Variable Sensitivity-Based Deterministic Robust Design for Nonlinear System

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In this paper, a novel robust design approach is proposed to design the robustness of the nonlinear system under large uncontrollable variation. First, a variable sensitivity approach is proposed to formulate the nonlinear effect into the variable sensitivity matrix. Then, a variable sensitivity-based robust design is developed to minimize the variable sensitivity matrix so that the influence of the uncontrollable variation to the performance will be minimized. Since the proposed robust design considers the influence of the nonlinear term in a large design region, it can effectively improve the robustness of the nonlinear system despite large uncontrollable variation. Simulation examples have demonstrated the effectiveness of the proposed design method.

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1 Introduction

Robust performance is one of the most important concerns in the design of any system since uncontrollable variations exist in the real industry, including manufacturing operations, variations in the material properties, and the operating environment. If these variations are not considered, they will degrade the performance and may result in a failure in practice. Robust design is one of the most important methods to achieve the robust performance of the system. Its fundamental principle is to improve the quality of a product by minimizing its performance sensitivity to variations [1]. In the past decades, much effort has been dedicated to robust design. All these works can be classified into two categories: experiment- and model-based robust design methods.

The experiment-based robust design methods use simulation or experiment to improve the system robustness, for example, the Monte Carlo method [2], Taguchi method [1], robust concept exploration method [3], and robust modeling method for modeling uncertainty [4]. These methods are fully based on the experimental data without considering the process knowledge. Thus, it could be inaccurate and needs large amounts of experimental data, which will lead to a high experimental cost, especially for the strongly nonlinear system.

The second class of methods is the model-based robust design methods, which use the model information to design the system robustness. The advantages of this kind of methods have low cost and high design accuracy compared with the experiment-based methods. In the past decades, much effort has been dedicated to this class of robust designs. These works mainly include two categories: probabilistic and deterministic robust designs [5].

The probabilistic robust design uses probabilistic information of the design variables and design parameters, usually their mean and variance, to improve the system robustness. Many relevant works have been reported recently. Rahman and Xu [6] proposed a univariate dimension reduction (UDR) to decompose a multidimensional response function into multiple one-dimensional functions for stochastic moments. Since this method is an approximate solution, it can produce large approximate error, especially for the strong nonlinear system. Parkinson [7] discussed seven robust design methods using engineering models. Du and Chen [8] checked several feasibility modeling techniques for the robust optimization. A formulation of robust design based on the mathematical model was proposed by Al-Widyan and Angeles [9]. Kalsi et al. [10] incorporated robust design concepts into the multidisciplinary design. Yu and Ishii [11] defined the manufacturing variation pattern to represent the characteristic patterns of the design variables. The main shortcoming of probabilistic approaches is that the essential information of probabilistic distributions may not be easy to obtain in practice [5].

The deterministic robust design is to minimize the worst case of the performance under variations. Li et al [5], and Gunawan and Azarm [12] proposed the sensitivity region measures for the robust design. However, this method is complex that costs a high computational, because the sensitivity region is difficult to find out, especially for the high dimensional system. Most of the deterministic approaches use the gradient information of variables and parameters to improve the system robustness due to its simplicity and low computational cost. Ting and Long [14] used the condition number of the sensitivity matrix to measure the robustness of the system. Zhu and Ting [15] used the theory of performance sensitivity distribution to study the system robustness. Caro et al. [16] compared two robust indexes—the Euclidean norm and condition number of the sensitivity matrix—and provided a two-consecutive-step synthesis method for the tolerance design. Recently, Lu and Li [17] proposed a hybrid data/model based robust design method to handle model uncertainty. However, these gradient based robust designs are only based on an approximate linear model through the local linearization. This approximation may cause the design to be less effective due to the larger approximate error, especially when the system has strong nonlinearity with larger uncontrollable variations. Thus, an effective deterministic robust design approach should be developed to design the strongly nonlinear system to be robust to larger uncontrollable variations.

