c}})^*, F]$. Then $U_{(\mathtt{f},\widetilde{\mathtt{f}})}$ is invertible. Let $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} := (U^{-1}_{(\mathtt{f},\widetilde{\mathtt{f}})})^*$. It is easy to show that $U_{(\mathtt{f},\widetilde{\mathtt{f}})}$ and $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})}$ are the desired matrices.

If $\mathtt{f}\widetilde{\mathtt{f}}^* = 0$, let $\{\mathtt{f}_3,\dots,\mathtt{f}_n\}$ be a basis of the orthogonal compliment of the linear span of $\{\mathtt{f},\widetilde{\mathtt{f}}\}$ in $\mathbb{F}^n$. Let $U_{(\mathtt{f},\widetilde{\mathtt{f}})} = [(\frac{\mathtt{f}}{\widetilde{c}_1})^*,(\frac{\widetilde{\mathtt{f}}}{c_2})^*,F]$ with $F := [\mathtt{f}_3^*,\dots,\mathtt{f}_n^*]$. Then $U_{(\mathtt{f},\widetilde{\mathtt{f}})}$ and $\widetilde{U}_{(\mathtt{f},\widetilde{\mathtt{f}})} := (U^{-1}_{(\mathtt{f},\widetilde{\mathtt{f}})})^*$ are the desired matrices. $\square$ $\square$

Thanks to Lemma 4, we can reduce the support lengths of a pair $(\mathsf{p},\widetilde{\mathsf{p}})$ of Laurent polynomials with symmetry by constructing a pair of biorthogonal matrices $(\mathsf{B},\widetilde{\mathsf{B}})$ of Laurent polynomials with symmetry as stated in the following lemma.

Lemma 5. *Let $(\mathsf{p},\widetilde{\mathsf{p}})$ be a pair of $1\times s$ vectors of Laurent polynomials with symmetry such that $\mathsf{p}\widetilde{\mathsf{p}}^* = 1$ and $\mathsf{S}\mathsf{p} = \mathsf{S}\widetilde{\mathsf{p}} = \varepsilon z^c[1_{s_1},-1_{s_2},z^{-1}1_{s_3},-z^{-1}1_{s_4}] =: \mathsf{S}\theta$ for some nonnegative integers $s_1,\dots,s_4$ satisfying $s_1+\cdots+s_4 = s$ and $\varepsilon\in\{1,-1\}, c\in\{0,1\}$. Suppose $|\mathrm{csupp}(\mathsf{p})|>0$. Then there exists a pair of $s\times s$ matrices $(\mathsf{B},\widetilde{\mathsf{B}})$ of Laurent polynomials with symmetry such that*

(1) $(\mathsf{B},\widetilde{\mathsf{B}})$ *is a pair of biorthogonal matrices:* $\mathsf{B}(z)\widetilde{\mathsf{B}}^*(z) = I_n$*;*
(2) $\mathsf{S}\mathsf{B} = \mathsf{S}\widetilde{\mathsf{B}} = (\mathsf{S}\theta)^*\mathsf{S}\theta_1$ *with* $\mathsf{S}\theta_1 = \varepsilon z^c[1_{s_1'},-1_{s_2'},z^{-1}1_{s_3'},-z^{-1}1_{s_4'}]$ *for some nonnegative integers* $s_1',\dots,s_4'$ *such that* $s_1'+\cdots+s_4' = s$*;*
(3) *the length of the coefficient support of* p *is reduced by that of* B. $\widetilde{\mathsf{B}}$ *does not increase the length of the coefficient support of* $\widetilde{\mathsf{p}}$*. That is,* $|\mathrm{csupp}(\mathsf{p}\mathsf{B})| \le |\mathrm{csupp}(\mathsf{p})| - |\mathrm{csupp}(\mathsf{B})|$ *and* $|\mathrm{csupp}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}})| \le |\mathrm{csupp}(\widetilde{\mathsf{p}})|$.

Proof. We shall only prove the case that $\mathsf{S}\theta = [1_{s_1},-1_{s_2},z^{-1}1_{s_3},-z^{-1}1_{s_4}]$. The proofs for other cases are similar. By their symmetry patterns, p and $\widetilde{\mathsf{p}}$ must take the forms as follows with $\ell>0$ and $\mathrm{coeff}(\mathsf{p},-\ell)\ne 0$:

$$\begin{aligned}
\mathsf{p} &= [\mathtt{f}_1,-\mathtt{f}_2,\mathtt{g}_1,-\mathtt{g}_2]z^{-\ell} + [\mathtt{f}_3,-\mathtt{f}_4,\mathtt{g}_3,-\mathtt{g}_4]z^{-\ell+1} + \sum_{k=-\ell+2}^{\ell-2}\mathrm{coeff}(\mathsf{p},k)z^k \\
&\quad + [\mathtt{f}_3,\mathtt{f}_4,\mathtt{g}_1,\mathtt{g}_2]z^{\ell-1} + [\mathtt{f}_1,\mathtt{f}_2,\mathbf{0},0]z^{\ell}; \\
\widetilde{\mathsf{p}} &= [\widetilde{\mathtt{f}}_1,-\widetilde{\mathtt{f}}_2,\widetilde{\mathtt{g}}_1,-\widetilde{\mathtt{g}}_2]z^{-\widetilde{\ell}} + [\widetilde{\mathtt{f}}_3,-\widetilde{\mathtt{f}}_4,\widetilde{\mathtt{g}}_3,-\widetilde{\mathtt{g}}_4]z^{-\widetilde{\ell}+1} + \sum_{k=-\widetilde{\ell}+2}^{\widetilde{\ell}-2}\mathrm{coeff}(\widetilde{\mathsf{p}},k)z^k \\
&\quad + [\widetilde{\mathtt{f}}_3,\widetilde{\mathtt{f}}_4,\widetilde{\mathtt{g}}_1,\widetilde{\mathtt{g}}_2]z^{\widetilde{\ell}-1} + [\widetilde{\mathtt{f}}_1,\widetilde{\mathtt{f}}_2,\mathbf{0},0]z^{\widetilde{\ell}}.
\end{aligned} \tag{66}$$

Then, either $\|\mathtt{f}_1\|+\|\mathtt{f}_2\|\ne 0$ or $\|\mathtt{g}_1\|+\|\mathtt{g}_2\|\ne 0$. Considering $\|\mathtt{f}_1\|+\|\mathtt{f}_2\|\ne 0$, due to $\mathsf{p}\widetilde{\mathsf{p}}^* = 1$ and $|\mathrm{csupp}(\mathsf{p})|>0$, we have $\mathtt{f}_1\widetilde{\mathtt{f}}_1^* - \mathtt{f}_2\widetilde{\mathtt{f}}_2^* = 0$. Let $C := \mathtt{f}_1\widetilde{\mathtt{f}}_1^* = \mathtt{f}_2\widetilde{\mathtt{f}}_2^*$. There are at most three cases: (a) $C\ne 0$; (b) $C=0$ but both $\mathtt{f}_1,\mathtt{f}_2$ are nonzero vectors; (c) $C=0$ and one of $\mathtt{f}_1,\mathtt{f}_2$ is 0.

Case (a). In this case, we have $\mathtt{f}_1\widetilde{\mathtt{f}}_1^*\ne 0$ and $\mathtt{f}_2\widetilde{\mathtt{f}}_2^*\ne 0$. By Lemma 4, we can construct two pairs of biorthogonal matrices $(U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)},\widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)})$ and $(U_{(\mathtt{f}_2,\widetilde{\mathtt{f}}_2)},\widetilde{U}_{(\mathtt{f}_2,\widetilde{\mathtt{f}}_2)})$ with respect to the pairs $(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)$ and $(\mathtt{f}_2,\widetilde{\mathtt{f}}_2)$ such that

$$\begin{aligned}
&U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)} = \left[\left(\frac{\widetilde{\mathtt{f}}_1}{\widetilde{c}_1}\right)^*,F_1\right],\quad \widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)} = \left[\left(\frac{\mathtt{f}_1}{c_1}\right)^*,\widetilde{F}_1\right],\quad \mathtt{f}_1U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)} = c_1\varepsilon_1,\quad \widetilde{\mathtt{f}}_1\widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)} = \widetilde{c}_1\varepsilon_1,\\
&U_{(\mathtt{f}_2,\widetilde{\mathtt{f}}_2)} = \left[\left(\frac{\widetilde{\mathtt{f}}_2}{\widetilde{c}_1}\right)^*,F_2\right],\quad \widetilde{U}_{(\mathtt{f}_2,\widetilde{\mathtt{f}}_2)} = \left[\left(\frac{\mathtt{f}_2}{c_1}\right)^*,\widetilde{F}_2\right],\quad \mathtt{f}_2U_{(\mathtt{f}_2,\widetilde{\mathtt{f}}_2)} = c_1\varepsilon_1,\quad \widetilde{\mathtt{f}}_2\widetilde{U}_{(\mathtt{f}_2,\widetilde{\mathtt{f}}_2)} = \widetilde{c}_1\varepsilon_1,
\end{aligned}$$

where $c_1, \widetilde{c}_1$ are constants in $\mathbb{F}$ such that $C = c_1\overline{\widetilde{c}_1}$. Define $\mathsf{B}_0(z), \widetilde{\mathsf{B}}_0(z)$ as follows:

$$\begin{aligned}
\mathsf{B}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{\widetilde{\mathtt{f}}_1}{\widetilde{c}_1})^* & F_1 & -\frac{1-z^{-1}}{2}(\frac{\widetilde{\mathtt{f}}_1}{\widetilde{c}_1})^* & 0 & 0 \\ -\frac{1-z^{-1}}{2}(\frac{\widetilde{\mathtt{f}}_2}{\widetilde{c}_1})^* & 0 & \frac{1+z^{-1}}{2}(\frac{\widetilde{\mathtt{f}}_2}{\widetilde{c}_1})^* & F_2 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{s_3+s_4} \end{array}\right], \\
\widetilde{\mathsf{B}}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_1}{c_1})^* & \widetilde{F}_1 & -\frac{1-z^{-1}}{2}(\frac{\mathtt{f}_1}{c_1})^* & 0 & 0 \\ -\frac{1-z^{-1}}{2}(\frac{\mathtt{f}_2}{c_1})^* & 0 & \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_2}{c_1})^* & \widetilde{F}_2 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{s_3+s_4} \end{array}\right].
\end{aligned} \tag{67}$$

