Linear Multiscale Transforms Based on Even-Reversible Subdivision Operators

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Abstract Multiscale transforms for real-valued data, based on interpolatory subdivision operators, have been studied in recent years. They are easy to define and can be extended to other types of data, for example, to manifold-valued data. In this chapter, we define linear multiscale transforms, based on certain linear, non-interpolatory subdivision operators, termed "even-reversible." For such operators, we prove, using Wiener's lemma, the existence of an inverse to the linear operator defined by the even part of the subdivision mask and term it "even-inverse." We show that the non-interpolatory subdivision operators, with spline or pseudo-spline masks, are even-reversible and derive explicitly, for the quadratic and cubic spline subdivision operators, the symbols of the corresponding even-inverse operators. We also analyze properties of the multiscale transforms based on even-reversible subdivision operators, in particular, their stability and the rate of decay of the details.

1 Introduction

A multiscale transform is a ubiquitous way for representing data of the form $\{(t_k, f_k),\ k \in \mathbb{Z}\}$, with $t_k = kh$ for some positive h, and f_k a real value associated with the point t_k. A general approach to multiscale transforms was introduced in [21]. Such a linear representation can be obtained by discrete wavelet transforms (see e.g. [3]). Another common way for generating a multiscale transform is based on linear or non-linear interpolatory subdivision schemes (see e.g. [1, 6, 10, 19]). For a detailed analysis of several non-linear multiscale transforms, see [12] and references therein. The main applications of multiscale transforms are in data compression and denoising (see e.g. [1, 2, 6, 18–20, 28]). Multiscale transforms

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based on interpolatory subdivision operators are simple and rather intuitive. The linear ones were extended to manifold-valued data in [14, 15] and [24].

In this chapter, we extend the construction of multiscale transforms from interpolatory subdivision operators to a wider class of subdivision operators termed *even-reversible* operators and show that a large class of the subdivision operators studied in the literature are even-reversible. We derive properties of the multiscale transforms based on even-reversible subdivision operators, such as decay rates of the data generated by the transforms, and the stability of the transforms.

1.1 Multiscale Transform: From Interpolatory to Even-Reversible Subdivision

First, we present the multiscale transform based on an interpolatory subdivision operator $\mathcal{S}$. Let $\mathbf{f}$ denote a bi-infinite sequence with elements $\{f_k \in \mathbb{R},\ k \in \mathbb{Z}\}$. Since $\mathcal{S}$ is interpolatory,

$$(\mathcal{S}\mathbf{f})_{2k} = f_k, \quad k \in \mathbb{Z},$$

and it is straightforward to decimate data at level j, $\mathbf{f}^j$, to that at the coarser level $j-1$, by taking every second element. Then, the refinement of $\mathbf{f}^{j-1}$ by $\mathcal{S}$ is exact for the even elements of $\mathbf{f}^j$. Here is the pyramid of data generated by j *decomposition* steps of the multiscale transform based on $\mathcal{S}$,

$$\mathbf{f}^{\ell-1} = \{f_k^{(\ell-1)} = f_{2k}^{(\ell)},\ k \in \mathbb{Z}\},\ \ \mathbf{d}^\ell = \mathbf{f}^\ell - \mathcal{S}\mathbf{f}^{\ell-1},\ \ \ell = j, j-1, \ldots, 1. \tag{1.1}$$

The elements of $\mathbf{d}^\ell$ are termed *details* at level ℓ. The data $\mathbf{f}^j$ can be obtained exactly from the data of the pyramid by the *reconstruction* steps

$$\mathbf{f}^\ell = \mathcal{S}\mathbf{f}^{\ell-1} + \mathbf{d}^\ell, \quad \ell = 1, 2, \ldots, j. \tag{1.2}$$

Note that every second element of $\mathbf{d}^\ell$ vanishes, since $\mathcal{S}$ is interpolatory. Thus, this method yields details satisfying

$$\mathbf{d}_{2k}^\ell = 0, \quad k \in \mathbb{Z}. \tag{1.3}$$

Multiscale transforms based on non-interpolatory subdivision operators were studied before, with the decimation from level j to level $j-1$, defined by "reverse subdivision." In [26], $\mathbf{f}^{j-1} = \mathcal{D}\mathbf{f}^j$, where the decimation operator $\mathcal{D}$ is related to the subdivision operator $\mathcal{S}$ by a least squares fit of $\mathcal{S}\mathbf{f}^{j-1}$ to $\mathbf{f}^j$. This method is improved in [25], by minimizing a functional consisting of two terms; one is $\|\mathcal{S}(\mathcal{D}\mathbf{f}^j) - \mathbf{f}^j\|_2$, and the other is a roughness measure of $\mathbf{f}^{j-1} = (\mathcal{D}\mathbf{f}^j)$. It is clear that in this approach, the details do not satisfy (1.3). Specific ways to reverse Chaikin scheme and Catmull–Clark scheme are presented in [22] and [23], respectively.

Here, we propose to reverse only the even elements of $\mathbf{f}^j$, as in the case of interpolatory subdivision operators, namely for a given subdivision operator $\mathcal{S}$ to use a decimation operator $\mathcal{D}$ such that

$$(\mathcal{S}(\mathcal{D}\mathbf{f}^j))_{2k} = f_{2k}^{(j)}, \quad k \in \mathbb{Z}. \tag{1.4}$$

This is achieved by using Wiener's lemma [16], which guarantees the existence of such a decimation operator, under mild conditions on the mask of $\mathcal{S}$. We term such subdivision operators, for which $\mathcal{D}$ satisfying (1.4) exists, even-reversible and refer to $\mathcal{D}$ as the even-inverse of $\mathcal{S}$.

1.2 Outline

Section 2 consists of mathematical tools and results used in the chapter, such as Wiener's lemma. In Sect. 3, we first give a sufficient condition on the mask of a subdivision operator guaranteeing that the operator is even-reversible, then show that pseudo-spline subdivision operators are even-reversible, and derive explicit expression of the symbols of the even-inverse operators corresponding to the quadratic and cubic spline subdivision operators. In Sect. 4, we derive the decay rate of the details in the pyramid generated by the multiscale transform and analyze the stability of the transform. Some final remarks are given in Sect. 5, and some technical proofs are postponed to the Appendix.

2 Preliminaries

In this section, we introduce some necessary notation and known results related to subdivision operators and Wiener algebra, which are used in this chapter.

We denote by $l(\mathbb{Z})$ the space of real-valued sequences $\alpha : \mathbb{Z} \to \mathbb{R}$ and by $l_0(\mathbb{Z}) \subseteq l(\mathbb{Z})$ the space of sequences of finite support. For $p \in [1, \infty]$, $l_p(\mathbb{Z})$ denotes the usual space of l_p sequences. That is, $l_p(\mathbb{Z}) := \left\{\alpha \in l(\mathbb{Z}) : \|\alpha\|_p := \left(\sum_{k\in\mathbb{Z}} |\alpha_k|^p\right)^{1/p} < \infty\right\}$, $1 \le p < \infty$, and $l_\infty(\mathbb{Z}) := \{\alpha \in l(\mathbb{Z}) : \|\alpha\|_\infty := \sup_{k\in\mathbb{Z}} |\alpha_k| < \infty\}$. Note that the inclusion $l_0(\mathbb{Z}) \subseteq l_1(\mathbb{Z}) \subseteq l_p(\mathbb{Z}) \subseteq l_q(\mathbb{Z}) \subseteq l_\infty(\mathbb{Z})$ holds for any $1 < p < q < \infty$.

We say that $\alpha \in l(\mathbb{Z})$ is a *mask* if $\alpha \in l_1(\mathbb{Z})$. A mask α is of finite support ($\alpha \in l_0(\mathbb{Z})$) if $\alpha_k = 0$ for all $|k| \ge N$ for some integer N. Given a mask α, we can define the (dyadic) *upscaling rule* or *subdivision operator* $\mathcal{S}_\alpha : l_\infty(\mathbb{Z}) \to l_\infty(\mathbb{Z})$ associated with α by

$$(\mathcal{S}_\alpha c)_k := \sum_{\ell\in\mathbb{Z}} \alpha_{k-2\ell} c_\ell, \quad k \in \mathbb{Z},\ c \in l_\infty(\mathbb{Z}). \tag{2.1}$$

It is easily seen that $\mathcal{S}_\alpha$ is a bounded linear operators on $l_\infty(\mathbb{Z})$ provided that $\alpha \in l_1(\mathbb{Z})$. The *subdivision scheme* based on $\mathcal{S}_\alpha$ is the repeated application of (2.1), generating a sequence of sequences $\mathcal{S}_\alpha^j c$, $j \in \mathbb{N}$, for $c \in l_\infty(\mathbb{Z})$. The subdivision scheme is said to be *convergent* if the sequence of piecewise linear interpolants to the data $\{(k2^{-j}, (\mathcal{S}_\alpha^j c)_k), k \in \mathbb{Z}\}$, $j \in \mathbb{N}$, is uniformly convergent for any $c \in l_\infty(\mathbb{Z})$. For more about subdivision schemes, see e.g. [9].