In this paper, a novel deterministic robust design is proposed to improve the robustness of the nonlinear system despite larger uncontrollable variations. The nonlinear system is first formulated into a linear structure, which will be easy to handle using the well-developed robust design methods. This linear structure has a variable sensitivity matrix due to the influence of all nonlinear terms. Then, the bounds of both the variable sensitivity matrix and
its singular values can be calculated in a larger design region. Finally, with the variable sensitivity information incorporated, the influence of the uncontrollable variation to the performance can be minimized under the framework of the traditional robust design. Both the fundamental analysis and numerical simulation demonstrate effectiveness of the proposed robust design method.

2 Problem Description

Many systems in real-world are often of strong nonlinearity with larger uncontrollable variations; for example, the thermal process and the fluid process in industry are described as a complex nonlinear partial differential equation with larger uncontrollable boundary and environment variations [18]. This kind of systems can be expressed as

\[ Y = f(s) \] (1)

where \( Y = [y_1 \cdots y_n]^T \) represents the performance vector; the variable \( s = [d, p]^T \) includes two parts: the controllable design variable vector \( d = [d_1 \cdots d_m]^T \), whose nominal value can be selected between the range of upper and lower bounds, and the model parameter vector \( p = [p_1 \cdots p_n]^T \) with the uncontrollable variation \( \Delta p \) around its nominal value \( p_0 \); and \( f(s) = [f_1(s) \cdots f_n(s)]^T \) is the nonlinear model and its nonlinear term is defined as \( N(s) \). For convenience, \( f(s) \) is simply denoted as \( f \).

There are some difficulties in designing the robustness of the system (Eq. (1)) under the larger variation \( \Delta s \) due to following reasons:

(a) influence of the nonlinear term \( N(s) \) to the performance
(b) interaction of the larger uncontrollable variation \( \Delta s \) and the nonlinear term \( N(s) \)

Usually, the traditional deterministic robust designs, such as the Euclidean norm method [15,16] or the condition number method [14], are used to minimize the influence of the uncontrollable variation to the performance based on a linear model. This linear model is obtained by the local linearization approach, where the Taylor series expansion method is usually used, and expressed as

\[ \Delta Y = J_{0s} \cdot \Delta s \] (2)

where the sensitivity matrix \( J_0 = (\partial f/\partial s)|_{s_0} \) and \( s_0 = [d_0, p_0]^T \) with the nominal values \( d_0 \) and \( p_0 \). It is well-known that this linear approximation is only effective around the neighborhood of the design point.

Obviously, the traditional deterministic robust design methods can work well for the linear system or the weakly nonlinear system because the linear model (Eq. (2)) can approximate the practical system well, as shown in Fig. 1 (a). However, when the system has strong nonlinearity, it will produce a larger approximate error \( \Delta e \) between the linear model (Eq. (2)) and the practical system (Eq. (1)), especially when the uncontrollable variation \( \Delta s \) is larger. This error \( \Delta e \) can lead to the traditional deterministic robust design methods less effective because these traditional methods only use the performance variation \( \Delta Y \) of the linear model (Eq. (2)) to measure the robustness. For example, for point \( A \) in Fig. 1 (b), the traditional robust designs use the performance variation \( \Delta Y \) obtained from the linear model (Eq. (2)) to measure the robustness. However, the practical performance variation \( \Delta Y \) is \( \Delta Y + \Delta e \) that is much larger than the performance variation \( \Delta Y \). Thus, the traditional robust designs will be less effective for this kind of case due to the larger approximate error \( \Delta e \). An effective robust design approach should be developed to design the robustness of the nonlinear system under larger uncontrollable variation \( \Delta s \).