Direct computation shows that $\mathsf{B}_0(z)\widetilde{\mathsf{B}}_0(z)^* = I_s$ due to the special structures of the pairs $(U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)}, \widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)})$ and $(U_{(\mathtt{f}_2,\widetilde{\mathtt{f}}_2)}, \widetilde{U}_{(\mathtt{f}_2,\widetilde{\mathtt{f}}_2)})$ constructed by Lemma 4. The symmetry patterns of $\mathsf{p}\mathsf{B}_0$ and $\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_0$ satisfies

$$\mathsf{S}(\mathsf{p}\mathsf{B}_0) = \mathsf{S}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_0) = [z^{-1}, 1_{s_1-1}, -z^{-1}, -1_{s_2-1}, z^{-1}1_{s_3}, -z^{-1}1_{s_4}].$$

Moreover, $\mathsf{B}_0(z)$, $\widetilde{\mathsf{B}}_0(z)$ reduce the lengths of the coefficient support of p and $\widetilde{\mathsf{p}}$ by 1, respectively.

In fact, due to the above symmetry pattern and the structures of $\mathsf{B}_0, \widetilde{\mathsf{B}}_0$, we only need to show that $\mathrm{coeff}([\mathsf{p}\mathsf{B}_0]_j, \ell) = \mathrm{coeff}([\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_0]_j, \ell) = 0$ for $j = 1, s_1+1$. Note that $\mathrm{coeff}([\mathsf{p}\mathsf{B}_0]_j, \ell) = \mathrm{coeff}(\mathsf{p}, \ell)\mathrm{coeff}([\mathsf{B}_0]_{:,1}, 0) = \frac{1}{2\overline{\widetilde{c}_1}}(\mathtt{f}_1\widetilde{\mathtt{f}}_1^* - \mathtt{f}_2\widetilde{\mathtt{f}}_2^*) = 0$. Similar computations apply for other terms. Thus, $|\mathrm{csupp}(\mathsf{p}\mathsf{B}_0)| < \mathrm{csupp}(\mathsf{p})$ and $|\mathrm{csupp}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_0)| < |\mathrm{csupp}(\widetilde{\mathsf{p}})|$. Let E be a permutation matrix such that

$$\mathsf{S}(\mathsf{p}\mathsf{B}_0)E = \mathsf{S}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_0)E = [1_{s_1-1}, -1_{s_2-1}, z^{-1}1_{s_3+1}, -z^{-1}1_{s_4+1}] =: \mathsf{S}\theta_1.$$

Define $\mathsf{B}(z) = \mathsf{B}_0(z)E$ and $\widetilde{\mathsf{B}}(z) = \widetilde{\mathsf{B}}_0(z)E$. Then $\mathsf{B}(z)$ and $\widetilde{\mathsf{B}}(z)$ are the desired matrices.

Case (b). In this case, $\mathtt{f}_1\widetilde{\mathtt{f}}_1^* = \mathtt{f}_2\widetilde{\mathtt{f}}_2^* = 0$ and both $\mathtt{f}_1, \mathtt{f}_2$ are nonzero vectors. We have $\mathtt{f}_1\mathtt{f}_1^* \neq 0$ and $\mathtt{f}_2\mathtt{f}_2^* \neq 0$. Again, by Lemma 4, we can construct two pairs of biorthogonal matrices $(U_{(\mathtt{f}_1,\mathtt{f}_1)}, \widetilde{U}_{(\mathtt{f}_1,\mathtt{f}_1)})$ and $(U_{(\mathtt{f}_2,\mathtt{f}_2)}, \widetilde{U}_{(\mathtt{f}_2,\mathtt{f}_2)})$ with respect to the pairs $(\mathtt{f}_1, \mathtt{f}_1)$ and $(\mathtt{f}_2, \mathtt{f}_2)$ such that

$$\begin{aligned}
U_{(\mathtt{f}_1,\mathtt{f}_1)} &= \left[\left(\frac{\mathtt{f}_1}{\widetilde{c}_1}\right)^*, F_1\right], \quad \widetilde{U}_{(\mathtt{f}_1,\mathtt{f}_1)} = \left[\left(\frac{\mathtt{f}_1}{c_0}\right)^*, F_1\right], \quad \mathtt{f}_1 U_{(\mathtt{f}_1,\mathtt{f}_1)} = c_0\varepsilon_1, \\
U_{(\mathtt{f}_2,\mathtt{f}_2)} &= \left[\left(\frac{\mathtt{f}_2}{\widetilde{c}_2}\right)^*, F_2\right], \quad \widetilde{U}_{(\mathtt{f}_2,\mathtt{f}_2)} = \left[\left(\frac{\mathtt{f}_2}{c_0}\right)^*, F_2\right], \quad \mathtt{f}_2 U_{(\mathtt{f}_2,\mathtt{f}_2)} = c_0\varepsilon_1,
\end{aligned}$$

where $c_0, \widetilde{c}_1, \widetilde{c}_2$ are constants in $\mathbb{F}$ such that $\mathtt{f}_1\mathtt{f}_1^* = c_0\overline{\widetilde{c}_1}$ and $\mathtt{f}_2\mathtt{f}_2^* = c_0\overline{\widetilde{c}_2}$. Let $\mathsf{B}_0, \widetilde{\mathsf{B}}_0(z)$ be defined as follows:

$$
\begin{aligned}
\mathsf{B}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_1}{c_1})^* & F_1 & -\frac{1-z^{-1}}{2}(\frac{\mathtt{f}_1}{c_1})^* & 0 & 0 \\ -\frac{1-z^{-1}}{2}(\frac{\mathtt{f}_2}{\tilde{c}_2})^* & 0 & \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_2}{\tilde{c}_2})^* & F_2 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{s_3+s_4} \end{array}\right], \\
\widetilde{\mathsf{B}}_0(z) &= \left[\begin{array}{cc|cc|c} \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_1}{c_0})^* & F_1 & -\frac{1-z^{-1}}{2}(\frac{\mathtt{f}_1}{c_0})^* & 0 & 0 \\ -\frac{1-z^{-1}}{2}(\frac{\mathtt{f}_2}{c_0})^* & 0 & \frac{1+z^{-1}}{2}(\frac{\mathtt{f}_2}{c_0})^* & F_2 & 0 \\ \hline 0 & 0 & 0 & 0 & I_{s_3+s_4} \end{array}\right].
\end{aligned} \tag{68}
$$

We can show that $\mathsf{B}_0(z)$ reduces the length of the coefficient support of p by 1, while $\widetilde{\mathsf{B}}_0(z)$ does not increase the support length of $\widetilde{\mathsf{p}}$. Moreover, similar to case (a), we can find a permutation matrix E such that

$$
\mathsf{S}(\mathsf{p}\mathsf{B}_0)E = \mathsf{S}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_0)E = [1_{s_1-1}, -1_{s_2-1}, z^{-1}1_{s_3+1}, -z^{-1}1_{s_4+1}] =: \mathsf{S}\theta_1.
$$

Define $\mathsf{B}(z) = \mathsf{B}_0(z)E$ and $\widetilde{\mathsf{B}}(z) = \widetilde{\mathsf{B}}_0(z)E$. Then $\mathsf{B}(z)$ and $\widetilde{\mathsf{B}}(z)$ are the desired matrices.