We next introduce convolution, downsampling, and upsampling operations. For two sequences $\alpha \in l_1(\mathbb{Z})$ and $c \in l_\infty(\mathbb{Z})$, we defined the *convolution* $\alpha * c \in l_\infty(\mathbb{Z})$ to be

$$(\alpha * c)_k := \sum_{\ell\in\mathbb{Z}} \alpha_\ell c_{k-\ell}, \quad k \in \mathbb{Z},$$

and the *downsampling operator* $\downarrow 2$ applied to a sequence c to be $(c \downarrow 2)_k := c_{2k}$, $k \in \mathbb{Z}$, as well as the *upsampling operator* $\uparrow 2$:

$$(c \uparrow 2)_k := \begin{cases} c_{k/2} & k \text{ even} \\ 0 & \text{otherwise} \end{cases}, \qquad k \in \mathbb{Z}.$$

The subdivision in (2.1) can be restated as $\mathcal{S}_\alpha c = \alpha * (c \uparrow 2)$.

For a sequence $c \in l(\mathbb{Z})$, we define its *symbol* to be

$$c(z) := \sum_{k\in\mathbb{Z}} c_k z^k, \quad z \in \mathbb{C}.$$

Its even part c_{ev} to be $(c_{ev})_k = c_{2k} = (c \downarrow 2)_k$, $k \in \mathbb{Z}$, and its odd part c_{od} to be $(c_{od})_k = c_{2k+1}$, $k \in \mathbb{Z}$. The symbols of the even and odd parts can be determined by

$$c_{ev}(z^2) = \frac{c(z) + c(-z)}{2} \quad \text{and} \quad c_{od}(z^2) = \frac{c(z) - c(-z)}{2z}.$$

In terms of symbolic computation, it is easily shown that

$$\begin{aligned} &[c \downarrow 2](z) = c_{ev}(z), && [c \uparrow 2](z) = c(z^2), \\ &c(z) = c_{ev}(z^2) + z\, c_{od}(z^2), && [\alpha * c](z) = \alpha(z)c(z). \end{aligned}$$

Moreover, we have $[\mathcal{S}_\alpha c](z) = \alpha(z)c(z^2)$.

In this chapter, we investigate the following multiscale transform based on a subdivision operator $\mathcal{S}_\alpha$:

$$c^{(j-1)} = \mathcal{D}_\gamma c^{(j)} := \gamma * (c^{(j)} \downarrow 2), \quad d^{(j)} := c^{(j)} - \mathcal{S}_\alpha c^{(j-1)}, \tag{2.2}$$

where $\mathcal{D}_\gamma$ is a *decimation operator* associated with a mask γ to be determined. Iterating (2.2) yields a pyramid consisting of the data $\{c^{(0)}; d^{(1)}, \dots, d^{(j)}\}$, where $c^{(0)}$ is the coarse approximation coefficients and $d^{(\ell)}$ are the detail coefficients at level $\ell = 1, \dots, j$. The reconstruction (backward transform) from $\{c^{(0)}; d^{(1)}, \dots, d^{(j)}\}$ is straightforward:

$$c^{(\ell)} = \mathcal{S}_\alpha c^{(\ell-1)} + d^{(\ell)}, \quad \ell = 1, \dots, j, \tag{2.3}$$

and it has the perfect reconstruction property for any pair (γ, α) of masks. However, the detail coefficients $d^{(\ell)}$, $\ell = 1, \dots, j$ do not necessarily satisfy (1.3).

Given a subdivision operator $\mathcal{S}_\alpha$ associated with a finitely supported mask $\alpha \in l_0(\mathbb{Z})$, we investigate the decimation operator $\mathcal{D}_\gamma$ associated with a mask γ so that $d^{(\ell)}$ satisfies $d^{(\ell)}_{2k} = 0$, $k \in \mathbb{Z}$, for $\ell = 1, \dots, j$, as in the case when $\mathcal{S}_\alpha$ is an interpolatory subdivision operator and $\mathcal{D}_\gamma c = c \downarrow 2$. Thus, we are seeking a mask γ such that

$$[(\mathcal{I} - \mathcal{S}_\alpha \mathcal{D}_\gamma)c] \downarrow 2 = 0 \tag{2.4}$$

for any sequence $c \in l_\infty(\mathbb{Z})$, where $\mathcal{I}$ is the identity operator. In such a case, the detail coefficients can be downsampled by a factor of 2 without loss of information, which is a desirable property in data compression.

To solve γ from (2.4), we use Wiener's lemma [16]. First, we introduce Wiener algebra.

Let $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$. The *Wiener algebra* $W(\mathbb{T})$ consists of all complex-valued functions f on $[-\pi, \pi]$ such that f has absolutely convergent Fourier series. That is, $W(\mathbb{T}) := \{f \in C([-\pi, \pi]) : \|f\| := \sum_{n\in\mathbb{Z}} |\widehat{f}(n)| < \infty\}$, where $\widehat{f}(n) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx$ is the nth Fourier coefficient of f. The Wiener algebra $W(\mathbb{T})$ is closed under pointwise multiplication of functions. It is easy to show that

$$\|fg\| \le \|f\| \cdot \|g\| \quad \forall f, g \in W(\mathbb{T}).$$

Thus, the Wiener algebra is a commutative unitary Banach algebra. The Wiener algebra $W(\mathbb{T})$ is isomorphic to the Banach algebra $l_1(\mathbb{Z})$ with the isomorphism given by the Fourier transform: $f \mapsto \{\widehat{f}(n)\}_{n\in\mathbb{Z}}$.

Theorem 2.1 (Wiener's Lemma) *If $f \in W(\mathbb{T})$ and $f(x) \neq 0$ for all $x \in [-\pi, \pi]$, then $1/f \in W(\mathbb{T})$.*

Consequently, for $\alpha \in l_0(\mathbb{Z})$, if $\alpha(z) \neq 0$ for all $z \in \mathbb{T}$, then α has an inverse $\alpha^{-1} =: \gamma \in l_1(\mathbb{Z})$ determined by $\gamma(z) = 1/\alpha(z)$, $z \in \mathbb{T}$.

Any mask α in the Wiener algebra defines a *bi-infinite Toeplitz matrix* of the form $A_\alpha = (\alpha_{j-k})_{j,k\in\mathbb{Z}}$. Then, A_α has an inverse if and only if α has an inverse. In this case, we have that the inverse $(A_\alpha)^{-1}$ of A_α satisfies $(A_\alpha)^{-1} = A_{\alpha^{-1}}$. A Toeplitz matrix A_α is also a linear operator on $l_2(\mathbb{Z})$. One can show that the l_2-operator norm of A_α is given by [13, Section 2.2]

$$\|A_\alpha\|_2 := \sup_{\|c\|_2=1} \|A_\alpha c\|_2 = \sup_{z\in\mathbb{T}} |\alpha(z)|. \tag{2.5}$$

We say that a finitely supported mask $\alpha \in l_0(\mathbb{Z})$ is *s-banded* if $\alpha_k = 0$ for all $|k| > s$. The following theorem gives the decay of the inverse of a banded Hermitian Toeplitz matrix.

Theorem 2.2 (Theorem 2.1 in [27]) *Let A be a bi-infinite Toeplitz matrix acting on $l_2(\mathbb{Z})$ and assume A to be Hermitian, positive definite, and s-banded (i.e. $A_{k,\ell} = 0$ if $|k-\ell| > s$). Set $\kappa = \|A\|_2\, \|A^{-1}\|_2$, $q = (\sqrt{\kappa}-1)/(\sqrt{\kappa}+1)$, and $\lambda = q^{1/s}$. Then,*

$$(A^{-1})_{k,\ell} \le K\lambda^{|k-\ell|}, \quad k, \ell \in \mathbb{Z},$$

where $K = \|A^{-1}\|_2 \max\left\{1, \frac{(1+\sqrt{\kappa})^2}{2\kappa}\right\}$.

A direct consequence of the above results is the decay property of the inverse γ of an invertible mask $\alpha \in l_0(\mathbb{Z})$. For more general results on the decay property of the inverse of a mask, we refer to [16].