3 A Novel Variable Sensitivity-Based Robust Design

3.1 Robust Design Methodology. In this paper, a new robust design approach as indicated in Fig. 2 is proposed to design the nonlinear system to be robust to large uncontrollable variation. First, the nonlinear system is modeled as a linear structure using the variable sensitivity approach, which will make the design relatively easier using well-developed robust design theories. Moreover, since the sensitivity matrix of this linear structure is formulated to consider the influence of the nonlinear terms, it would be a variable matrix, which is different to the traditional linearization approach with a constant sensitivity matrix \( J_0 \). Then, a variable sensitivity-based robust design is proposed to minimize the influence of the uncontrollable variation to the performance. Since the proposed robust design considers the effect of the nonlinearity in a larger design region, the designed system will be robust under larger uncontrollable variations.

3.2 Variable Sensitivity Approach. The proposed variable sensitivity approach is based on the following idea. As illustrated in Fig. 3, given the nonlinear system (Eq. (1)), the variable sensitivity \( J \) is found out such that \( \Delta Y = \Delta f(s)/\Delta s \) with \( J \in [J_{\min}, J_{\max}] \) under \( s \in [s_l, s_u] \). The performance variation \( \Delta Y \) of any given point in \( s \in [s_l, s_u] \) may be expressed as the product of the variation \( \Delta s \) and the sensitivity \( J \). This sensitivity \( J \) is equal to the gradient of the straight line between the given point and the design point. For example, the performance variation \( \Delta Y \) at point

![Fig. 1](image1.png)

![Fig. 2](image2.png)
A relative to the design point B is equal to \( \Delta Y = J_{AB}(s_A - s_B) \), where \( J_{AB} \) is the sensitivity to be same as the gradient of the line AB. Thus, all performance variations may be expressed as \( \Delta Y = J \Delta s \) in \( s \in [s_l, s_u] \), where the sensitivity J at different points may have different values. Obviously, the maximal and minimal values of the variable sensitivity in \( s \in [s_l, s_u] \) can be figured out through the system model and denoted as \( J = [J_{\min}, J_{\max}] \). This variable sensitivity approach has the following advantages:

- There is no approximation because all nonlinear terms are fully formulated into the variable sensitivity matrix.
- It has a linear structure, which is easy to handle using the well-developed robust design approaches.

### 3.3 Design for Single Performance/Single Variable

Under the single performance function \( Y \) and single variable \( s \), the sensitivity matrix \( J \) is a scalar. The robust design for this case is relatively simple.

#### 3.3.1 Variable Sensitivity Approach-Based Modeling

First, the original system (Eq. (1)) can be easily formulated as the following linear structure around the design point \( s_0 \):

\[
\Delta Y = J \Delta s
\]

with

\[
J = \left( \frac{f(s) - f(s_0)}{s - s_0} \right)
\]

When \( s \) is close to \( s_0 \), \( J \) is equal to \( J_0 \) as defined in Eq. (2) so that the proposed method is the same with the traditional robust design methods. However, when \( s \) has a slightly larger variation around \( s_0 \), \( J \) will be variable under different variable \( s \). It is clear that this variable sensitivity matrix \( J \) incorporates the nonlinear information of the system. Thus, this linear structure (Eq. (3)) can well express the nonlinear system in a larger design region. Also, it is clear that the traditional deterministic robust design becomes a special case of the proposed design method.

Then, the minimal \( J_{\min} \) and maximal \( J_{\max} \) values of \( J \) are obtained by solving the following optimizations when \( \Delta s \in [\Delta s_l, \Delta s_u] \):

\[
J_{\min} = \min_{\Delta s \in [\Delta s_l, \Delta s_u]} J, \quad J_{\max} = \max_{\Delta s \in [\Delta s_l, \Delta s_u]} J
\]

Thus, \( J \) will be bounded in \([J_{\min}, J_{\max}]\).