Case (c). In this case, $\mathtt{f}_1\widetilde{\mathtt{f}}_1^* = \mathtt{f}_2\widetilde{\mathtt{f}}_2^* = 0$ and one of $\mathtt{f}_1$ and $\mathtt{f}_2$ is nonzero. Without loss of generality, we assume that $\mathtt{f}_1 \neq 0$ and $\mathtt{f}_2 = 0$. Construct a pair of matrices $(U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)}, \widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)})$ by Lemma 4 such that $\mathtt{f}_1 U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)} = c_1\varepsilon_1$ and $\widetilde{\mathtt{f}}_1\widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)} = c_2\varepsilon_2$ (when $\widetilde{\mathtt{f}}_1 = 0$, the pair of matrices is given by $(U_{(\mathtt{f}_1,\mathtt{f}_1)}, \widetilde{U}_{(\mathtt{f}_1,\mathtt{f}_1)})$). Extend this pair to a pair of $s \times s$ matrices $(U, \widetilde{U})$ by $U := \operatorname{diag}(U_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)}, I_{s_3+s_4})$ and $\widetilde{U} := \operatorname{diag}(\widetilde{U}_{(\mathtt{f}_1,\widetilde{\mathtt{f}}_1)}, I_{s_3+s_4})$. Then $\mathsf{p}U$ and $\widetilde{\mathsf{p}}\widetilde{U}$ must be of the form:

$$
\begin{aligned}
\mathsf{q} := \mathsf{p}U &= [c_1, 0, \ldots, 0, -\mathtt{f}_2, \mathtt{g}_1, -\mathtt{g}_2]z^{-\ell} + [\mathtt{f}_3, -\mathtt{f}_4, \mathtt{g}_3, -\mathtt{g}_4]z^{-\ell+1} \\
&\quad + \sum_{k=-\ell+2}^{\ell-2} \operatorname{coeff}(\mathsf{q}, k)z^k + [\mathtt{f}_3, \mathtt{f}_4, \mathtt{g}_1, \mathtt{g}_2]z^{\ell-1} + [c_1, 0, \ldots, 0, \mathtt{f}_2, \mathbf{0}, 0]z^{\ell};
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\mathsf{q}} := \widetilde{\mathsf{p}}\widetilde{U} &= [0, c_2, \ldots, 0, -\widetilde{\mathtt{f}}_2, \widetilde{\mathtt{g}}_1, -\widetilde{\mathtt{g}}_2]z^{-\widetilde{\ell}} + [\widetilde{\mathtt{f}}_3, -\widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_3, -\widetilde{\mathtt{g}}_4]z^{-\widetilde{\ell}+1} \\
&\quad + \sum_{k=-\widetilde{\ell}+2}^{\widetilde{\ell}-2} \operatorname{coeff}(\widetilde{\mathsf{q}}, k)z^k + [\widetilde{\mathtt{f}}_3, \widetilde{\mathtt{f}}_4, \widetilde{\mathtt{g}}_1, \widetilde{\mathtt{g}}_2]z^{\widetilde{\ell}-1} + [0, c_2, \ldots, 0, , \widetilde{\mathtt{f}}_2, \mathbf{0}, 0]z^{\widetilde{\ell}}.
\end{aligned}
$$

If $[\widetilde{\mathsf{q}}]_1 \equiv 0$, we choose k such that $k = \arg\min_{\ell \neq 1}\{|\operatorname{csupp}([\mathsf{q}]_1)| - |\operatorname{csupp}([\mathsf{q}]_\ell)|\}$, i.e., k is an integer such that the length of coefficient support of $|\operatorname{csupp}([\mathsf{q}]_1)| - |\operatorname{csupp}([\mathsf{q}]_k)|$ is minimal among those of all $|\operatorname{csupp}([\mathsf{q}]_1)| - |\operatorname{csupp}([\mathsf{q}]_\ell)|$, $\ell = 2, \ldots, s$; otherwise, due to $\mathsf{q}\widetilde{\mathsf{q}}^* = 0$, there must exist k such that

$$
|\operatorname{csupp}([\mathsf{q}]_1)| - |\operatorname{csupp}([\mathsf{q}]_k)| \le \max_{2 \le j \le s} |\operatorname{csupp}([\widetilde{\mathsf{q}}]_j)| - |\operatorname{csupp}([\widetilde{\mathsf{q}}]_1)|,
$$

(k might not be unique, we can choose one of such k so that $|\operatorname{csupp}([\mathsf{q}]_1)| - |\operatorname{csupp}([\mathsf{q}]_k)|$ is minimal among all $|\operatorname{csupp}([\mathsf{q}]_1)| - |\operatorname{csupp}([\mathsf{q}]_\ell)|, \ell = 2, \ldots, s$).

For such k (in the case of either $[\widetilde{\mathsf{q}}]_1 = 0$ or $[\widetilde{\mathsf{q}}]_1 \neq 0$), define two matrices $\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ as follows:

$$\mathsf{B}(z) = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ -b(z) & 0 & \cdots & 1 & \\ \hline & & & & I_{s-k} \end{array}\right], \widetilde{\mathsf{B}}(z) = \left[\begin{array}{cccc|c} 1 & 0 & \cdots & b^*(z) & \\ 0 & 1 & \cdots & 0 & \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & \cdots & 1 & \\ \hline & & & & I_{s-k} \end{array}\right],$$

where $b(z)$ in $\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ is a Laurent polynomial with symmetry such that $\mathsf{S}b(z) = \mathsf{S}([\mathsf{q}]_1/[\mathsf{q}]_k)$, $|\mathrm{csupp}([\mathsf{q}]_1 - b(z)[\mathsf{q}]_k)| < |\mathrm{csupp}([\mathsf{q}]_k)|$, and $|\mathrm{csupp}([\widetilde{\mathsf{q}}]_k - b^*(z)[\widetilde{\mathsf{q}}]_1)| \le \max_{1\le \ell \le s} |\mathrm{csupp}([\widetilde{\mathsf{q}}]_\ell)|$. Such $b(z)$ can be easily obtained by long division.

It is straightforward to show that $\mathsf{B}(z)\widetilde{\mathsf{B}}^*(z) = I_s$. $\mathsf{B}(z)$ reduces the length of the coefficient support of q by that of $b(z)$ due to $|\mathrm{csupp}([\mathsf{q}]_1 - b(z)[\mathsf{q}]_k)| < |\mathrm{csupp}([\mathsf{q}]_k)|$. And by our choice of k, $\widetilde{\mathsf{B}}(z)$ does not increase the length of the coefficient support of $\widetilde{\mathsf{q}}$. Moreover, the symmetry patterns of both q and $\widetilde{\mathsf{q}}$ are preserved.

In summary, for all cases (a), (b), and (c), we can always find a pair of biorthogonal matrices $(\mathsf{B}, \widetilde{\mathsf{B}})$ of Laurent polynomials such that B reduces the length of the coefficient support of p while $\widetilde{\mathsf{B}}$ does not increase the length of the coefficient support of $\widetilde{\mathsf{p}}$.

For $\|\mathtt{f}_1\| + \|\mathtt{f}_2\| = 0$, we must have $\|\mathsf{g}_1\| + \|\mathsf{g}_2\| \neq 0$. The discussion for this case is similar to above. We can find two matrices $\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ such that all items in the lemma hold. In the case that $\mathsf{g}_1\widetilde{\mathsf{g}}_1^* = \mathsf{g}_2\widetilde{\mathsf{g}}_2^* = c_1\overline{\widetilde{c}_1} \neq 0$, the pair $(\mathsf{B}_0(z), \widetilde{\mathsf{B}}_0(z))$ similar to (67) is of the form

$$\begin{aligned} \mathsf{B}_0(z) &= \left[\begin{array}{c|cc|cc} I_{s_1+s_2} & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1+z}{2}\left(\frac{\widetilde{\mathsf{g}}_1}{\widetilde{c}_1}\right)^* & G_1 & -\frac{1-z}{2}\left(\frac{\widetilde{\mathsf{g}}_1}{\widetilde{c}_1}\right)^* & 0 \\ 0 & -\frac{1-z}{2}\left(\frac{\widetilde{\mathsf{g}}_2}{\widetilde{c}_1}\right)^* & 0 & \frac{1+z}{2}\left(\frac{\widetilde{\mathsf{g}}_2}{\widetilde{c}_1}\right)^* & G_2 \end{array}\right], \\ \widetilde{\mathsf{B}}_0(z) &= \left[\begin{array}{c|cc|cc} I_{s_1+s_2} & 0 & 0 & 0 & 0 \\ \hline 0 & \frac{1+z}{2}\left(\frac{\mathsf{g}_1}{c_1}\right)^* & \widetilde{G}_1 & -\frac{1-z}{2}\left(\frac{\mathsf{g}_1}{c_1}\right)^* & 0 \\ 0 & -\frac{1-z}{2}\left(\frac{\mathsf{g}_2}{c_1}\right)^* & 0 & \frac{1+z}{2}\left(\frac{\mathsf{g}_2}{c_1}\right)^* & \widetilde{G}_2 \end{array}\right]. \end{aligned} \tag{69}$$

The pairs for other cases can be obtained similarly. We are done. □ □

Now, we can prove Theorem 4 using Lemma 5.

Proof (of Theorem 4). First, we normalize the symmetry patterns of $\mathbb{P}$ and $\widetilde{\mathbb{P}}$ to the standard form as in (22). Let $\mathsf{Q} := \mathsf{U}^*_{\mathsf{S}\theta_1}\mathbb{P}\mathsf{U}_{\mathsf{S}\theta_2}$ and $\widetilde{\mathsf{Q}} := \mathsf{U}^*_{\mathsf{S}\theta_1}\widetilde{\mathbb{P}}\mathsf{U}_{\mathsf{S}\theta_2}$ (given θ, $\mathsf{U}_{\mathsf{S}\theta}$ is obtained by (23)). Then the symmetry of each row of Q or $\widetilde{\mathsf{Q}}$ is of the form $\varepsilon z^c[1_{s_1}, -1_{s_2}, z^{-1}1_{s_3}, -z^{-1}1_{s_4}]$ for some $\varepsilon \in \{-1,1\}$ and $c \in \{0,1\}$.