Corollary 2.1 *Let $\alpha \in l_0(\mathbb{Z})$ be s-banded (i.e. $\alpha_k = 0$ if $|k| > s$) with $\alpha(z) > 0$ for all $z \in \mathbb{T}$. Set $\kappa = \frac{\sup_{z\in\mathbb{T}} |\alpha(z)|}{\inf_{z\in\mathbb{T}} |\alpha(z)|}$, $q = (\sqrt{\kappa}-1)/(\sqrt{\kappa}+1)$, and $\lambda = q^{1/s}$. Then, $\gamma = \alpha^{-1}$ exists and*

$$|\gamma_\ell| \le K\lambda^{|\ell|}, \quad \ell \in \mathbb{Z},$$

where $K = \frac{1}{\inf_{z\in\mathbb{T}} |\alpha(z)|} \max\left\{1, \frac{(1+\sqrt{\kappa})^2}{2\kappa}\right\}$.

Proof Since $\alpha(z) > 0$ for all $z \in \mathbb{T}$, by Wiener's lemma, $\gamma = \alpha^{-1}$ exists. Now, by Fejér–Riesz lemma [3], there exists a real-valued mask $\beta \in l_0(\mathbb{Z})$ such that $\alpha(z) = \beta(z)\beta(1/z)$ for all $z \in \mathbb{T}$. Then, one can easily show that $A_\alpha = A_\beta(A_\beta)^\top$. Hence, from $\alpha(z) > 0$ for all $z \in \mathbb{T}$, we conclude that A_α is symmetric positive definite. Now, by Theorem 2.2 and the fact that $\|A_\alpha\|_2 = \sup_{z\in\mathbb{T}} |\alpha(z)|$ and $\|A_\alpha^{-1}\|_2 = \|A_{\alpha^{-1}}\|_2 = \inf_{z\in\mathbb{T}} |\alpha(z)|$, we conclude the result. □

3 Even-Reversible Subdivision

In this section, we give a sufficient condition on the mask of a subdivision operator for the existence of the linear multiscale transform defined in (2.2) that satisfies property (2.4). We call such a subdivision operator *even-reversible*. We show that for a large class of even-reversible subdivision operators, the elements of the mask of the decimation operator in (2.4) decay exponentially, and we prove that most of the subdivision operators in the literature are in that class.

3.1 Existence of the Multiscale Transform

We first deduce a sufficient condition on the mask of a subdivision operator for the existence of a linear multiscale transform satisfying (2.4).

Theorem 3.1 *Let $\mathcal{S}_\alpha$ be a subdivision operator with a mask α defined as in* (2.1)*, and let $\mathcal{D}_\gamma$ be a decimation operator with a mask γ defined as in* (2.2)*. Then,* (2.4) *holds, i.e., $d := (\mathcal{I} - \mathcal{S}_\alpha \mathcal{D}_\gamma)c$ satisfies $(d \downarrow 2) = 0$ for any $c \in l_\infty(\mathbb{Z})$, if and only if γ is the inverse of α_{ev}.*

Proof Note that (2.4) is equivalent to $\frac{c(z)+c(-z)}{2} - \frac{[\mathcal{S}_\alpha \mathcal{D}_\gamma c](z)+[\mathcal{S}_\alpha \mathcal{D}_\gamma c](-z)}{2} = 0$; that is,

$$c_{ev}(z^2) = [\mathcal{S}_\alpha \mathcal{D}_\gamma c]_{ev}(z^2) = \alpha_{ev}(z^2)[\mathcal{D}_\gamma c](z^2) = \alpha_{ev}(z^2)\gamma(z^2)c_{ev}(z^2), \quad z \in \mathbb{T}.$$

Consequently, (2.4) is equivalent to $\alpha_{ev}(z)\gamma(z) = 1$ for all $z \in \mathbb{T}$, i.e., γ is the inverse of the even part α_{ev} of α. $\square$

Now, using Wiener's lemma, we get

Corollary 3.1 *For any mask $\alpha \in l_0(Z)$ satisfying $\alpha_{ev}(z) \neq 0$ for all $z \in \mathbb{T}$,* (2.4) *holds with $\gamma \in l_1(Z)$ determined by $\gamma(z) = \frac{1}{\alpha_{ev}(z)}$, $z \in \mathbb{T}$.*

We call a subdivision operator $\mathcal{S}_\alpha$ with a mask α such that $\alpha_{ev}(z) \neq 0$ for all $z \in \mathbb{T}$ *even-reversible*, and we call the decimation operator $\mathcal{D}_\gamma$ satisfying (2.4) the *even-inverse* of $\mathcal{S}_\alpha$. It is shown in the next subsection that for a large class of finitely supported masks, the condition $\alpha_{ev}(z) \neq 0$ for all $z \in \mathbb{T}$ does hold.

In view of Theorem 3.1, the multiscale transform (2.2) becomes

$$\begin{cases} c^{(\ell-1)}(z) & = \alpha_{ev}^{-1}(z)c_{ev}^{(\ell)}(z) \\ d_{od}^{(\ell)}(z) & = c_{od}^{(\ell)}(z) - \alpha_{od}(z)\alpha_{ev}^{-1}(z)c_{ev}^{(\ell)}(z)\ , \quad \ell = j, \ldots, 1. \\ d_{ev}^{\ell}(z) & \equiv 0 \end{cases} \tag{3.1}$$

We term (3.1) *multiscale transform based on an even-reversible subdivision* (*MTER*).

In terms of matrix computation, the MTER can be written as

$$\begin{cases} c^{(\ell-1)} & = A_{\alpha_{ev}^{-1}} c_{ev}^{(\ell)} \\ d_{od}^{(\ell)} & = c_{od}^{(\ell)} - A_{\alpha_{od}} A_{\alpha_{ev}^{-1}} c_{ev}^{(\ell)}\ , \quad \ell = j, \ldots, 1. \\ d_{ev}^{(\ell)} & \equiv 0 \end{cases}$$

In particular, when α is *interpolatory*,

$$\alpha_{ev}(z) \equiv 1 \text{ and } \gamma(z) = \alpha_{ev}^{-1}(z) \equiv 1, \tag{3.2}$$

and the MTER is reduced to an interpolatory pyramid:

$$\begin{cases} c^{(\ell-1)} & = (c^{(\ell)} \downarrow 2) \\ d_{od}^{(\ell)} & = c_{od}^{(\ell)} - \alpha_{od} * (c^{(\ell)} \downarrow 2)\ , \\ d_{ev}^{(\ell)} & \equiv 0 \end{cases} \quad \ell = j, \ldots, 1.$$

3.2 The Even-Inverse of the Subdivision Operator

In case $\alpha_{ev} > 0$ for all $z \in \mathbb{T}$ and α_{ev} is s-banded, then by Corollary 2.1, $\gamma = \alpha_{ev}^{-1}$ exists, and the elements of the mask γ decay exponentially, which is an important feature for the computation of the decimation operation $\mathcal{D}_\gamma c$. More precisely,

$$|\gamma_\ell| \le K\lambda^{|\ell|} \qquad \forall \ell \in \mathbb{Z},$$

where K and λ are defined as in Corollary 2.1 with α there replaced by α_{ev}.

3.3 Examples

In this subsection, we provide some examples of commonly used subdivision operators with finitely supported masks. We show that for a large class of masks, the corresponding subdivision operators are even-reversible.

First, we give two examples of spline subdivision operators of low order for which we can compute explicitly the even-inverse. We omit the case of the linear spline subdivision operator because it is interpolatory.

Example 1 (Quadratic Spline) Consider the mask α for the (centered) cubic B-spline of order 3: $\alpha(z) = \frac{z^{-1}(1+z)^3}{2^2}$. That is, $\alpha = \{\alpha_{-1}, \alpha_0, \alpha_1, \alpha_2\} = \frac{1}{4}\{1, 3, 3, 1\}_{[-1,2]}$ (i.e. supp$(\alpha) = [-1, 2] \cap \mathbb{Z}$ with $\alpha_{-1} = 1/4$, $\alpha_0 = \alpha_1 = 3/4$, and $\alpha_2 = 1/4$). Then,

$$\alpha_{ev}(z^2) = \frac{1}{2}(\alpha(z) + \alpha(-z)) = \frac{1}{4}(3 + z^2),$$

(continued)

Example 1 (continued)
or as a sequence, $\alpha_{ev} = \frac{1}{4}\{3, 1\}_{[0,1]}$. We have $|\alpha_{ev}(z)| = |\frac{1}{4}(3+z)| \geq \frac{1}{2}$ for all $z \in \mathbb{T}$. Consequently, the inverse γ of α_{ev} exists and is given by

$$\gamma(z) = \frac{1}{\alpha_{ev}(z)} = \frac{4}{3+z} = \frac{4}{3}\left(1 - \frac{1}{3}z + \frac{1}{9}z^2 + \cdots\right) = \frac{4}{3}\sum_{k=0}^{\infty}\left(-\frac{1}{3}\right)^k z^k,$$

which is the symbol of the even-inverse operator $\mathcal{D}_\gamma$. Thus, the quadratic spline subdivision operator is even-reversible. Direct computations show that $\|\gamma\|_1 = 2$, $\|A_\gamma\|_2 = 2$, and $\|\gamma\|_\infty = 4/3$.