**Example 1.** This simple example is used to show how the variable sensitivity approach works. The system is described as follows:

\[
Y = f(s) = 3s^2 \quad \text{and} \quad s \in [1, 3]
\]

From (Eq. (5)), the performance variation \( \Delta Y \) at the design point \( s_0 = 2 \) can be derived as

\[
\Delta Y = f(s) - f(s_0) = 3s^2 - 3s_0^2 = (3s + 3s_0)\Delta s
\]

Thus, its sensitivity matrix \( J \) is

\[
J = 3s + 3s_0
\]

and the maximal and minimal values of the sensitivity matrix \( J \) is obtained from the optimization (Eq. (4)) as

\[
J_{\min} = 9 \quad \text{and} \quad J_{\max} = 15
\]

Note: From Eq. (7), it is clear that \( J \) is equal to \( J_0 \) defined in Eq. (2) when \( s \) is close to \( s_0 \). Thus, the linear model (Eq. (2)) is a special case of the proposed variable sensitivity model (Eq. (3)).

#### 3.3.2 Variable Sensitivity-Based Robust Design

The variable sensitivity-based robust design aims to minimize the worst case of the performance variation \( \Delta Y \). Two different cases are discussed as follows.

**3.3.2.1 Robust design for the same sign of \( J_{\max} \) and \( J_{\min} \).** In this case, the worst case \( \Delta Y_{\max} \) of the performance variation \( \Delta Y \) is equal to \( \max(\{J_{\min} \cdot (\Delta s), J_{\max}\} \cdot \Delta s \), as shown in Fig. 4, where \( \max(\{J_{\min} \cdot (\Delta s), J_{\max}\} \) is actually the maximal value between \( J_{\min} \) and \( J_{\max} \). For example, if \( J_{\max} \) and \( J_{\min} \) are minus, then \( \max(\{J_{\min} \cdot (\Delta s), J_{\max}\} \) is \( J_{\max} \). Otherwise, \( \max(\{J_{\min} \cdot (\Delta s), J_{\max}\} \) is \( J_{\max} \). Thus, the robust design is equivalent to the minimization of \( \max(\{J_{\min} \cdot (\Delta s), J_{\max}\} \) without controlling \( \Delta s \)

\[
\min_{\Delta s} \max(\{J_{\min} \cdot (\Delta s), J_{\max}\})
\]

**3.3.2.2 Robust design for the different signs of \( J_{\max} \) and \( J_{\min} \).** Similarly, the worst case \( \Delta Y_{\max} \) is equal to \( J_{\max} - J_{\min} \cdot \Delta s \), as shown in Fig. 5. Thus, the robust design problem is equivalent to the minimization of \( |J_{\max} - J_{\min}| \)

\[
\min_{\Delta s} |J_{\max} - J_{\min}|
\]

**The solution can make the nonlinear system robust to larger uncontrollable variation.**

**Fig. 3 Variable sensitivity approach**

![Variable sensitivity approach](image)

**Fig. 4 Robust design for the same sign of \( J_{\max} \) and \( J_{\min} \)**

\[
J = 3s + 3s_0
\]

and the maximal and minimal values of the sensitivity matrix \( J \) is obtained from the optimization (Eq. (4)) as

\[
J_{\min} = 9 \quad \text{and} \quad J_{\max} = 15
\]

**Fig. 5 Robust design for the different signs of \( J_{\max} \) and \( J_{\min} \)**

\[
J = 3s + 3s_0
\]
3.4 Design for Multiperformances/Multivariables. When there are the multiperformance functions \( Y \) and multivariables \( s \), the sensitivity matrix \( J \) becomes a \( m \times (n+l) \) matrix.

3.4.1 Variable Sensitivity Approach-Based Modeling. When there are the multiperformance functions \( Y \) and multivariables \( s \), the performance variations \( \Delta Y \) can be easily expressed as

\[
\Delta Y = \Delta f(s) = J \Delta s \tag{11}
\]

where the variable sensitivity matrix \( J \) becomes an \( m \times (n+l) \) matrix.

