BOUNDARY ELEMENT METHOD

A case study: the Helmholtz equation.

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Abstract

This is a brief introduction to the boundary element method (BEM) for solving elliptic partial differential equations (PDEs) when they are reduced to boundary integral equations (BIEs), either using Green’s formulae directly or using other indirect methods. Up to now, the BEM has been well analyzed and become a mature method just like the FEM. In this notes, we discuss briefly both the direct and indirect approach of the formulation of boundary integral equations. After we have solved the boundary integral equation with BEM, we obtain both the solution and its normal derivative on the boundary, therefore the interior values of the solution are given by the boundary integral representation. We sketch the frame work of Galerkin BEM as well as the collocation BEM in 2D. Some problems of implementation of the collocation BEM in 3D was discussed. Most of the content in this notes can be found in the lecture notes [7] written in 2006. The matrix generated by BEM is full, which can be solved by using iterative solvers such as GMRES with matrix-vector multiplication computed with the fast multipole method (FMM). For simplicity, we focus on the discussion of Helmholtz equation. Other elliptic equations can be analyzed similarly provided the fundamental solution is available.

Key words: Helmholtz equation, fundamental solution, Green’s formula, boundary integral representation, boundary integral equation, boundary element method, multipole expansion.

1 Introduction

The boundary element method (BEM) is a numerical computational method of solving linear elliptic PDEs which have been formulated as equivalent boundary integral equations. It can be applied in many areas of engineering and science including fluid mechanics, acoustics, electromagnetics, and fracture mechanics. The boundary element method attempts to use the given boundary conditions to fit boundary values into the integral equation, rather than values throughout the space defined by a partial differential equation. Once this is done, in the post-processing stage, the integral equation can then be used again to calculate numerically the solution directly at any desired point in the interior of the solution domain. The BEM is often more efficient than other methods, including finite elements method, if the solution is only desired at some particular point of the domain. Moreover, the BEM is especially efficient to solve exterior problems, which reduce the infinite domain problem to a finite domain problem of lower dimension. However, for many problems boundary element methods are significantly less efficient than volume-discretisation methods (finite element method, finite difference method, finite volume method). Boundary element formulations typically give rise to fully populated matrices. Compression techniques such as multipole expansions can be used to ameliorate these problems, though at the cost of added complexity and with a success-rate that depends heavily on the nature of the problem being solved and the geometry involved. The linear system can
be solved by using the iterative solver GMRES with matrix-vector product computed with the
multipole expansion method. Then the cost of a matrix-vector product can be reduced to \(O(N)\),
where \(N\) is the size of the linear system.

The BEM is only applicable to problems for which Green’s functions is available and is
only efficient to problems that no nonlinearities appear in the domain, though in principle,
nonlinearities can be included in the formulation by introducing volume integrals which then
require the volume to be discretised before solution can be attempted, removing one of
the most often cited advantages of BEM. These usually involve fields in linear homogeneous media.
This places considerable restrictions on the range and generality of problems to which boundary
elements can usefully be applied.

The Green’s functions, or fundamental solutions, are often problematic to integrate as they
contains singularity at the source point. Integrating such singular fields is not easy. For simple
element geometries, analytical integration can be used. For more general elements, it is possible
to design purely numerical schemes that adapt to the singularity, but at great computational
cost.

In the next section, we give a direct formulation of boundary integral representation of the
solution for the Helmholtz equation as well as the conventional and hypersingular boundary
integral equations.

2 Direct formulation of BIEs

The direct boundary integral formulation uses Green’s formula to reduce the differential equation
in the domain to a boundary integral equation. For instance, we consider the electromagnetic
scattering problem outside a bounded Lipschitz domain \(\Omega\) in \(\mathbb{R}^n\), which can be described as an
exterior boundary value problem:

\[
\begin{align*}
\Delta u + k^2 u &= 0, \quad \text{in } \mathbb{R}^n \setminus \bar{\Omega}, \\
\partial_n u &= g, \quad \text{on } \partial \Omega,
\end{align*}
\]

(2.1)

with the Sommerfeld radiation condition

\[
\lim_{r \to \infty} \sqrt{r} \left( \frac{\partial u}{\partial r} - iku \right) = 0, \quad r = |x|
\]

(2.2)

uniformly in all directions. The Sommerfeld radiation condition is to ensure that the solution
decreases fast enough as \(|x| \to \infty\). For details of the existence and uniqueness analysis for
this boundary value problem we refer to \([1, 5]\). Here and follows, we assume that the problem
(2.1)-(2.2) has a unique solution.