Example 2 (Cubic Spline) Consider the mask α for the (centered) B-spline of order 4: $\alpha(z) = \frac{z^{-2}(1+z)^4}{2^3}$. That is, $\alpha = \frac{1}{8}\{1, 4, 6, 4, 1\}_{[-2,2]}$. Then,

$$\alpha_{ev}(z^2) = \frac{1}{2}(\alpha(z) + \alpha(-z)) = \frac{1}{8}(z^{-2} + 6 + z^2),$$

or as a sequence, $\alpha_{ev} = \frac{1}{8}\{1, 6, 1\}_{[-1,1]}$. Note that $\alpha_{ev}(z) = \frac{1}{8}(z^{-1}+6+z) \geq \frac{1}{2}$ for all $z \in \mathbb{T}$. Consequently, the inverse γ of α_{ev} exists. Hence, the cubic spline subdivision operator is even-reversible, and

$$\gamma(z) = \frac{1}{\alpha_{ev}(z)} = \frac{8}{z^{-1}+6+z} = \frac{4}{3} \times \frac{1}{1 + \frac{z^{-1}+z}{6}} = \frac{4}{3}\sum_{n=0}^{\infty}\left(\frac{-1}{6}\right)^n (z^{-1}+z)^n$$
$$= \frac{4}{3}\left(\sum_{n=0}^{\infty} 6^{-2n}(z^{-1}+z)^{2n} - \sum_{n=0}^{\infty} 6^{-2n-1}(z^{-1}+z)^{2n+1}\right).$$

By (2.5), we have $\|A_\gamma\|_2 = 2$. We succeeded to obtain an explicit expression of $\gamma(z)$ of the form

$$\gamma(z) = \sqrt{2} + \sqrt{2}\sum_{k=1}^{\infty}\left(\frac{-1}{3+2\sqrt{2}}\right)^k \times \left(z^k + z^{-k}\right). \tag{3.3}$$

With this explicit form of $\gamma(z)$, the decimation operation in the MTER can be implemented. Moreover, from (3.3), we get $\|\gamma\|_1 = 2$, $\|\gamma\|_\infty = \sqrt{2}$, and the exponential decay of the elements of γ. The proof of (3.3) is not straightforward and is given in the Appendix.

From the above examples, one can expect that spline subdivision operators are even-reversible. In fact, it holds more generally. The class of spline subdivision operators can be regarded as a special subclass of a larger class called *pseudo-spline* subdivision operators, which we introduce next.

Let $n, \nu \in \mathbb{N} \cup \{0\}$ satisfy $0 \le \nu \le \lfloor n/2 \rfloor - 1$. Define

$$\alpha^{n,\nu}(z) = \frac{z^{-\lfloor n/2 \rfloor}(1+z)^n}{2^{n-1}} \sum_{j=0}^{\nu} \binom{n/2+j-1}{j} \left(\frac{1}{2} - \frac{z+z^{-1}}{4}\right)^j. \tag{3.4}$$

When $\nu = 0$, the mask $\alpha^{n,0} = \alpha^n$ belongs to the family of masks of spline subdivision operators. When $n = 2k$ and $\nu = k-1$, the mask $\alpha^{2k,k-1}$ is the Deslauries–Dubuc's interpolatory mask [5]. For $n = 2k$ and $0 \le \nu \le k-1$, the masks $\alpha^{2k,\nu}$ are the masks of the primal pseudo-splines (pseudo-splines of type II), while for $n = 2k+1$, the masks $\alpha^{2k+1,\nu}$, $0 \le \nu \le k-1$ are the masks of the dual pseudo-splines. For more about pseudo-splines, see [4, 7, 8, 11] and references therein.

The next theorem states that pseudo-spline subdivision operators are even-reversible, which implies that all spline subdivision operators are even-reversible.

Theorem 3.2 *Let $n, \nu \in \mathbb{N} \cup \{0\}$ satisfying $0 \le \nu \le \lfloor n/2 \rfloor - 1$ and $\alpha^{n,\nu}$ be the pseudo-spline mask defined as in (3.4). Then, the following holds:*

(1) $\alpha_{ev}^{n,\nu}(1) = 1$ *and* $\|A_{\alpha_{ev}^{n,\nu}}\|_2 = \max_{z \in \mathbb{T}} |\alpha_{ev}^{n,\nu}(z)| = 1$.
(2) $\gamma = (\alpha_{ev}^{n,\nu})^{-1}$ *exists with* $\gamma(1) = 1$.
(3) *The l_2-norm of A_γ is given by* $\|A_\gamma\|_2 = \frac{2^{\lfloor \frac{n-1}{2} \rfloor + \nu}}{\sum_{j=0}^{\nu} \binom{n/2+\nu}{j}}$.
(4) *The elements of γ decay exponentially.*

The proof of Theorem 3.2 is postponed to the Appendix. We remark that (i) when $\nu = 0$, we have $\|A_\gamma\|_2 = 2^{\lfloor (n-1)/2 \rfloor}$ and (ii) when $n = 2k$ and $\nu = k-1$, we have $\gamma(z) \equiv 1$.

4 Decay and Stability

We now turn to the study of the decay property of the pyramid sequences and the stability of the transforms.

4.1 Decay

Let $\Delta : l(\mathbb{Z}) \to l(Z)$ denote the *difference operator*: $(\Delta c)_k = c_{k+1} - c_k$ for $c \in l(\mathbb{Z})$. Then, Δ commutes with convolution operators. Indeed, by $[\Delta c](z) =$

$(z^{-1}-1)c(z)$ and $[\gamma * c](z) = \gamma(z)c(z)$, we have

$$[\Delta(\gamma * c)](z) = (z^{-1}-1)(\gamma(z)c(z)) = \gamma(z)[(z^{-1}-1)c(z)] = [\gamma * (\Delta c)](z).$$

Suppose the data sequence $c^{(j)} = f\big|_{2^{-j}\mathbb{Z}}$, where f is a function in $C^1(\mathbb{R})$ with bounded first derivative. Then,

$$[\Delta c^{(j)}]_k = f(2^{-j}(k+1)) - f(2^{-j}k) = f'(\xi)\cdot 2^{-j}$$

for some $\xi \in (2^{-j}k, 2^{-j}(k+1))$. Thus, $\|\Delta c^{(j)}\|_\infty \le K2^{-j}$ with $K = \|f'\|_\infty$.

Applying the scheme in (2.2), we have a pyramid of data consisting of approximation coefficient sequences $c^{(\ell)}$ and detail coefficient sequences $d^{(\ell)}$. The following two results concern the decay property of $\Delta c^{(\ell)}$ and $d^{(\ell)}$.

Theorem 4.1 (Difference of Approximation Coefficients) *Let $c^{(j)}$ be such that $\|\Delta c^{(j)}\|_\infty \le K2^{-j}$. Let $c^{(\ell)} := \gamma * (c^{(\ell+1)} \downarrow 2)$ with $\gamma \in l_1(\mathbb{Z})$ and $0 \le \ell \le j-1$. Then,*

$$\|\Delta c^{(\ell)}\|_\infty \le K\,\|\gamma\|_1^j \cdot (2\|\gamma\|_1)^{-\ell}, \quad 0 \le \ell \le j-1. \tag{4.1}$$

Proof For $c_{ev} = c \downarrow 2$, we have

$$(\Delta c_{ev})_k = c_{2k+2} - c_{2k} = (c_{2k+2} - c_{2k+1}) + (c_{2k+1} - c_{2k}) = (\Delta c)_{2k+1} + (\Delta c)_{2k}, \quad k \in \mathbb{Z},$$

and hence, $\|\Delta c_{ev}\|_\infty \le 2\|\Delta c\|_\infty$. Since Δ commutes with convolution operators, we have

$$\begin{aligned}\|\Delta c^{(j-1)}\|_\infty &= \|\Delta(\gamma * (c^{(j)} \downarrow 2))\|_\infty = \|\gamma * \Delta(c^{(j)} \downarrow 2))\|_\infty \\ &\le \|\gamma\|_1 \cdot \|\Delta(c^{(j)} \downarrow 2)\|_\infty \le \|\gamma\|_1 \cdot 2\|\Delta c^{(j)}\|_\infty.\end{aligned}$$

Iterating the above inequality starting with $\|\Delta c^{(j)}\|_\infty \le K2^{-j}$, we conclude (4.1). □

Under some very mild conditions on the masks α, γ, we can show that the detail coefficients have the same decay property as the differences of the approximation coefficients.