Let \( \Delta s \in [\Delta s_1, \Delta s_n] \) and then the system model may be used to calculate the minimal value \( f_{\text{min}}^{ij} \) and the maximal value \( f_{\text{max}}^{ij} \) of \( J_{ij} \), which is an element of \( J \)

\[
f_{\text{min}}^{ij} = \max_{\Delta s \in [\Delta s_1, \Delta s_n]} J_{ij} \quad f_{\text{max}}^{ij} = \min_{\Delta s \in [\Delta s_1, \Delta s_n]} J_{ij} \tag{12}
\]

Then, the performance variations \( \Delta Y \) in Eq. (11) may be rewritten as

\[
(\Delta f(s))^T \Delta f(s) = (\Delta s)^T B \Delta s \tag{13}
\]

with

\[
B = J^T J
\]

From Eqs. (12) and (13), the matrix \( B \) is a bound matrix with its bound derived as

\[
B_{ij} = \begin{bmatrix} f_{\text{min}}^{ij} & f_{\text{max}}^{ij} \end{bmatrix} \tag{14}
\]

Moreover, according to the singular value decomposition (SVD) theory, any real symmetric matrix \( B \) may be decomposed as

\[
B = \xi \text{diag}(\sigma_1, \ldots, \sigma_m) \xi^T \tag{15}
\]

where \( \sigma_i \) is the singular value of \( J \), and the corresponding orthogonal eigenvector is denoted as \( \xi \), which is one element of the vector \( \bar{\xi} = [\xi_1, \cdots, \xi_m] \).

Since the matrix \( B \) varies within its bound, its singular values are also bounded as

\[
\sigma_i \in [\sigma_{ij}^{\text{min}}, \sigma_{ij}^{\text{max}}] \tag{16}
\]

This bound can be calculated from the matrix \( B \).

Example 2. The nonlinear system is described as

\[
\begin{align*}
Y &= f(s) = \begin{bmatrix} s_1^2 + s_2^2 \\ 3s_1 + 2s_2^2 \end{bmatrix} \\
&= \begin{bmatrix} s_1 + s_1, 0 \\ s_2 + s_2, 0 \end{bmatrix} \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}
\end{align*}
\]

and

\[
s = [s_1, s_2]^T
\]

with

\[
s_1 \in [1, 3], \quad s_2 \in [1, 3]
\]

From Eq. (17), the performance variations \( \Delta Y \) around the design point \( s_0 = [2, 2]^T \) can be formulated as

\[
\Delta Y = f(s) - f(s_0) = \begin{bmatrix} s_1^2 + s_2^2 - s_1, 0 - s_2, 0 \\ 3s_1 + 2s_2^2 - 3s_1, 0 - 2s_2, 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} s_1 + s_1, 0 \\ s_2 + s_2, 0 \end{bmatrix} \begin{bmatrix} \Delta s_1 \\ \Delta s_2 \end{bmatrix}
\]

Thus, its sensitivity matrix is

\[
J = \begin{bmatrix} s_1 + s_1, 0 \\ s_2 + s_2, 0 \end{bmatrix}
\]

Thus, the bound of the matrix \( J \) can be calculated as

\[
J_{1,1} \in [3, 5], \quad J_{1,2} \in [7, 19], \quad J_{2,2} \in [6, 10]
\]

Then, the matrix \( B \) may be figured out

\[
B = \begin{bmatrix} J_{1,1} + J_{1,2} & J_{1,1} + J_{1,2} \\ J_{1,1} + J_{1,2} & J_{1,2} + J_{2,2} \end{bmatrix}
\]

Thus, the bound of the matrix \( B \) can be calculated as

\[
B_{1,1} \in [18, 34], \quad B_{1,2} = B_{2,1} \in [39, 125], \quad B_{2,2} \in [85, 461]
\]

Moreover, the singular value of matrix \( B \) can be expressed as

\[
\sigma_1 = 0.5(B_{2,2} + B_{1,1} + \sqrt{(B_{2,2} + B_{1,1})^2 - 4(B_{2,2}B_{1,1} - B_{1,2}^2)}),
\]

\[
\sigma_2 = 0.5(B_{2,2} + B_{1,1} - \sqrt{(B_{2,2} + B_{1,1})^2 - 4(B_{2,2}B_{1,1} - B_{1,2}^2)}),
\]

Thus, the bound of the singular values can be calculated as

\[
\sigma_1 \in [97, 490.6], \quad \sigma_2 \in [3.96, 7.72]
\]

Note: From Eq. (19), it is clear that \( J \) is equal to \( J_0 \) defined in (Eq. (2)) when \( s \) is close to \( s_0 \). Thus, the linear model (Eq. (2)) is a special case of the proposed variable sensitivity model (Eq. (11)) even for the system with multiperformances/multivariables.