It is not difficult to verify that the Helmholtz equation in \(\mathbb{R}^n\) has a fundamental solution

\[
G(x, y) = \begin{cases} 
\frac{i}{4} H_0^{(1)}(k|x - y|), & \text{for } n = 2, \\
e^{ik|x - y|}, & \text{for } n = 3,
\end{cases}
\]

which satisfies

\[-(\Delta x + k^2)G(x, y) = \delta(x - y)\]
in the sense of distributions for any fixed \( y \in \mathbb{R}^n \). If \( u \in C^2(\mathbb{R}^2 \setminus \bar{\Omega}) \cap H^1_{\text{loc}}(\mathbb{R}^n \setminus \bar{\Omega}) \) satisfies the Helmholtz equation in the exterior domain \( \mathbb{R}^n \setminus \bar{\Omega} \) and the Sommerfeld radiation condition uniformly as \( |x| \to \infty \), then by using Green’s formulae formally, i.e.

\[
-k^2 \int_{\mathbb{R}^n \setminus \bar{\Omega}} uv dx + \int_{\mathbb{R}^n \setminus \bar{\Omega}} \nabla u \cdot \nabla v dx = \int_{\mathbb{R}^n \setminus \bar{\Omega}} u(-\Delta v - k^2 v) dx - \int_{\partial \Omega} u \partial_n v d\sigma,
\]

(2.3)

\[
-k^2 \int_{\mathbb{R}^n \setminus \bar{\Omega}} uv dx + \int_{\mathbb{R}^n \setminus \bar{\Omega}} \nabla u \cdot \nabla v dx = \int_{\mathbb{R}^n \setminus \bar{\Omega}} v(-\Delta u - k^2 u) dx - \int_{\partial \Omega} v \partial_n u d\sigma,
\]

(2.4)

with \( v(x) = G(x, y) \), where \( n \) is the unit inner normal vector on the boundary \( \partial \Omega \), we obtain the boundary integral representation

\[
u(y) = \int_{\partial \Omega} \left( \frac{\partial G(x, y)}{\partial n(x)} u(x) - G(x, y) \partial_n u(x) \right) d\sigma(x), \quad y \in \Omega,
\]

(2.5)

By differentiating the equation (2.5), we derive the boundary integral representation of the potential gradients

\[
\frac{\partial u(y)}{\partial y_i} = \int_{\partial \Omega} \left( \frac{\partial^2 G(x, y)}{\partial y_i \partial n(x)} u(x) - \frac{\partial G(x, y)}{\partial y_i} \partial_n u(x) \right) d\sigma(x), \quad y \in \Omega.
\]

(2.6)

Taking trace and normal derivative of the equation (2.5), we obtain the conventional boundary integral equation (CBIE)

\[
\frac{1}{2} u(y) = \int_{\partial \Omega} \left( \frac{\partial G(x, y)}{\partial n(x)} u(x) - G(x, y) \partial_n u(x) \right) d\sigma(x), \quad y \in \partial \Omega
\]

(2.7)

and the hypersingular boundary integral equation (HBIE)

\[
\frac{1}{2} \partial_n u(y) = \int_{\partial \Omega} \frac{\partial^2 G(x, y)}{\partial n(x) \partial n(y)} u(x) d\sigma(x) - \int_{\partial \Omega} \frac{\partial G(x, y)}{\partial n(y)} \partial_n u(x) d\sigma(x), \quad y \in \partial \Omega,
\]

(2.8)

The above equations are valid everywhere except at the corner of the boundary \( \partial \Omega \). For weak solutions \( u \in H^1(\Omega) \), the CBIE is valid in \( H^{1/2}(\partial \Omega) \) and the HBIE is valid in \( H^{-1/2}(\partial \Omega) \).