Theorem 4.2 (Detail Coefficients) *Let $c^{(j)} \in l_\infty(\mathbb{Z})$ be a sequence. Define $c^{(\ell-1)}$ and $d^{(\ell)}$ to be*

$$c^{(\ell-1)} := \gamma * (c^{(\ell)} \downarrow 2), \quad d^{(\ell)} := c^{(\ell)} - \mathcal{S}_\alpha c^{(\ell-1)}, \quad 1 \le \ell \le j,$$

where $\alpha, \gamma \in l_1(\mathbb{Z})$ are masks satisfying

$$\sum_k \alpha_{2k} = \sum_k \alpha_{2k+1} = \sum_k \gamma_k = 1 \tag{4.2}$$

and

$$\sum_{k\in\mathbb{Z}} |\alpha_k|\,|k| = K_\alpha < \infty, \quad 2\sum_{k\in\mathbb{Z}} |\gamma_k|\,|k| = K_\gamma < \infty. \tag{4.3}$$

Then,

$$\|d^{(\ell)}\|_\infty \le K_{\alpha,\gamma}\|\Delta c^{(\ell)}\|_\infty, \quad 1 \le \ell \le j \tag{4.4}$$

with $K_{\alpha,\gamma} := K_\gamma\|\alpha\|_1 + K_\alpha\|\gamma\|_1$. *In particular, if* $\|\Delta c^{(j)}\|_\infty \le K \cdot 2^{-j}$ *for some* K *independent of* j, *then*

$$\|d^{(\ell)}\|_\infty \le \left(K \cdot K_{\alpha,\gamma} \cdot \|\gamma\|_1^j\right) \cdot (2\|\gamma\|_1)^{-\ell}, \quad 1 \le \ell \le j. \tag{4.5}$$

Proof Let $\eta = \{\eta_k\}_{k\in\mathbb{Z}} := c^{(\ell-1)} = \gamma * (c^{(\ell)} \downarrow 2)$. Then, $\eta_k = \sum_s \gamma_{k-s}c_{2s}^{(\ell)}$. By (4.2), we have $d_k^{(\ell)} = c_k^{(\ell)} - [\mathcal{S}_\alpha\eta]_k = c_k^{(\ell)} - \sum_s \alpha_{k-2s}\eta_s = \sum_s \alpha_{k-2s}\Big[\sum_n \gamma_{s-n}(c_k^{(\ell)} - c_{2n}^{(\ell)})\Big]$. Consequently,

$$\begin{aligned}
\|d^{(\ell)}\|_\infty &\le \sum_s |\alpha_{k-2s}|\Big[\sum_n |\gamma_{s-n}||2n-k|\Big]\|\Delta c^{(\ell)}\|_\infty \\
&\le \sum_s |\alpha_{k-2s}|\Big[\sum_n |\gamma_{s-n}|(|2n-2s| + |k-2s|)\Big]\|\Delta c^{(\ell)}\|_\infty \\
&\le \sum_s |\alpha_{k-2s}|\Big[K_\gamma + |k-2s|\|\gamma\|_1\Big]\|\Delta c^{(\ell)}\|_\infty \\
&= (K_\gamma\|\alpha\|_1 + K_\alpha\|\gamma\|_1)\|\Delta c^{(\ell)}\|_\infty = K_{\alpha,\gamma}\|\Delta c^{(\ell)}\|_\infty \\
&\le K \cdot K_{\alpha,\gamma} \cdot \|\gamma\|_1^j \cdot (2\|\gamma\|_1)^{-\ell},
\end{aligned}$$

where the last inequality follows from (4.1). □

Theorems 4.1 and 4.2 imply the following corollary.

Corollary 4.1 *Let* $\alpha \in l_0(\mathbb{Z})$ *be a mask of a convergent subdivision scheme satisfying* $\alpha_{ev}(z) > 0$ *for* $z \in \mathbb{T}$, *and let* $c^{(j)}$ *be a sequence satisfying* $\|\Delta c^{(j)}\|_\infty \le K2^{-j}$. *Then, the pyramid generated by the MTER in* (3.1) *satisfies* (4.1) *and* (4.5).

We remark that in the interpolatory case, (3.2) holds. Therefore, inequalities (4.1) and (4.5) depend on the level ℓ and are independent of j. For non-interpolatory subdivision operators and for a fixed j, the decay of the details is faster, but the constant is bigger since $\|\gamma\|_1 > 1$.

4.2 Stability of Reconstruction

In this subsection, we show that the reconstruction is stable.

Theorem 4.3 *Suppose* $\sup_{j\in\mathbb{N}} \|\mathcal{S}_\alpha^j\|_\infty \le K$ *for some constant* $K > 0$*. Then, the reconstructed data* $c^{(j)}$ *at level* j *from coarse data* $c^{(0)}$ *and details* $d^1, \dots, d^{(j)}$ *via* (2.3) *are stable; that is,*

$$\|c^{(j)} - \widetilde{c}^{(j)}\|_\infty \le K \left(\|c^{(0)} - \widetilde{c}^{(0)}\|_\infty + \sum_{\ell=1}^{j} \|d^{(\ell)} - \widetilde{d}^{(\ell)}\|_\infty \right),$$

where $\widetilde{c}^{(j)}$ *is reconstructed from the data* $\widetilde{c}^{(0)}$, $\widetilde{d}^1, \dots, \widetilde{d}^{(j)}$ *via* (2.3).

Proof From $c^{(\ell)} = \mathcal{S}_\alpha c^{(\ell-1)} + d^{(\ell)}$, we have

$$\begin{aligned}
\|c^{(j)} - \widetilde{c}^{(j)}\|_\infty &= \|\mathcal{S}_\alpha(c^{(j-1)} - \widetilde{c}^{(j-1)}) + (d^{(j)} - \widetilde{d}^{(j)})\|_\infty \\
&= \|\mathcal{S}_\alpha^2(c^{(j-2)} - \widetilde{c}^{(j-2)}) + \mathcal{S}_\alpha(d^{(j-1)} - \widetilde{d}^{(j-1)}) + (d^{(j)} - \widetilde{d}^{(j)})\|_\infty \\
&\vdots \\
&= \left\| \mathcal{S}_\alpha^j(c^{(0)} - \widetilde{c}^{(0)}) + \sum_{\ell=1}^{j} \mathcal{S}_\alpha^{j-\ell}(d^{(\ell)} - \widetilde{d}^{(\ell)}) \right\|_\infty \\
&\le \|\mathcal{S}_\alpha^j\|_\infty \|c^{(0)} - \widetilde{c}^{(0)}\|_\infty + \sum_{\ell=1}^{j} \|\mathcal{S}_\alpha^{j-\ell}\|_\infty \|d^{(\ell)} - \widetilde{d}^{(\ell)}\|_\infty \\
&\le K \left(\|c^{(0)} - \widetilde{c}^{(0)}\|_\infty + \sum_{\ell=1}^{j} \|d^{(\ell)} - \widetilde{d}^{(\ell)}\|_\infty \right).
\end{aligned}$$

□

It is well known and easy to see from the uniform boundedness principle that the condition $\sup_{j\in\mathbb{N}} \|\mathcal{S}_\alpha^j\|_\infty \le K$ holds whenever the subdivision scheme based on $\mathcal{S}_\alpha$ is convergent.

4.3 Stability of Decomposition

We now show stability of the decomposition of the MTER in (3.1) for fixed j.