Let $\mathsf{p} := [\mathsf{Q}]_{1,:}$ and $\widetilde{\mathsf{p}} := [\widetilde{\mathsf{Q}}]_{1,:}$ be the first row of $\mathsf{Q}, \widetilde{\mathsf{Q}}$, respectively. Applying Lemma 5 recursively, we can find pairs of biorthogonal matrices of Laurent polynomials $(\mathsf{B}_1, \widetilde{\mathsf{B}}_1)$, ..., $(\mathsf{B}_K, \widetilde{\mathsf{B}}_K)$ such that $\mathsf{p}\mathsf{B}_1\cdots\mathsf{B}_K = [1,0,\ldots,0]$ and $\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}_1\cdots\widetilde{\mathsf{B}}_K = [1,\mathsf{q}(z)]$ for some $1\times(s-1)$ vector of Laurent polynomials with symmetry. Note that by Lemma 5, all pairs $(\mathsf{B}_j, \mathsf{B}_{j+1})$ and $(\widetilde{\mathsf{B}}_j, \widetilde{\mathsf{B}}_{j+1})$ for $j = 1, \ldots, K-1$ have mutually compatible symmetry. Now construct $\mathsf{B}_{K+1}(z), \widetilde{\mathsf{B}}_{K+1}(z)$ as follows:

$$\mathsf{B}_{K+1}(z) = \begin{bmatrix} 1 & 0 \\ \mathsf{q}^*(z) & I_{s-1} \end{bmatrix}, \widetilde{\mathsf{B}}_{K+1}(z) = \begin{bmatrix} 1 & -\mathsf{q}(z) \\ 0 & I_{s-1} \end{bmatrix}.$$

B_{K+1} and $\widetilde{\mathsf{B}}_{K+1}$ are biorthogonal. Let $\mathsf{A} := \mathsf{B}_1 \cdots \mathsf{B}_K \mathsf{B}_{K+1}$ and $\widetilde{\mathsf{A}} := \widetilde{\mathsf{B}}_1 \cdots \widetilde{\mathsf{B}}_K \widetilde{\mathsf{B}}_{K+1}$. Then, $\mathsf{p}\mathsf{A} = \widetilde{\mathsf{p}}\widetilde{\mathsf{A}} = \varepsilon_1$.

Note that $\mathsf{Q}\mathsf{A}$ and $\widetilde{\mathsf{Q}}\widetilde{\mathsf{A}}$ are of the forms

$$\mathsf{Q}\mathsf{A} = \begin{bmatrix} 1 & 0 \\ 0 & \mathsf{Q}_1(z) \end{bmatrix}, \widetilde{\mathsf{Q}}\widetilde{\mathsf{A}} = \begin{bmatrix} 1 & 0 \\ 0 & \widetilde{\mathsf{Q}}_1(z) \end{bmatrix}$$

for some $(r-1) \times s$ matrices $\mathsf{Q}_1, \widetilde{\mathsf{Q}}_1$ of Laurent polynomials with symmetry. Moreover, due to Lemma 5, the symmetry patterns of Q_1 and $\widetilde{\mathsf{Q}}_1$ are compatible and satisfies $\mathsf{S}\mathsf{Q}_1 = \mathsf{S}\widetilde{\mathsf{Q}}_1$. The rest of the proof is completed by employing the standard procedure of induction. □ □

According to the proof of Theorem 4, we have an extension algorithm for Theorem 4. See Algorithm 4.

Algorithm 4 Biorthogonal matrix extension with symmetry

(a) **Input**: $\mathbb{P}, \widetilde{\mathbb{P}}$ as in Theorem 4 with $\mathsf{S}\mathbb{P} = \mathsf{S}\widetilde{\mathbb{P}} = (\mathsf{S}\theta_1)^*\mathsf{S}\theta_2$ for two $1 \times r$, $1 \times s$ row vectors θ_1, θ_2 of Laurent polynomials with symmetry.

(b) **Initialization**: Let $\mathsf{Q} := \mathsf{U}^*_{\mathsf{S}\theta_1}\mathbb{P}\mathsf{U}_{\mathsf{S}\theta_2}$ and $\widetilde{\mathsf{Q}} := \mathsf{U}^*_{\mathsf{S}\theta_1}\widetilde{\mathbb{P}}\mathsf{U}_{\mathsf{S}\theta_2}$. Then both Q and $\widetilde{\mathsf{Q}}$ have the same symmetry pattern as follows:

$$\mathsf{S}\mathsf{Q} = \mathsf{S}\widetilde{\mathsf{Q}} = [1_{r_1}, -1_{r_2}, z1_{r_3}, -z1_{r_4}]^{\mathrm{T}}[1_{s_1}, -1_{s_2}, z^{-1}1_{s_3}, -z^{-1}1_{s_4}], \tag{70}$$

where all nonnegative integers $r_1, \ldots, r_4, s_1, \ldots, s_4$ are uniquely determined by $\mathsf{S}\mathbb{P}$. Note that this step does not increase the lengths of the coefficient support of both $\mathbb{P}$ and $\widetilde{\mathbb{P}}$.

(c) **Support Reduction**:

1: Let $\mathsf{U}_0 := \mathsf{U}^*_{\mathsf{S}\theta_2}$ and $\mathsf{A} = \widetilde{\mathsf{A}} := I_s$.
2: **for** $k = 1$ to r **do**
3: Let $\mathsf{p} := [\mathsf{Q}]_{k,k:s}$ and $\widetilde{\mathsf{p}} := [\widetilde{\mathsf{Q}}]_{k,k:s}$.
4: **while** $|\mathrm{csupp}(\mathsf{p})| > 0$ and $|\mathrm{csupp}(\widetilde{\mathsf{p}})| > 0$ **do**
5: Construct a pair of biorthogonal matrices $(\mathsf{B}, \widetilde{\mathsf{B}})$ with respect to the pair $(\mathsf{p}, \widetilde{\mathsf{p}})$ by Lemma 5 such that $|\mathrm{csupp}(\mathsf{p}\mathsf{B})| + |\mathrm{csupp}(\widetilde{\mathsf{p}}\widetilde{\mathsf{B}})| < |\mathrm{csupp}(\mathsf{p})| + |\mathrm{csupp}(\widetilde{\mathsf{p}})|$.
6: Replace $\mathsf{p}, \widetilde{\mathsf{p}}$ by $\mathsf{p}\mathsf{B}$, $\widetilde{\mathsf{p}}\widetilde{\mathsf{B}}$, respectively.
7: Set $\mathsf{A} := \mathsf{A}\,\mathrm{diag}(I_{k-1}, \mathsf{B})$ and $\widetilde{\mathsf{A}} := \widetilde{\mathsf{A}}\,\mathrm{diag}(I_{k-1}, \widetilde{\mathsf{B}})$.
8: **end while**
9: The pair $(\mathsf{p}, \widetilde{\mathsf{p}})$ is of the form: $([1, 0, \ldots, 0], [1, \mathsf{q}(z)])$ for some $1 \times (s-k)$ vector of Laurent polynomials $\mathsf{q}(z)$. Construct $\mathsf{B}(z), \widetilde{\mathsf{B}}(z)$ as follows:

$$\mathsf{B}(z) = \begin{bmatrix} 1 & 0 \\ \mathsf{q}^*(z) & I_{s-k} \end{bmatrix}, \widetilde{\mathsf{B}}(z) = \begin{bmatrix} 1 & -\mathsf{q}(z) \\ 0 & I_{s-k} \end{bmatrix}.$$

10: Set $\mathsf{A} := \mathsf{A}\,\mathrm{diag}(I_{k-1}, \mathsf{B})$ and $\widetilde{\mathsf{A}} := \widetilde{\mathsf{A}}\,\mathrm{diag}(I_{k-1}, \widetilde{\mathsf{B}})$.
11: Set $\mathsf{Q} := \mathsf{Q}\mathsf{A}$ and $\widetilde{\mathsf{Q}} := \widetilde{\mathsf{Q}}\widetilde{\mathsf{A}}$.
12: **end for**

(d) **Finalization**: Let $\mathsf{U}_1 := \mathrm{diag}(\mathsf{U}_{\mathsf{S}\theta_1}, I_{s-r})$. Set $\mathbb{P}_e := \mathsf{U}_1\mathsf{A}^*\mathsf{U}_0$ and $\widetilde{\mathbb{P}}_e := \mathsf{U}_1\widetilde{\mathsf{A}}^*\mathsf{U}_0$.

(e) **Output**: A pair of desired matrices $(\mathbb{P}_e, \widetilde{\mathbb{P}}_e)$ satisfying all the properties in Theorem 4.

4.2 Application to Construction of Biorthogonal Multiwavelets with Symmetry

For the construction of biorthogonal refinable function vectors (a pair of biorthogonal low-pass filters), the CBC (*coset by coset*) algorithm proposed in [11] provides a systematic way of constructing a desirable dual mask from a given primal mask that satisfies certain conditions. More precisely, given a mask (low-pass filter) satisfying the condition that a dual mask exists, following the CBC algorithm, one can construct a dual mask with any preassigned orders of sum rules, which is closely related to the regularity of the refinable function vectors. Furthermore, if the primal mask has symmetry, then the CBC algorithm also guarantees that the dual mask has symmetry. Thus, the first part of MRA corresponding to the construction of biorthogonal multiwavelets is more or less solved. However, how to derive the wavelet generators (high-pass filters) with symmetry remains open even for the scalar case ($r=1$). We shall see that using our extension algorithm for the biorthogonal case, the wavelet generators do have symmetry once the given refinable function vectors possess certain symmetry patterns.