We see that the solution of the exterior Neumann problem satisfies the hypersingular boundary integral equation

\[
Hu = -(\frac{1}{2} I + T^*) g,
\]

(2.9)

where \( H \) is the hypersingular operator

\[
Hu(y) = -\int_{\partial \Omega} \frac{\partial^2 G(x, y)}{\partial n(x) \partial n(y)} u(x) d\sigma(x), \quad y \in \partial \Omega
\]

and \( T^* \) is defined by

\[
T^* g(y) = \int_{\partial \Omega} \frac{\partial G(x, y)}{\partial n(y)} g(x) d\sigma(x), \quad y \in \partial \Omega.
\]

The integral equation (2.9) has a unique solution if and only if the original exterior problem has a unique solution. In that case,

\[
u = -H^{-1}(\frac{1}{2} I + T^*) g.
\]
The operator on the right hand side relates the boundary value of the solution to its normal derivative and hence called the Dirichlet-to-Neumann map. However, the DtN mapping is not known analytically for domains with general geometry. The integral equation (2.9) has to be solved numerically. After the boundary integral equation (2.9) is solved, the values of the solution and its gradients in the domain $\mathbb{R}^n \setminus \bar{\Omega}$ are given by the boundary integral representations (2.5) and (2.6). Therefore, the $n$-dimensional problem in the exterior domain $\mathbb{R}^n \setminus \Omega$ is converted to a $(n-1)$-dimensional problem on the boundary $\partial \Omega$.

3 An indirect formulation

We see that the integral equation obtained by the direct formulation is somehow complicated. In another way, we try to look for a simple solution which can be expressed as

$$u(y) = -\int_{\partial \Omega} \frac{\partial G(x,y)}{\partial n(x)} \psi(x) d\sigma(x), \quad y \in \Omega. \quad (3.10)$$

By the property of the fundamental solution, for any function $\psi \in H^{1/2}(\partial \Omega)$, the function $u$ defined above satisfies Helmholtz equation in the domain $\mathbb{R}^n \setminus \Omega$ and the Sommerfeld radiation condition at infinity. The only restriction we have to impose on $\psi$ is that the solution $u$ satisfies the Neumann boundary condition, i.e.

$$-\int_{\partial \Omega} \frac{\partial^2 G(x,y)}{\partial n(y) \partial n(x)} \psi(x) d\sigma(x) = g(y), \quad y \in \partial \Omega. \quad (3.11)$$

By noting that the hypersingular operator is of Fredholm type with zero index [5], it is easy to observe that the existence and uniqueness of solution of the above integral equation is equivalent to that of the exterior problem (2.1)-(2.2). Therefore, we can solve the HBIE

$$H \psi = g \quad (3.12)$$

with an unknown density $\psi$ which does not has an explicit physical meaning. Then the solution values in the exterior domain is given by (3.10).

4 Galerkin BEM

The Galerkin method is similar for both 2D and 3D problems. For simplicity, we illustrate the idea in the 2D framework. We denote $\langle \phi, \psi \rangle = \int_{\partial \Omega} \phi \psi d\sigma$ for any $\phi, \psi$ defined on the boundary $\partial \Omega$ such that the integral has a meaning. The variational formulation of the integral equation (3.12) consists of finding $u \in H^{1/2}(\partial \Omega)$ such that

$$\langle H \psi, v \rangle = \langle g, v \rangle, \quad \forall v \in H^{1/2}(\partial \Omega). \quad (4.13)$$

Using the strong ellipticity of the hypersingular operator [5], i.e.

$$\langle H \psi, \psi \rangle \geq c \| \psi \|^2_{H^{1/2}(\partial \Omega)} - C \| \psi \|^2_{L^2(\partial \Omega)}, \quad \forall \psi \in H^{1/2}(\partial \Omega)$$

and the invertibility of the mapping

$$H : H^1(\partial \Omega) \to L^2(\partial \Omega),$$
a boundary element method can be designed to solve the problem, which goes the following way.