Theorem 4.4 *Suppose* $c^{(j)}, \widetilde{c}^{(j)} \in l_p(\mathbb{Z})$ *for* $p \in [1, \infty]$*. Let the pyramid data* $\{c^{(0)}; d^{(1)}, \dots, d^{(j)}\}$ *and* $\{\widetilde{c}^{(0)}; \widetilde{d}^{(1)}, \dots, \widetilde{d}^{(j)}\}$ *be obtained from* $c^{(j)}$ *and* $\widetilde{c}^{(j)}$*,* *respectively, by the scheme* (2.2)*. Then, we have for* $\ell = 1, \ldots, j$,

$$\begin{aligned} &\|c^{(0)} - \widetilde{c}^{(0)}\|_p \le \|\mathcal{D}_\gamma\|_p^j \cdot \|c^{(j)} - \widetilde{c}^{(j)}\|_p, \\ &\|d^{(\ell)} - \widetilde{d}^{(\ell)}\|_p \le \left(\|\mathcal{I} - \mathcal{S}_\alpha \mathcal{D}_\gamma\|_p \cdot \|\mathcal{D}_\gamma\|_p^j \cdot \|c^{(j)} - \widetilde{c}^{(j)}\|_p\right) \cdot \|\mathcal{D}_\gamma\|_p^{-\ell}. \end{aligned} \tag{4.6}$$

Proof By $\|c^{(\ell-1)} - \widetilde{c}^{(\ell-1)}\|_p = \|\mathcal{D}_\gamma(c^{(\ell)} - \widetilde{c}^{(\ell)})\|_p \le \|\mathcal{D}_\gamma\|_p \|c^{(\ell)} - \widetilde{c}^{(\ell)}\|_p$, we have

$$\|c^{(0)} - \widetilde{c}^{(0)}\|_p \le \|\mathcal{D}_\gamma\|_p^j \|c^{(j)} - \widetilde{c}^{(j)}\|_p.$$

Similarly, by

$$\|d^{(\ell)} - \widetilde{d}^{(\ell)}\|_p = \|(\mathcal{I} - \mathcal{S}_\alpha \mathcal{D}_\gamma)(c^{(\ell)} - \widetilde{c}^{(\ell)})\|_p \le \|\mathcal{I} - \mathcal{S}_\alpha \mathcal{D}_\gamma\|_p \|c^{(\ell)} - \widetilde{c}^{(\ell)})\|_p,$$

we have

$$\|d^{(\ell)} - \widetilde{d}^{(\ell)}\|_p \le \|(\mathcal{I} - \mathcal{S}_\alpha \mathcal{D}_\gamma)\|_p \|\mathcal{D}_\gamma\|_p^{j-\ell} \|c^{(j)} - \widetilde{c}^{(j)}\|_p.$$

We are done. □

Remarks

(i) For $\mathcal{S}_\alpha$ an interpolatory subdivision operator, (3.2) holds. Therefore, in the corresponding MTER, $\mathcal{D}_\gamma c$ is simply $\mathcal{D}_\gamma c = c \downarrow 2$ and $\|\mathcal{D}_\gamma\|_p \equiv 1$. It follows from Theorem 4.4 that the decomposition of MTERs based on interpolatory subdivision operators is stable for all $p \in [1, \infty]$; namely, the constants in (4.6) are independent of j.

(ii) For non-interpolatory subdivision operators corresponding to convergent even-reversible subdivision schemes, the even-inverse $\mathcal{D}_\gamma$ satisfies $\|\mathcal{D}_\gamma\|_p \ge 1$ since $\gamma(1) = 1$. In such a case, the constant $\|\mathcal{D}_\gamma\|_p^j$ depends on the level j of the data.

In the following, we give bounds on $\|\mathcal{D}_\gamma\|_\infty = \|\gamma\|_1$ for the family of primal pseudo-spline subdivision operators.

Theorem 4.5 *Let* $\alpha^{2k,\nu}$ *be the mask as defined in* (3.4) *with* $0 \le \nu \le k-1$ *and* $k \ge 2$*, and let* $\gamma = (\alpha_{ev}^{2k,\nu})^{-1}$*. Then,*

$$\|\mathcal{D}_\gamma\|_\infty = \|\gamma\|_1 \le C(k, \nu),$$

where

$$C(k, \nu) = \kappa \cdot \max\left\{1, \frac{(1+\sqrt{\kappa})^2}{2\kappa}\right\} \cdot \frac{1+\lambda}{1-\lambda} \tag{4.7}$$

is a constant with

$$\kappa = \frac{2^{k+\nu-1}}{\sum_{j=0}^{\nu}\binom{k+\nu}{j}}, \quad \lambda = \left(\frac{\sqrt{\kappa}-1}{\sqrt{\kappa}+1}\right)^{1/s}, \ \textit{and } s = \lfloor (k+\nu)/2 \rfloor.$$

The proof of Theorem 4.5 is given in the Appendix. We remark that

(i) For the case $\nu = k-1$, $\alpha^{2k,k-1}$ corresponds to the family of Deslauries–Dubuc's interpolatory masks [5]. In such a case, the constant $C(k, k-1)$ is exact in the sense that $C(k, k-1) = 1$.
(ii) For the case $\nu = 0$, $\alpha^{2k,0}$ is the mask of degree $2k-1$ (order $2k$) spline subdivision operator. In such a case, it can be shown that $C(k, 0) = O(k \cdot 2^{\frac{3k}{2}})$ for $k \to \infty$.

Combining the above results, we have the following result regarding the stability of decomposition of the MTER based on primal pseudo-spline masks in $l_p(\mathbb{Z})$ for two important cases $p = 2$ and $p = \infty$.

Corollary 4.2 *Suppose* $c^{(j)}, \widetilde{c}^{(j)} \in l_p(\mathbb{Z})$ *for* $p \in [1, \infty]$. *Let the pyramid data* $\{c^{(0)}; d^{(1)}, \ldots, d^{(j)}\}$ *and* $\{\widetilde{c}^{(0)}; \widetilde{d}^{(1)}, \ldots, \widetilde{d}^{(j)}\}$ *be obtained from* $c^{(j)}$ *and* $\widetilde{c}^{(j)}$, *respectively, by the scheme* (2.2) *with* $\alpha = \alpha^{2k,\nu}$ *for* $0 \le \nu \le k-2$ *and* $k \ge 2$. *Then,*

(1) *for* $p = \infty$, *we have*

$$\|c^{(0)} - \widetilde{c}^{(0)}\|_\infty \le C(k,\nu)^j \cdot \|c^{(j)} - \widetilde{c}^{(j)}\|_\infty,$$

$$\|d^{(\ell)} - \widetilde{d}^{(\ell)}\|_\infty \le \left(\|\mathcal{I} - \mathcal{S}_{\alpha^{2k,\nu}}\mathcal{D}_\gamma\|_\infty \cdot C(k,\nu)^j \cdot \|c^{(j)} - \widetilde{c}^{(j)}\|_\infty\right) \cdot C(k,\nu)^{-\ell} \tag{4.8}$$

for $\ell = 1, \ldots, j$, *where* $C(k,\nu)$ *is the constant defined in* (4.7);
(2) *for* $p = 2$, (4.6) *holds with* $\|\mathcal{D}_\gamma\|_2 = \|A_\gamma\|_2 = \frac{2^{k+\nu-1}}{\sum_{j=0}^{\nu}\binom{k+\nu}{j}}$.

Proof This is a direct consequence of Theorems 4.4 and 4.5. □

5 Final Remarks

In this section, we give some further remarks.

1. Most examples and results in this chapter deal with finitely supported masks for the purpose of simplicity of presentation. We point out that the MTER in (3.1) applies for any mask α provided $\alpha_{ev}(z) \neq 0$ for all $z \in \mathbb{T}$.
2. Using the weighted version of Wiener's lemma [16], one can study classes of masks of infinite support such as masks with polynomial or sub-exponential decay and masks corresponding to rational symbols. The extension of the class of masks might lead to even-inverse operators of smaller norm, and by that to the improvement of the stability of the decomposition and the decay rate of the details as given in Theorems 4.4 and 4.2, respectively.
3. The existence of MTER relies on Wiener's lemma. Since a high-dimensional version of Wiener's lemma holds, our MTER in (3.1) can be generalized to any dimension $d \in \mathbb{N}$.
4. Without involving Wiener's lemma, one can also analyze the decay property of the Laurent polynomials $\gamma(z) = \alpha_{ev}^{-1}(z)$ using the Fourier series techniques. One can show that $|\gamma(k)| := |\frac{1}{2\pi}\int_0^{2\pi} \gamma(e^{-i\xi})e^{-ik\xi}d\xi| \leq Ce^{-r_1|k|}$ for some $0 < r_1 < r$ with r being the radius of convergence of the Laurent polynomials of $\gamma(z)$. Such techniques were employed in [17].
5. One can consider more general decimation operator in (2.2), e.g.

$$c^{(j-1)} = \mathcal{D}_{\gamma_{ev},\gamma_{od}} c^{(j)} := \gamma_{ev} * c_{ev}^{(j)} + \gamma_{od} * c_{od}^{(j)}.$$

   When $\gamma_{od} = 0$, we see that $\mathcal{D}_{\gamma_{ev},\gamma_{od}} = \mathcal{D}_{\gamma_{ev}}$, which goes back to our setting. More general results could be obtained with more analysis.