Let $(\phi,\widetilde{\phi})$ be a pair of dual d-refinable function vectors associated with *a pair of biorthogonal low-pass filters* $(a_0,\widetilde{a}_0)$, that is, $\phi,\widetilde{\phi}$ are d-refinable function vectors associated with $a_0,\widetilde{a}_0$, respectively, and

$$\langle \phi,\widetilde{\phi}(\cdot-k)\rangle=\delta(k)I_r,\quad k\in\mathbb{Z}. \tag{71}$$

It is easy to show that the pair of biorthogonal low-pass filters $(a_0,\widetilde{a}_0)$ satisfies

$$\sum_{\gamma=0}^{\mathsf{d}-1}\mathsf{a}_{0;\gamma}(z)\widetilde{\mathsf{a}}_{0;\gamma}^{*}(z)=I_r,\qquad z\in\mathbb{C}\backslash\{0\}, \tag{72}$$

where $\mathsf{a}_{0;\gamma}$ and $\widetilde{\mathsf{a}}_{0;\gamma}$ are d-*band subsymbols (polyphase components)* of a_0 and $\widetilde{\mathsf{a}}_0$ defined similar to (15) by

$$\begin{array}{l}\mathsf{a}_{0;\gamma}(z):=d_1\sum_{k\in\mathbb{Z}}a_0(k+\mathsf{d}k)z^k,\\ \widetilde{\mathsf{a}}_{0;\gamma}(z):=d_2\sum_{k\in\mathbb{Z}}\widetilde{a}_0(k+\mathsf{d}k)z^k,\end{array}\quad \gamma\in\mathbb{Z}. \tag{73}$$

Here, d_1,d_2 are two constants in $\mathbb{F}$ such that $\mathsf{d}=d_1d_2$.

To construct biorthogonal multiwavelets in $L_2(\mathbb{R})$, we need to design high-pass filters $a_1,\ldots,a_{\mathsf{d}-1}:\mathbb{Z}\to\mathbb{F}^{r\times r}$ and $\widetilde{a}_1,\ldots,\widetilde{a}_{\mathsf{d}-1}:\mathbb{Z}\to\mathbb{F}^{r\times r}$ such that the polyphase matrices with respect to the filter banks $\{\mathsf{a}_0,\mathsf{a}_1,\ldots,\mathsf{a}_{\mathsf{d}-1}\}$ and $\{\widetilde{\mathsf{a}}_0,\widetilde{\mathsf{a}}_1,\ldots,\widetilde{\mathsf{a}}_{\mathsf{d}-1}\}$

$$\mathbf{P}(z)=\begin{bmatrix}\mathsf{a}_{0;0}(z)&\cdots&\mathsf{a}_{0;\mathsf{d}-1}(z)\\ \mathsf{a}_{1;0}(z)&\cdots&\mathsf{a}_{1;\mathsf{d}-1}(z)\\ \vdots&\vdots&\vdots\\ \mathsf{a}_{\mathsf{d}-1;0}(z)&\cdots&\mathsf{a}_{\mathsf{d}-1;\mathsf{d}-1}(z)\end{bmatrix},\ \widetilde{\mathbf{P}}(z)=\begin{bmatrix}\widetilde{\mathsf{a}}_{0;0}(z)&\cdots&\widetilde{\mathsf{a}}_{0;\mathsf{d}-1}(z)\\ \widetilde{\mathsf{a}}_{1;0}(z)&\cdots&\widetilde{\mathsf{a}}_{1;\mathsf{d}-1}(z)\\ \vdots&\vdots&\vdots\\ \widetilde{\mathsf{a}}_{\mathsf{d}-1;0}(z)&\cdots&\widetilde{\mathsf{a}}_{\mathsf{d}-1;\mathsf{d}-1}(z)\end{bmatrix} \tag{74}$$

are biorthogonal, that is, $\mathbf{P}(z)\widetilde{\mathbf{P}}^*(z) = I_{\mathsf{d}r}$, where $\mathsf{a}_{m;\gamma}, \widetilde{\mathsf{a}}_{m;\gamma}$ are subsymbols of $\mathsf{a}_m, \widetilde{\mathsf{a}}_m$ defined similar to (73) for $m, \gamma = 0, \dots, \mathsf{d}-1$, respectively. The pair of filter banks $(\{\mathsf{a}_0, \dots, \mathsf{a}_{\mathsf{d}-1}\}, \{\widetilde{\mathsf{a}}_0, \dots, \widetilde{\mathsf{a}}_{\mathsf{d}-1}\})$ satisfying $\mathbf{P}\widetilde{\mathbf{P}}^* = I_{\mathsf{d}r}$ is called *a pair of biorthogonal filter banks with the perfect reconstruction property*.

Let $(a_0, \widetilde{a}_0)$ be a pair of biorthogonal low-pass filters such that a_0 and $\widetilde{a}_0$ have the same symmetry satisfying (36). By a slight modification of Lemma 1 (more precisely, by modifying (40)), one can easily show that there exists a suitable *invertible* matrix U, i.e., $\det(\mathsf{U})$ is a monomial, of Laurent polynomials in $\mathbb{F}$ acting on $\mathbb{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]$ so that $\mathbb{P}_{\mathsf{a}_0}\mathsf{U}$ and $\mathbb{P}_{\widetilde{\mathsf{a}}_0}\widetilde{\mathsf{U}}$ have compatible symmetry ($\widetilde{\mathsf{U}} = (\mathsf{U}^*)^{-1}$). Note that $\mathbb{P}_{\mathsf{a}_0}$ itself may not have compatible symmetry.

Now, for a pair of biorthogonal d-band low-pass filters $(\mathsf{a}_0, \widetilde{\mathsf{a}}_0)$ with multiplicity r satisfying (36), we have an algorithm (see Algorithm 5) to construct high-pass filters $\mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}$ and $\widetilde{\mathsf{a}}_1, \dots, \widetilde{\mathsf{a}}_{\mathsf{d}-1}$ such that the polyphase matrices $\mathbf{P}(z)$ and $\widetilde{\mathbf{P}}(z)$ defined as in (74) satisfy $\mathbf{P}(z)\widetilde{\mathbf{P}}^*(z) = I_{\mathsf{d}r}$. Here, $\mathbb{P}_{\mathsf{a}_0} := [\mathsf{a}_{0;0}, \dots, \mathsf{a}_{0;\mathsf{d}-1}]$ and $\widetilde{\mathbb{P}}_{\widetilde{\mathsf{a}}_0} := [\widetilde{\mathsf{a}}_{0;0}, \dots, \widetilde{\mathsf{a}}_{0;\mathsf{d}-1}]$ are the polyphase vectors of $\mathsf{a}_0, \widetilde{\mathsf{a}}_0$ obtained by (73), respectively.

Algorithm 5 Construction of biorthogonal multiwavelets with symmetry

(a) **Input**: A pair of biorthogonal d-band filters $(\mathsf{a}_0, \widetilde{\mathsf{a}}_0)$ with multiplicity r and with the same symmetry as in (36).

(b) **Initialization**: Construct a pair of biorthogonal matrices $(\mathsf{U}, \widetilde{\mathsf{U}})$ in $\mathbb{F}$ by Lemma 1 such that both $\mathbb{P} := \mathbb{P}_{\mathsf{a}_0}\mathsf{U}$ and $\widetilde{\mathbb{P}} = \widetilde{\mathbb{P}}_{\widetilde{\mathsf{a}}_0}\widetilde{\mathsf{U}}$ ($\widetilde{\mathsf{U}} = (\mathsf{U}^*)^{-1}$) are matrices of Laurent polynomials with coefficients in $\mathbb{F}$ having compatible symmetry: $\mathsf{S}\mathbb{P} = \mathsf{S}\widetilde{\mathbb{P}} = [\varepsilon_1 z^{k_1}, \dots, \varepsilon_r z^{k_r}]^{\mathrm{T}}\mathsf{S}\theta$ for some $k_1, \dots, k_r \in \mathbb{Z}$ and some $1 \times \mathsf{d}r$ row vector θ of Laurent polynomials with symmetry.

(c) **Extension**: Derive $(\mathbb{P}_e, \widetilde{\mathbb{P}}_e)$ with all the properties as in Theorem 4 from $(\mathbb{P}, \widetilde{\mathbb{P}})$ by Algorithm 4.

(d) **High-pass Filters**: Let $\mathbf{P} := \mathbb{P}_e\widetilde{\mathsf{U}}^* =: (\mathsf{a}_{m;\gamma})_{0 \le m,\gamma \le \mathsf{d}-1}$, $\widetilde{\mathbf{P}} := \widetilde{\mathbb{P}}_e\mathsf{U}^* =: (\widetilde{\mathsf{a}}_{m;\gamma})_{0 \le m,\gamma \le \mathsf{d}-1}$ as in (74). For $m = 1, \dots, \mathsf{d}-1$, define high-pass filters

$$\mathsf{a}_m(z) := \frac{1}{d_1}\sum_{\gamma=0}^{\mathsf{d}-1}\mathsf{a}_{m;\gamma}(z^{\mathsf{d}})z^{\gamma}, \quad \widetilde{\mathsf{a}}_m(z) := \frac{1}{d_2}\sum_{\gamma=0}^{\mathsf{d}-1}\widetilde{\mathsf{a}}_{m;\gamma}(z^{\mathsf{d}})z^{\gamma}. \tag{75}$$