First, we partition the boundary into elements and define basis functions $\phi_j$, $j = 1, \cdots, N$, e.g. see Figure 1 for piecewise linear basis functions. Let $V_h$ be the finite dimensional space spanned by the basis functions so that $V_h \subset H^{1/2}(\partial \Omega)$. Then we turn to seek $u_h \in V_h$ which satisfies the finite dimensional problem:

$$\langle Hu_h, v \rangle = \langle g, v \rangle, \quad \forall v \in V_h.$$  

(4.14)

This gives us a linear system of equations, say, in matrix form,

$$Au_h = b,$$

with

$$A = [a_{ij}]_{N \times N}, \quad b = [b_i]_N, \quad a_{ij} = \langle H\phi_j, \phi_i \rangle, \quad \text{and} \quad b_i = \langle g, \phi_i \rangle.$$ 

5 Collocation BEM in 2D

For 2D problems, there is an easier way to solve the boundary integral equation (3.12) without calculating double integrals, i.e. to use collocation method.

In doing so, we partition the boundary into elements and define the basis functions $\phi_j$, $j = 1, \cdots, N$, either global or piecewise polynomials. Then we collocate the equation at $N$ different points, e.g. see Figure 2 for piecewise linear basis functions and collocation points.

The linear system is given by

$$Au_h = b$$

with

$$A = [a_{ij}]_{N \times N}, \quad a_{ij} = H\phi_j(y_i) \quad \text{and} \quad b = [g(y_i)]_N, \quad i, j = 1, 2, \cdots, N.$$ 

6 Collocation BEM in 3D

Generally speaking, engineers prefer the collocation method to the Galerkin method. The essential reason is that we have simpler integrals to compute or to approximate and that the deduction
is somewhat simpler in the sense that we are neither averaging on the cells nor considering an
intermediate variational formulation.

There are however some aspects to be taken into account for solving 3D problems with
collocation methods:

1. Not every choice of collocation nodes is going to work. For instance, taking a vertex per
triangle is not a judicious choice.

2. There is not a satisfactory theory for collocation method working even for smooth surfaces.
This should not worry too much a practitioner of the method: most people are convinced that
the theory will arrive in due time. However, the Galerkin setting gives more confidence to
mathematically oriented users of the boundary element method.

3. In some instances, practitioners of the method use many more collocation nodes than
elements and solve the incompatible equations by mean squares. This has the advantage that
we have to solve a Hermitian positive definite system.

7 Fast multipole method

Although the BEM enjoyed the reputation of ease in modeling problems with complicated ge-
ometries or infinite domains, it has been limited to solving large-scale problems for many years.
This is because the conventional BEM, as described in the previous chapter, produced dense and
nonsymmetric matrices that although smaller in sizes, require $O(N^2)$ operations for computing
the coefficients and $O(N^3)$ operations for solving the system by using direct solvers.

In the 1980s, Rokhlin and Greengard [4, 3, 2] pioneered the innovative fast multipole method
(FMM) that can be used to accelerate the solutions of BEM to $O(N)$. With the help of the
FMM, the BEM can now solve large-scale problems that are beyond the reach of other methods.
A comprehensive review of the fast multipole BEM can be found in [6].

8 Further readings

There are not many books dealing with the mathematical and numerical analysis of boundary
element methods. The literature is however much more important in the engineering world,
where you will be able to find many details on algorithms, implementation and especially different problems where boundary element techniques apply.

The book


details the boundary integral formulations for the Laplace, Helmholtz, Navier-Lamé (linear elasticity) and biharmonic (Kirchhoff plate) equations. The Sobolev theory is explained with care in the case of smooth interfaces. The fundamentals of Sobolev theory and finite element method are carefully explained. There are also some explanations on pseudodifferential operators, a theory that allows for a generalization of the behavior of all the boundary integral operators for smooth boundaries. The section on numerical analysis is not very long and right now it is not up-to-date.

The whole theory on boundary integral formulations based on the theory of elliptic operators is explained with an immense care and taste for mathematical detail in


This is a book of hard mathematics, where you will learn a lot but are asked to have patience. It does not cover numerical analysis.

Following is a book with discussions of how to implement FMM in solving the various boundary integral equations.


References