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Appendix

Proof of (3.3) Note that

$$\gamma(z) = \frac{8}{z^{-1}+6+z} = \frac{4}{3}\left(\sum_{n=0}^{\infty} 6^{-2n}(z^{-1}+z)^{2n} - \sum_{n=0}^{\infty} 6^{-2n-1}(z^{-1}+z)^{2n+1}\right).$$

For $\sum_{n=0}^{\infty}(\frac{1}{6})^{2n}(z^{-1}+z)^{2n}$, we have

$$\sum_{n=0}^{\infty} 6^{-2n}(z^{-1}+z)^{2n} = \sum_{n=0}^{\infty} 6^{-2n} \sum_{k=0}^{2n} \binom{2n}{k} z^{2(n-k)} = \sum_{n=0}^{\infty} 6^{-2n} \sum_{k=-n}^{n} \binom{2n}{n-k} z^{2k}$$
$$= \sum_{n=0}^{\infty} \binom{2n}{n} 6^{-2n} + \sum_{k=1}^{\infty} \left[\sum_{n=k}^{\infty} \binom{2n}{n-k} 6^{-2n} \right] (z^{2k} + z^{-2k})$$
$$= \sum_{n=0}^{\infty} \binom{2n}{n} 6^{-2n} + \sum_{k=1}^{\infty} \left[\sum_{n=0}^{\infty} \binom{2(n+k)}{n} 6^{-2(n+k)} \right] (z^{2k} + z^{-2k}).$$

Similarly, for the second summation, we have

$$\sum_{n=0}^{\infty} 6^{-2n-1}(z^{-1}+z)^{2n+1} = \sum_{k=0}^{\infty} \left[\sum_{n=0}^{\infty} \binom{2(n+k)+1}{n} 6^{-2(n+k)-1} \right] (z^{2k+1} + z^{-2k-1}).$$

Define for $k = 0, 1, 2, \ldots,$

$$a_k := \sum_{n=0}^{\infty} \binom{2(n+k)}{n} 6^{-2(n+k)} \quad \text{and} \quad b_k := \sum_{n=0}^{\infty} \binom{2(n+k)+1}{n} 6^{-2(n+k)-1}.$$

Then,

$$\gamma(z) = \frac{4}{3} \sum_{n=0}^{\infty} \left(\frac{-1}{6} \right)^n (z^{-1}+z)^n$$
$$= \frac{4}{3} \left[a_0 + \sum_{k=1}^{\infty} a_k (z^{2k} + z^{-2k}) - \sum_{k=0}^{\infty} b_k (z^{2k+1} + z^{-2k-1}) \right].$$

Using the formula $\binom{n+1}{m} = \binom{n}{m} + \binom{n}{m-1}$, it is easy to check that

$$a_k = \frac{1}{6}(b_{k-1} + b_k) \quad \text{and} \quad b_k = \frac{1}{6}(a_k + a_{k+1}).$$

Hence,

$$a_{k+1} = 6b_k - a_k \quad \text{and} \quad b_k = 6a_k - b_{k-1}. \tag{A.1}$$

Next, we proceed to prove the following identities using the recurrence relations (A.1):

$$a_k = \frac{3\sqrt{2}}{4}(3-2\sqrt{2})^{2k} \quad \text{and} \quad b_k = \frac{3\sqrt{2}}{4}(3-2\sqrt{2})^{2k+1}, \quad k = 0, 1, 2, \ldots. \tag{A.2}$$

First, we compute directly a_0 and b_0. Using the identities

$$(1-x)^{-t} = \sum_{n=0}^{\infty} \binom{t-1+n}{n} x^n \tag{A.3}$$

and $\binom{2n}{n} = 4^n \binom{n-1/2}{n}$, we have

$$a_0 = \sum_{n=0}^{\infty} \binom{2n}{n} 6^{-2n} = \sum_{n=0}^{\infty} \binom{1/2-1+n}{n} 9^{-n} = (1-1/9)^{-1/2} = \frac{3\sqrt{2}}{4}.$$

Similarly,

$$b_0 = \sum_{n=0}^{\infty} \binom{2n+1}{n} 6^{-2n-1} = 3\left[\sum_{n=0}^{\infty} \binom{1/2-1+n}{n} 9^{-n} - 1\right] = \frac{3\sqrt{2}}{4}(3-2\sqrt{2}).$$

Now, recursively using the relation of a_k and b_k in (A.1), we get

$$\begin{aligned} a_{k+1} = 6b_k - a_k &= \frac{3\sqrt{2}}{4}(6(3-2\sqrt{2})^{2k+1} - (3-2\sqrt{2})^{2k}) \\ &= \frac{3\sqrt{2}}{4}(3-2\sqrt{2})^{2k}(6(3-2\sqrt{2})-1) = \frac{3\sqrt{2}}{4}(3-2\sqrt{2})^{2k}(3-2\sqrt{2})^2 \\ &= \frac{3\sqrt{2}}{4}(3-2\sqrt{2})^{2(k+1)}, \end{aligned}$$

and

$$\begin{aligned} b_k = 6a_k - b_{k-1} &= \frac{3\sqrt{2}}{4}(6(3-2\sqrt{2})^{2k} - (3-2\sqrt{2})^{2k-1}) \\ &= \frac{3\sqrt{2}}{4}(3-2\sqrt{2})^{2k-1}(6(3-2\sqrt{2})-1) = \frac{3\sqrt{2}}{4}(3-2\sqrt{2})^{2k-1}(3-2\sqrt{2})^2 \\ &= \frac{3\sqrt{2}}{4}(3-2\sqrt{2})^{2k+1}. \end{aligned}$$

Therefore, (A.2) holds. In summary, we conclude that

$$\begin{aligned} \gamma(z) &= \frac{4}{3}\left[a_0 + \sum_{k=1}^{\infty} a_k (z^{2k} + z^{-2k}) - \sum_{k=0}^{\infty} b_k (z^{2k+1} + z^{-2k-1})\right] \\ &= \sqrt{2}\left[1 + \sum_{k=1}^{\infty} (3-2\sqrt{2})^{2k}(z^{2k} + z^{-2k}) - \sum_{k=0}^{\infty} (3-2\sqrt{2})^{2k+1}(z^{2k+1} + z^{-2k-1})\right], \end{aligned}$$

which proves (3.3).

Alternative, one can use standard complex analysis technique to compute the Laurent series of the function $\frac{8}{z^{-1}+6+z}$. Note that $z^2+6z+1=0$ has two real roots $-3\pm 2\sqrt{2}$ with one outside $\mathbb{T}$ and the other inside $\mathbb{T}$. Hence, one can write

$$\frac{8}{z^{-1}+6+z}=\frac{8z}{z^2+6z+1}=\frac{\sqrt{2}z}{z+3-2\sqrt{2}}-\frac{\sqrt{2}z}{z+3+2\sqrt{2}}.$$

Since $|-3+2\sqrt{2}|<1$, we have

$$\frac{\sqrt{2}z}{z+3-2\sqrt{2}}=\frac{\sqrt{2}}{1-(-3+2\sqrt{2})z^{-1}}=\sqrt{2}\sum_{k=0}^{\infty}(-3+2\sqrt{2})^k z^{-k} \tag{A.4}$$

holds for $|z|>|-3+2\sqrt{2}|$. Similarly, since $|-3-2\sqrt{2}|>1$, we have

$$-\frac{\sqrt{2}z}{z+3+2\sqrt{2}}=-\frac{\sqrt{2}z(3-2\sqrt{2})}{1-(-3+2\sqrt{2})z}=\sqrt{2}\sum_{k=1}^{\infty}(-3+2\sqrt{2})^k z^{k} \tag{A.5}$$

holds for $|z|<3+2\sqrt{2}$. Putting (A.4) and (A.5) together, we get (12). □

Proof of Theorem 3.2 Let $z=e^{-i\theta}$ and $x=\sin^2(\theta/2)$. One can show that

$$\alpha^{n,\nu}(e^{-i\theta})=2e^{i(\lfloor n/2\rfloor-n/2)\theta}\cos^n(\theta/2)Q_{n,\nu}(\sin^2(\theta/2)),$$

with

$$Q_{n,\nu}(x):=\sum_{j=0}^{\nu}\binom{n/2-1+j}{j}x^j=\frac{1}{(1-x)^{n/2}}+O(x^{\nu+1}),$$

where in the last equality, we use (A.3). Note that $Q_{n,\nu}(x)\geq 1$ for all $x\geq 0$. By that $\alpha_{ev}^{n,\nu}(z^2)=\frac{1}{2}(\alpha^{n,\nu}(z)+\alpha^{n,\nu}(-z))$, we have

$$\alpha_{ev}^{n,\nu}(e^{-2i\theta})=e^{i(\lfloor n/2\rfloor-n/2)\theta}\Big[(1-x)^{n/2}Q_{n,\nu}(x)+i^{2\lfloor n/2\rfloor-n}(-1)^n x^{n/2}Q_{n,\nu}(1-x)\Big].$$

It is easy to see that $\alpha_{ev}^{n,\nu}(1)=1$.