(e) **Output**: A pair of biorthogonal filter banks $(\{\mathsf{a}_0, \mathsf{a}_1, \dots, \mathsf{a}_{\mathsf{d}-1}\}, \{\widetilde{\mathsf{a}}_0, \widetilde{\mathsf{a}}_1, \dots, \widetilde{\mathsf{a}}_{\mathsf{d}-1}\})$ with symmetry and with the perfect reconstruction property, i.e. $\mathbf{P}, \widetilde{\mathbf{P}}$ in (74) are biorthogonal and all filters $\mathsf{a}_m, \widetilde{\mathsf{a}}_m$, $m = 1, \dots, \mathsf{d}-1$, have symmetry:

$$\begin{aligned}\mathsf{a}_m(z) &= \operatorname{diag}(\varepsilon_1^m z^{\mathsf{d}c_1^m}, \dots, \varepsilon_r^m z^{\mathsf{d}c_r^m})\mathsf{a}_m(1/z)\operatorname{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}),\\ \widetilde{\mathsf{a}}_m(z) &= \operatorname{diag}(\varepsilon_1^m z^{\mathsf{d}c_1^m}, \dots, \varepsilon_r^m z^{\mathsf{d}c_r^m})\widetilde{\mathsf{a}}_m(1/z)\operatorname{diag}(\varepsilon_1 z^{-c_1}, \dots, \varepsilon_r z^{-c_r}),\end{aligned} \tag{76}$$

where $c_\ell^m := (k_\ell^m - k_\ell) + c_\ell \in \mathbb{R}$ and all $\varepsilon_\ell^m \in \{-1, 1\}$, $k_\ell^m \in \mathbb{Z}$, for $\ell = 1, \dots, r$ and $m = 1, \dots, \mathsf{d}-1$, are determined by the symmetry pattern of $\mathbb{P}_e$ as follows:

$$[\varepsilon_1 z^{k_1}, \dots, \varepsilon_r z^{k_r}, \varepsilon_1^1 z^{k_1^1}, \dots, \varepsilon_r^1 z^{k_r^1}, \dots, z^{k_1^{\mathsf{d}-1}}, \dots, \varepsilon_r^{\mathsf{d}-1} z^{k_r^{\mathsf{d}-1}}]^{\mathrm{T}}\mathsf{S}\theta := \mathsf{S}\mathbb{P}_e. \tag{77}$$

Let $(\phi, \widetilde{\phi})$ be a pair of biorthogonal d-refinable function vectors in $L_2(\mathbb{R})$ associated with a pair of biorthogonal d-band filters $(\mathsf{a}_0, \widetilde{\mathsf{a}}_0)$ and with $\phi = [\phi_1, \dots, \phi_r]^{\mathrm{T}}$,

$\widetilde{\phi} = [\widetilde{\phi}_1, \ldots, \widetilde{\phi}_r]^{\mathrm{T}}$. Define multiwavelet function vectors $\psi^m = [\psi_1^m, \ldots, \psi_r^m]^{\mathrm{T}}$, $\widetilde{\psi}^m = [\widetilde{\psi}_1^m, \ldots, \widetilde{\psi}_r^m]^{\mathrm{T}}$ associated with the high-pass filters $\mathsf{a}_m, \widetilde{\mathsf{a}}_m$, $m = 1, \ldots, \mathsf{d}-1$, by

$$\widehat{\psi^m}(\mathsf{d}\xi) := \mathsf{a}_m(\mathrm{e}^{-\mathrm{i}\xi})\widehat{\phi}(\xi),\ \widehat{\widetilde{\psi}^m}(\mathsf{d}\xi) := \widetilde{\mathsf{a}}_m(\mathrm{e}^{-\mathrm{i}\xi})\widehat{\widetilde{\phi}}(\xi),\ \xi \in \mathbb{R}. \tag{78}$$

It is well known that $\{\psi^1, \ldots, \psi^{\mathsf{d}-1}; \widetilde{\psi}^1, \ldots, \widetilde{\psi}^{\mathsf{d}-1}\}$ generates a biorthonormal multiwavelet basis in $L_2(\mathbb{R})$. Moreover, since the high-pass filters $\mathsf{a}_1, \ldots, \mathsf{a}_{\mathsf{d}-1}$, $\widetilde{\mathsf{a}}_1, \ldots, \widetilde{\mathsf{a}}_{\mathsf{d}-1}$ satisfy (76), it is easy to verify that each $\psi^m = [\psi_1^m, \ldots, \psi_r^m]^{\mathrm{T}}$, $\widetilde{\psi}^m = [\widetilde{\psi}_1^m, \ldots, \widetilde{\psi}_r^m]^{\mathrm{T}}$ defined in (78) has the following symmetry:

$$\begin{array}{llll} \psi_1^m(c_1^m - \cdot) = \varepsilon_1^m \psi_1^m, & \psi_2^m(c_2^m - \cdot) = \varepsilon_2^m \psi_2^m, & \ldots, & \psi_r^m(c_r^m - \cdot) = \varepsilon_r^m \psi_r^m, \\ \widetilde{\psi}_1^m(c_1^m - \cdot) = \varepsilon_1^m \widetilde{\psi}_1^m, & \widetilde{\psi}_2^m(c_2^m - \cdot) = \varepsilon_2^m \widetilde{\psi}_2^m, & \ldots, & \widetilde{\psi}_r^m(c_r^m - \cdot) = \varepsilon_r^m \widetilde{\psi}_r^m. \end{array} \tag{79}$$

In the following, let us present an example to demonstrate our results and illustrate our algorithms.

Example 3. Let $\mathsf{d} = 3, r = 2$, and $a_0, \widetilde{a}_0$ be a pair of dual d-filters with symbols $\mathsf{a}_0(z), \widetilde{\mathsf{a}}_0(z)$ (cf. [13]) given by

$$\mathsf{a}_0(z) = \frac{1}{243}\begin{bmatrix} a_{11}(z) & a_{12}(z) \\ a_{21}(z) & a_{22}(z) \end{bmatrix}, \quad \widetilde{\mathsf{a}}_0(z) = \frac{1}{34884}\begin{bmatrix} \widetilde{a}_{11}(z) & \widetilde{a}_{12}(z) \\ \widetilde{a}_{21}(z) & \widetilde{a}_{22}(z) \end{bmatrix}.$$

where

$$\begin{aligned} a_{11}(z) &= -21z^{-2} + 30z^{-1} + 81 + 14z - 5z^2, \\ a_{12}(z) &= 60z^{-1} + 84 - 4z^2 + 4z^3, \\ a_{21}(z) &= 4z^{-2} - 4z^{-1} + 84z + 60z^2, \\ a_{22}(z) &= -5z^{-1} + 14 + 81z + 30z^2 - 21z^3, \end{aligned}$$

and

$$\begin{aligned} \widetilde{a}_{11}(z) &= 1292z^{-2} + 2,844z^{-1} + 17,496 + 2,590z - 1,284z^2 + 1,866z^3, \\ \widetilde{a}_{12}(z) &= -4,773z^{-2} + 9,682z^{-1} + 8,715 - 2,961z + 386z^2 - 969z^3, \\ \widetilde{a}_{21}(z) &= -969z^{-2} + 386z^{-1} - 2,961 + 8,715z + 9,682z^2 - 4,773z^3, \\ \widetilde{a}_{22}(z) &= 1,866z^{-2} - 1,284z^{-1} + 2,590 + 17,496z + 2,844z^2 + 1,292z^3. \end{aligned}$$

The low-pass filters a_0 and $\widetilde{a}_0$ do not satisfy (36). However, we can employ a very simple orthogonal transform $E := \left[\begin{smallmatrix} 1 & 1 \\ 1 & -1 \end{smallmatrix}\right]$ to $\mathsf{a}_0, \widetilde{\mathsf{a}}_0$ so that the symmetry in (36) holds. That is, for $\mathsf{b}_0(z) := E\mathsf{a}_0(z)E^{-1}$ and $\widetilde{\mathsf{b}}_0(z) := E^{-1}\widetilde{\mathsf{a}}_0(z)E$, it is easy to verify that b_0 and $\widetilde{\mathsf{b}}_0$ satisfy (36) with $c_1 = c_2 = 1/2$ and $\varepsilon_1 = 1, \varepsilon_2 = -1$. Let $\mathsf{d} = d_1 d_2$ with $d_1 = 1$ and $d_2 = 3$. Construct $\mathbb{P}_{\mathsf{b}_0} := [\mathsf{b}_{0;0}, \mathsf{b}_{0;1}, \mathsf{b}_{0;2}]$ and $\widetilde{\mathbb{P}}_{\widetilde{\mathsf{b}}_0} := [\widetilde{\mathsf{b}}_{0;0}, \widetilde{\mathsf{b}}_{0;1}, \widetilde{\mathsf{b}}_{0;2}]$ from b_0 and $\widetilde{\mathsf{b}}_0$. Let U be given by

$$\mathsf{U} = \begin{bmatrix} z^{-1} & 0 & z^{-1} & 0 & 0 & 0 \\ 0 & z^{-1} & 0 & z^{-1} & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

and define $\widetilde{\mathsf{U}} := (\mathsf{U}^*)^{-1}$. Let $\mathbb{P} := \mathbb{P}_{\mathsf{b}_0}\mathsf{U}$ and $\widetilde{\mathbb{P}} := \widetilde{\mathbb{P}}_{\widetilde{\mathsf{b}}_0}\widetilde{\mathsf{U}}$. Then we have $\mathsf{S}\mathbb{P} = \mathsf{S}\widetilde{\mathbb{P}} = [z^{-1}, -z^{-1}]^{\mathrm{T}}[1,-1,-1,1,1,-1]$ and $\mathbb{P}, \widetilde{\mathbb{P}}$ are given by

$$\mathbb{P} = c\begin{bmatrix} t_{11}(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & t_{14} & t_{15}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\ t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \end{bmatrix},$$

$$\widetilde{\mathbb{P}} = \widetilde{c}\begin{bmatrix} \widetilde{t}_{11}(1+\frac{1}{z}) & \widetilde{t}_{12}(1-\frac{1}{z}) & \widetilde{t}_{13}(1-\frac{1}{z}) & \widetilde{t}_{14} & \widetilde{t}_{15}(1+\frac{1}{z}) & \widetilde{t}_{16}(1-\frac{1}{z}) \\ \widetilde{t}_{21}(1-\frac{1}{z}) & \widetilde{t}_{22}(1+\frac{1}{z}) & \widetilde{t}_{23}(1+\frac{1}{z}) & \widetilde{t}_{24}(1-\frac{1}{z}) & \widetilde{t}_{25}(1-\frac{1}{z}) & \widetilde{t}_{26}(1+\frac{1}{z}) \end{bmatrix},$$

where $c = \frac{1}{486}, \widetilde{c} = \frac{3}{34,884}$ and t_{jk}'s, $\widetilde{t}_{jk}$'s are constants defined as follows:

$$\begin{array}{llllll} t_{11} = 162, & t_{12} = 34, & t_{13} = -196, & t_{14} = 0, & t_{15} = 81, & t_{16} = 29, \\ t_{21} = -126, & t_{22} = -14, & t_{13} = 176, & t_{24} = -36, & t_{15} = -99, & t_{16} = -31, \\ \widetilde{t}_{11} = 5,814, & \widetilde{t}_{12} = -1,615, & \widetilde{t}_{13} = -7,160, & \widetilde{t}_{14} = 0, & \widetilde{t}_{15} = 5,814, & \widetilde{t}_{16} = 2,584, \\ \widetilde{t}_{21} = -5,551, & \widetilde{t}_{22} = 5,808, & \widetilde{t}_{13} = 7,740, & \widetilde{t}_{24} = -1,358, & \widetilde{t}_{15} = -6,712, & \widetilde{t}_{16} = -4,254. \end{array}$$

Applying Algorithm 2, we obtain $\mathbb{P}_e$ and $\widetilde{\mathbb{P}}_e$ as follows:

$$\mathbb{P}_e = c\left[\begin{array}{cccccc} t_{11}(1+\frac{1}{z}) & t_{12}(1-\frac{1}{z}) & t_{13}(1-\frac{1}{z}) & t_{14} & t_{15}(1+\frac{1}{z}) & t_{16}(1-\frac{1}{z}) \\ t_{21}(1-\frac{1}{z}) & t_{22}(1+\frac{1}{z}) & t_{23}(1+\frac{1}{z}) & t_{24}(1-\frac{1}{z}) & t_{25}(1-\frac{1}{z}) & t_{26}(1+\frac{1}{z}) \\ \hline t_{31}(1+\frac{1}{z}) & t_{32}(1-\frac{1}{z}) & t_{33}(1-\frac{1}{z}) & t_{34}(1+\frac{1}{z}) & t_{35}(1+\frac{1}{z}) & t_{36}(1-\frac{1}{z}) \\ t_{41} & 0 & 0 & t_{44} & t_{45} & 0 \\ \hline 0 & t_{52} & t_{53} & 0 & 0 & t_{56} \\ t_{61}(1-\frac{1}{z}) & t_{62}(1+\frac{1}{z}) & t_{63}(1+\frac{1}{z}) & t_{64}(1-\frac{1}{z}) & t_{65}(1-\frac{1}{z}) & t_{66}(1+\frac{1}{z}) \end{array}\right],$$

where all t_{jk}'s are constants given by

$$\begin{array}{llll} t_{31} = 24, & t_{32} = \dfrac{472}{27}, & t_{33} = -\dfrac{148}{27}, & \\ t_{34} = -36, & t_{35} = -24, & t_{36} = -\dfrac{112}{27}, & \\ t_{41} = \dfrac{1,09,998}{533}, & t_{44} = \dfrac{94,041}{533}, & t_{45} = -\dfrac{1,09,989}{533}, & \\ t_{52} = 406c_0, & t_{53} = 323c_0, & t_{56} = 1,142c_0, & c_0 = \dfrac{16,09,537}{13,122}, \\ t_{61} = 24,210c_1, & t_{62} = 14,318c_1, & t_{63} = -11,807c_1, & t_{64} = -26,721c_1, \\ t_{65} = -14,616c_1, & t_{66} = -1,934c_1, & c_1 = 200/26,163. & \end{array}$$

And

$$\widetilde{\mathbb{P}}_e = \widetilde{c}\left[\begin{array}{cccccc} \widetilde{t}_{11}(1+\frac{1}{z}) & \widetilde{t}_{12}(1-\frac{1}{z}) & \widetilde{t}_{13}(1-\frac{1}{z}) & \widetilde{t}_{14} & \widetilde{t}_{15}(1+\frac{1}{z}) & \widetilde{t}_{16}(1-\frac{1}{z}) \\ \widetilde{t}_{21}(1-\frac{1}{z}) & \widetilde{t}_{22}(1+\frac{1}{z}) & \widetilde{t}_{23}(1+\frac{1}{z}) & \widetilde{t}_{24}(1-\frac{1}{z}) & \widetilde{t}_{25}(1-\frac{1}{z}) & \widetilde{t}_{26}(1+\frac{1}{z}) \\ \hline \widetilde{t}_{31}(1+\frac{1}{z}) & \widetilde{t}_{32}(1-\frac{1}{z}) & \widetilde{t}_{33}(1-\frac{1}{z}) & \widetilde{t}_{34}(1+\frac{1}{z}) & \widetilde{t}_{35}(1+\frac{1}{z}) & \widetilde{t}_{36}(1-\frac{1}{z}) \\ \widetilde{t}_{41} & 0 & 0 & \widetilde{t}_{44} & \widetilde{t}_{45} & 0 \\ \hline 0 & \widetilde{t}_{52} & \widetilde{t}_{53} & 0 & 0 & \widetilde{t}_{56} \\ \widetilde{t}_{61}(1-\frac{1}{z}) & \widetilde{t}_{62}(1+\frac{1}{z}) & \widetilde{t}_{63}(1+\frac{1}{z}) & \widetilde{t}_{64}(1-\frac{1}{z}) & \widetilde{t}_{65}(1-\frac{1}{z}) & \widetilde{t}_{66}(1+\frac{1}{z}) \end{array}\right],$$

where all $\widetilde{t}_{jk}$'s are constants given by

$$
\begin{array}{llll}
\widetilde{t}_{31} = 3,483\widetilde{c}_0, & \widetilde{t}_{32} = 37,427\widetilde{c}_0, & \widetilde{t}_{33} = 4,342\widetilde{c}_0, & \widetilde{t}_{34} = -12,222\widetilde{c}_0, \\
\widetilde{t}_{35} = -3,483\widetilde{c}_0, & \widetilde{t}_{36} = -7,267, & \widetilde{c}_0 = \dfrac{8,721}{4,264}, & \\
\widetilde{t}_{41} = 5,814, & \widetilde{t}_{44} = 1,1628, & \widetilde{t}_{45} = -1,1628, & \\
\widetilde{t}_{52} = 3\widetilde{c}_1, & \widetilde{t}_{53} = 2\widetilde{c}_1, & \widetilde{t}_{56} = 10\widetilde{c}_1, & \widetilde{c}_1 = \dfrac{12,680,011}{243}; \\
\widetilde{t}_{61} = 18,203\widetilde{c}_2, & \widetilde{t}_{62} = 1,01,595\widetilde{c}_2, & \widetilde{t}_{63} = 1,638\widetilde{c}_2, & \widetilde{t}_{64} = -33,950\widetilde{c}_2, \\
\widetilde{t}_{65} = -10,822\widetilde{c}_2, & \widetilde{t}_{66} = -36,582\widetilde{c}_2, & \widetilde{c}_2 = \dfrac{26,163}{2,13,200}. &
\end{array}
$$

Note that $\mathbb{P}_e$ and $\widetilde{\mathbb{P}}_e$ satisfy

$$\mathsf{S}\mathbb{P}_e = \mathsf{S}\mathbb{P}_e = [z^{-1}, -z^{-1}, z^{-1}, 1, -1, -z^{-1}]^{\mathrm{T}}[1, -1, -1, 1, 1, -1].$$

From the polyphase matrices $\mathbf{P} := \mathbb{P}_e\widetilde{\mathsf{U}}^*$ and $\widetilde{\mathbf{P}} := \widetilde{\mathbb{P}}_e\mathsf{U}^*$, we derive high-pass filters $\mathsf{b}_1, \mathsf{b}_2$ and $\widetilde{\mathsf{b}}_1, \widetilde{\mathsf{b}}_2$ as follows:

$$\mathsf{b}_1(z) = \begin{bmatrix} b^1_{11}(z) & b^1_{12}(z) \\ b^1_{21}(z) & b^1_{22}(z) \end{bmatrix}, \mathsf{b}_2(z) = \begin{bmatrix} b^2_{11}(z) & b^2_{12}(z) \\ b^2_{21}(z) & b^2_{22}(z) \end{bmatrix},$$

where

$$
\begin{aligned}
b^1_{11}(z) &= \frac{199}{6,561} + \frac{125}{6,561}z^3 - \frac{4}{81}z^2 + \frac{199}{6,561}z - \frac{4}{81}z^{-1} + \frac{125}{6,561}z^{-2}, \\
b^1_{12}(z) &= -\frac{361}{6,561} - \frac{125}{6,561}z^3 - \frac{56}{6,561}z^2 + \frac{361}{6,561}z + \frac{56}{6,561}z^{-1} + \frac{125}{6,561}z^{-2}, \\
b^1_{21}(z) &= \frac{679}{3,198}z^3 + \frac{679}{3,198}z - \frac{679}{1,599}z^2, \quad b^1_{22}(z) = \frac{387}{2,132}z^3 - \frac{387}{2,132}z, \\
b^2_{11}(z) &= c_3(323z^3 - 323z), \\
b^2_{12}(z) &= c_3(406z^3 + 2,284z^2 + 406z), \\
b^2_{21}(z) &= c_4(-36,017 + 12,403z^3 - 29,232z^2 + 36,017z + 29,232z^{-1} - 12,403z^{-2}), \\
b^2_{22}(z) &= c_4(41,039 - 12,403z^3 - 3,868z^2 + 41,039z - 3,868z^{-1} - 12,403z^{-2}), \\
c_3 &= \frac{27}{32,19,074}, \quad c_4 = \frac{50}{63,57,609}.
\end{aligned}
$$