For $n=2k$, we have

$$\alpha_{ev}^{2k,\nu}(e^{-2i\theta})=(1-x)^kQ_{2k,\nu}(x)+x^kQ_{2k,\nu}(1-x)=:R(x)+R(1-x), \tag{A.6}$$

where $R(x):=(1-x)^kQ_{2k,\nu}(x)=(1-x)^k\sum_{j=0}^{\nu}\binom{k-1+j}{j}x^j$. Define $g(x):=R(x)+R(1-x)$. By using $(j+1)\binom{k+j}{j+1}-j\binom{k-1+j}{j}=k\binom{k-1+j}{j}$, one can show that

$$R'(x) = -(k+\nu)\binom{k-1+\nu}{\nu}(1-x)^{k-1}x^{\nu}.$$

Thus,

$$g'(x) = R'(x) - R'(1-x) = (k+\nu)\binom{k-1+\nu}{\nu}x^{\nu}(1-x)^{\nu}\Big[x^{k-1-\nu} - (1-x)^{k-1-\nu}\Big].$$

It is easily seen that $g'(x) \leq 0$ for $x \in [0, 1/2]$ and $g'(x) \geq 0$ for $x \in [1/2, 1]$. Consequently,

$$\begin{aligned}\min_{z\in\mathbb{T}} |\alpha_{ev}^{2k,\nu}(z)| &= \min_{z\in\mathbb{T}} \alpha_{ev}^{2k,\nu}(z) = \min_{x\in[0,1]} g(x) = g(1/2) = 2R(1/2)\\ &= 2^{1-k}\sum_{j=0}^{\nu}\binom{k-1+j}{j}2^{-j} > 0,\end{aligned}$$

where the last equation can be shown to be equivalent to

$$\min_{z\in\mathbb{T}} |\alpha_{ev}^{2k,\nu}(z)| = 2^{1-k-\nu}\sum_{j=0}^{\nu}\binom{k+\nu}{j} > 0. \tag{A.7}$$

Moreover, by (2.5), we have

$$\|A_{\alpha_{ev}^{2k,\nu}}\|_2 = \max_{z\in\mathbb{T}} |\alpha_{ev}^{2k,\nu}(z)| = \max_{x\in[0,1]} g(x) = g(0) = g(1) = 1.$$

For $n = 2k+1$, we have

$$\alpha_{ev}^{2k+1,\nu}(e^{-2i\theta}) = e^{-i\theta/2}\Big[(1-x)^{k+1/2}Q_{2k+1,\nu}(x) + i\cdot x^{k+1/2}Q_{2k+1,\nu}(1-x)\Big],$$

from which, we have

$$\begin{aligned}|\alpha_{ev}^{2k+1,\nu}(e^{-2i\theta})|^2 &= (1-x)^{2k+1}(Q_{2k+1,\nu}(x))^2 + x^{2k+1}(Q_{2k+1,\nu}(1-x))^2\\ &=: R(x)^2 + R(1-x)^2,\end{aligned}$$

where

$$R(x) := (1-x)^{k+1/2}Q_{2k+1,\nu}(x) = (1-x)^{k+1/2}\sum_{j=0}^{\nu}\binom{k-1/2+j}{j}x^j.$$

Define $g(x) = R(x)^2 + R(1-x)^2$. Then, by using $(j+1)\binom{k+1/2+j}{j+1} - j\binom{k-1/2+j}{j} = (k+1/2)\binom{k-1/2+j}{j}$, we can show similarly that

$$R'(x) = -(k+1/2+\nu)\binom{k-1/2+\nu}{\nu}(1-x)^{k-1/2}x^{\nu}.$$

Consequently,

$$\begin{aligned} g'(x) &= 2\big[R(x)R'(x) - R(1-x)R'(1-x)\big] \\ &= -2cx^{\nu}(1-x)^{\nu}\left[\sum_{j=0}^{\nu}\binom{k-1/2+j}{j}(1-x)^j x^j\Big[(1-x)^{2k-\nu-j} - x^{2k-\nu-j}\Big]\right], \end{aligned}$$

where $c = (k+1/2+\nu)\binom{k-1/2+\nu}{\nu}$. Now, it is easy to see that each term

$$t_j(x) := \binom{k-1/2+j}{j}(1-x)^j x^j\big[(1-x)^{2k-\nu-j} - x^{2k-\nu-j}\big]$$

in the above summation for $j = 0, \ldots, \nu$ satisfies

$$t_j(x) \geq 0 \text{ for } x \in [0, 1/2] \text{ and } t_j(x) \leq 0 \text{ for } x \in [1/2, 1].$$

Hence, $g'(x) \leq 0$ for $x \in [0, 1/2]$ and $g'(x) \geq 0$ for $x \in [1/2, 1]$. Consequently,

$$\begin{aligned} \min_{x\in[0,1]} g(x) = g(1/2) = 2R(1/2)^2 &= 2\left(2^{-k-1/2}\sum_{j=0}^{\nu}\binom{k-1/2+j}{j}2^{-j}\right)^2 \\ &= 2^{-2k-2\nu}\left(\sum_{j=0}^{\nu}\binom{k+1/2+\nu}{j}\right)^2. \end{aligned}$$

Therefore,

$$\min_{z\in\mathbb{T}} |\alpha_{ev}^{2k+1,\nu}(z)| = \min_{x\in[0,1]} \sqrt{g(x)} = 2^{-k-\nu}\sum_{j=0}^{\nu}\binom{k+1/2+\nu}{j} > 0,$$

and by (2.5), we have

$$\|A_{\alpha_{ev}^{2k+1,\nu}}\|_2 = \max_{z\in\mathbb{T}} |\alpha_{ev}^{2k+1,\nu}(z)| = g(0) = g(1) = 1.$$

Combining the above results for n even and odd, we see that $|\alpha_{ev}^{n,\nu}(z)| > 0$ for all $z \in \mathbb{T}$ (in particular, $\alpha_{ev}^{2k,\nu}(z) > 0$ for all $z \in \mathbb{T}$). Hence, by Wiener's lemma, its inverse $\gamma = (\alpha_{ev}^{n,\nu})^{-1}$ exists and $\gamma(1) = \frac{1}{\alpha_{ev}^{n,\nu}(1)} = 1$. Moreover, by (2.5), we have

$$\|A_{\alpha_{ev}^{n,\nu}}\|_2 = \max_{z\in\mathbb{T}} |\alpha_{ev}^{n,\nu}(z)| = 1$$

and

$$\|A_\gamma\|_2 = \max_{z\in\mathbb{T}} |\gamma(z)| = \frac{1}{\min_{z\in\mathbb{T}} |\alpha_{ev}^{n,\nu}(z)|} = \frac{2^{\lfloor \frac{n-1}{2} \rfloor + \nu}}{\sum_{j=0}^{\nu} \binom{n/2+\nu}{j}}.$$

The exponential decay of the elements of γ in the case that $n = 2k$ follows directly from Corollary 2.1 since $\alpha_{ev}^{2k,\nu}(z) > 0$ for all $z \in \mathbb{T}$. In case $n = 2k + 1$, the exponential decay of the elements of γ follows from the weighted version of Wiener's lemma [16]. $\square$

Proof of Theorem 4.5 By (A.6) and (A.7), we have $\alpha_{ev}^{2k,\nu}(z) > 0$ for all $z \in \mathbb{T}$. Applying Corollary 2.1 and Theorem 3.2, we have

$$|\gamma_n| \le K\lambda^{|n|}, \quad n \in \mathbb{Z}, \tag{A.8}$$

where $K = \kappa \cdot \max\left\{1, \frac{(1+\sqrt{\kappa})^2}{2\kappa}\right\}$ with $\kappa = \frac{\sup_{z\in\mathbb{T}} |\alpha_{ev}^{2k,\nu}(z)|}{\inf_{z\in\mathbb{T}} |\alpha_{ev}^{2k,\nu}(z)|} = \|A_\gamma\|_2 = \frac{2^{k+\nu-1}}{\sum_{j=0}^{\nu}\binom{k+\nu}{j}}$, and $\lambda = q^{1/s}$ with $q = (\sqrt{\kappa} - 1)/(\sqrt{\kappa} + 1)$, $\quad s = \lfloor (k + \nu)/2 \rfloor$. Thus, by (A.8), we have

$$\|\gamma\|_1 = |\gamma_0| + 2\sum_{n=1}^{\infty} |\gamma_n| \le K\Big(1 + 2\sum_{n=1}^{\infty} \lambda^n\Big) = K\frac{1+\lambda}{1-\lambda} = C(k,\nu).$$

$\square$

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