And

$$\widetilde{\mathsf{b}}_1(z) = \begin{bmatrix} \widetilde{b}^1_{11}(z) & \widetilde{b}^1_{12}(z) \\ \widetilde{b}^1_{21}(z) & \widetilde{b}^1_{22}(z) \end{bmatrix}, \widetilde{\mathsf{b}}_2(z) = \begin{bmatrix} \widetilde{b}^2_{11}(z) & \widetilde{b}^2_{12}(z) \\ \widetilde{b}^2_{21}(z) & \widetilde{b}^2_{22}(z) \end{bmatrix},$$

where

$$\widetilde{b}^1_{11}(z) = -\frac{859}{17,056} + \frac{7,825}{17,056}z^3 - \frac{3,483}{8,528}z^2 - \frac{859}{17,056}z - \frac{3,483}{8,528}z^{-1} + \frac{7,825}{17,056}z^{-2},$$

$$\widetilde{b}^1_{12}(z) = -\frac{49,649}{17,056} + \frac{25,205}{17,056}z^3 - \frac{559}{656}z^2 + \frac{49,649}{17,056}z + \frac{559}{656}z^{-1} - \frac{25,205}{17,056}z^{-2},$$
$$\widetilde{b}^1_{21}(z) = \frac{1}{6}(z^3 + z - 2z^2), \quad \widetilde{b}^1_{22}(z) = \frac{1}{3}(z^3 - z),$$
$$\widetilde{b}^2_{11}(z) = 2\widetilde{c}_3(z^3 - z),$$
$$\widetilde{b}^2_{12}(z) = \widetilde{c}_3(3z^3 + 10z^2 + 3z),\ \widetilde{c}_3 = \frac{39,257}{26,244};$$
$$\widetilde{b}^2_{21}(z) = -\frac{9,939}{1,70,560} + \frac{59,523}{8,52,800}z^3 - \frac{16,233}{4,26,400}z^2 + \frac{9,939}{1,70,560}z + \frac{16,233}{4,26,400}z^{-1} - \frac{59,523}{8,52,800}z^{-2},$$
$$\widetilde{b}^2_{22}(z) = \frac{81,327}{1,70,560} + \frac{40,587}{1,70,560}z^3 - \frac{4,221}{32,800}z^2 + \frac{81,327}{1,70,560}z - \frac{4,221}{32,800}z^{-1} + \frac{40,587}{1,70,560}z^{-2}.$$

Then the high-pass filters $\mathsf{b}_1, \mathsf{b}_2$ and $\widetilde{\mathsf{b}}_1, \widetilde{\mathsf{b}}_2$ satisfy (76) with $c^1_1 = c^1_2 = 1/2$, $\varepsilon^1_1 = 1, \varepsilon^1_2 = 1$ and $c^2_1 = c^2_2 = 3/2$, $\varepsilon^1_1 = -1, \varepsilon^1_2 = -1$, respectively. Using E, we can define $\mathsf{a}_1, \mathsf{a}_2$ and $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$ to be the high-pass filters constructed from $\mathsf{b}_1, \mathsf{b}_2$ and $\widetilde{\mathsf{b}}_1, \widetilde{\mathsf{b}}_2$ by $\mathsf{a}_1(z) := E^{-1}\mathsf{b}_1(z)E, \mathsf{a}_2 := E^{-1}\mathsf{b}_2E$ and $\widetilde{\mathsf{a}}_1(z) := E\widetilde{\mathsf{b}}_1(z)E^{-1}, \widetilde{\mathsf{a}}_2 := E\widetilde{\mathsf{b}}_2E^{-1}$.

See Fig. 4 for graphs of the 3-refinable function vectors $\phi, \widetilde{\phi}$ associated with the low-pass filters $\mathsf{a}_0, \widetilde{\mathsf{a}}_0$, respectively, and the biorthogonal multiwavelet function vectors ψ^1, ψ^2 and $\widetilde{\psi}^1, \widetilde{\psi}^2$ associated with the high-pass filters $\mathsf{a}_1, \mathsf{a}_2$ and $\widetilde{\mathsf{a}}_1, \widetilde{\mathsf{a}}_2$, respectively. Also, see Fig. 3 for graphs of the 3-refinable function vectors $\eta, \widetilde{\eta}$ associated with the low-pass filters $\mathsf{b}_0, \widetilde{\mathsf{b}}_0$, respectively, and the biorthogonal multiwavelet function vectors ζ^1, ζ^2 and $\widetilde{\zeta}^1, \widetilde{\zeta}^2$ associated with the high-pass filters $\mathsf{b}_1, \mathsf{b}_2$ and $\widetilde{\mathsf{b}}_1, \widetilde{\mathsf{b}}_2$, respectively.

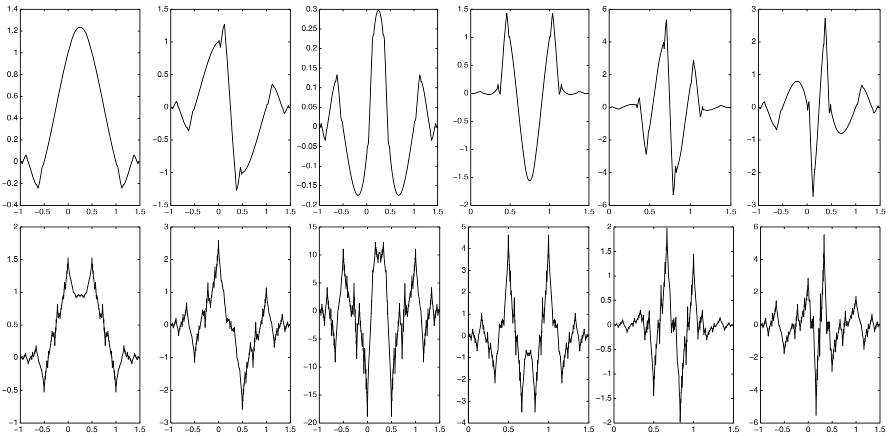

Fig. 3: Graphs of $\phi = [\phi_1, \phi_2]^{\mathrm{T}}$, $\psi^1 = [\psi^1_1, \psi^1_2]^{\mathrm{T}}$, and $\psi^2 = [\psi^2_1, \psi^2_2]^{\mathrm{T}}$ (*top, left to right*), and $\widetilde{\phi} = [\widetilde{\phi}_1, \widetilde{\phi}_2]^{\mathrm{T}}$, $\widetilde{\psi}^1 = [\widetilde{\psi}^1_1, \widetilde{\psi}^1_2]^{\mathrm{T}}$, and $\widetilde{\psi}^2 = [\widetilde{\psi}^2_1, \widetilde{\psi}^2_2]^{\mathrm{T}}$ (*bottom, left to right*)

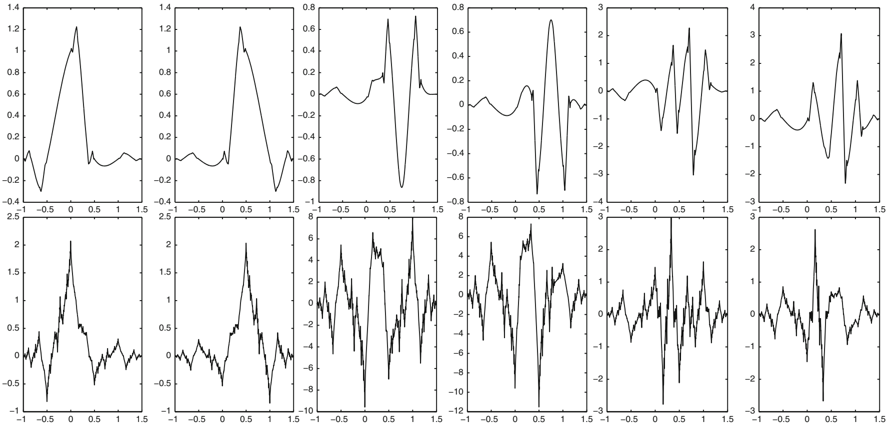

Fig. 4: Graphs of $\eta = [\eta_1, \eta_2]^{\mathrm{T}}$, $\zeta^1 = [\zeta_1^1, \zeta_2^1]^{\mathrm{T}}$, and $\zeta^2 = [\zeta_1^2, \zeta_2^2]^{\mathrm{T}}$ (*top, left to right*), and $\widetilde{\eta} = [\widetilde{\eta}_1, \widetilde{\eta}_2]^{\mathrm{T}}$, $\widetilde{\zeta}^1 = [\widetilde{\zeta}_1^1, \widetilde{\zeta}_2^1]^{\mathrm{T}}$, and $\widetilde{\zeta}^2 = [\widetilde{\zeta}_1^2, \widetilde{\zeta}_2^2]^{\mathrm{T}}$ (*bottom, left to right*)

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