Moreover, inserting Eq. (15) into Eq. (13), the performance \( Y \) may be expressed as follows:

\[
Y = \sum_{j=1}^{n+l} \sigma_j \xi_j^T
\]

with

\[
Y = \|\Delta Y\|_2 = (\Delta f(s))^T \Delta f(s)
\]

and

\[
[\xi_1, \cdots, \xi_{n+l}]^T = \xi^T \Delta s
\]

Since both the singular values and the eigenvectors in Eq. (25) vary within a bound, the performance \( Y \) in the \( m \)-dimensional space is a set of hyperellipsoids, as defined in Eq. (25). Its two-dimensional projection is depicted in Fig. 6.

Every hyperellipsoid has the following characteristics:

(a) The performance \( Y \) defined in Eq. (25) is the same for every point on the hyperellipsoid.

(b) The length of the \( i \)th principal axis of the \( j \)th hyperellipsoid is \( Y_i / \sigma_i \), where the singular value \( \sigma_i \) is a value bounded in \( [\sigma_i^{\text{min}}, \sigma_i^{\text{max}}] \). The smaller \( \sigma_i \) is, the longer the \( i \)th principal axis will be. The longest/shortest principal axis corresponds to the least/most sensitive direction for the given hyperellipsoid.

For all hyperellipsoids:

(1) The length of the \( i \)th principal axis of all hyperellipsoids is bounded in \( [Y_i / \sigma_i^{\text{max}}, Y_i / \sigma_i^{\text{min}}] \).
(2) The longest/shortest principal axis in all hyperellipsoids means the least/most sensitive direction for the nonlinear system.

3.4.2 Variable Sensitivity-Based Robust Design. Here, the Euclidean norm is used as the robust index because Caro et al. [16] has confirmed that the Euclidean norm is more suitable as the robust index than the condition number. This robust design method requires that all principal axes have the long length, especially the shortest principal axis in all hyperellipsoids. Since the shortest principal axis in all hyperellipsoids corresponds to the maximal singular value \( \sigma_{\text{max}} \) in all singular values, if the maximal singular value \( \sigma_{\text{max}} \) is minimized, then the shortest principal axis will have the relatively long length. So the design variables \( d \) for the robust performance can be figured out by solving the following min-max optimization problem:

\[
\begin{align*}
\min_{d_0} & \max_{i} (\sigma_{\text{max}}^i) \\
\text{s.t.} & \quad h(s) = 0, \quad l(s) \leq 0
\end{align*}
\]

The solution of the optimization (Eq. (26)) can achieve the robustness under the larger uncontrollable variation. However, when the dimensions of the problems become larger, this min-max optimization will cause a higher computational cost.

3.5 Summary. The proposed robust design methodology is summarized in Fig. 7. The nonlinear system is first formulated as a linear structure just as Eqs. (3) and (11) using the variable sensitivity method. Since its variable sensitivity matrix incorporates the effect of the nonlinear terms, it can express the nonlinear system better than the traditional linearization approach with a constant sensitivity matrix \( J_0 \). Then, the bound of the varying sensitivity matrix is calculated from the system model. For multiperformances/multivariables system, the maximal and minimal singular values should also be calculated as Eq. (16). Finally, based on the obtained sensitivity matrix, the variable sensitivity-based robust designs as Eqs. (9), (10), and (26) can design the robustness of the nonlinear system despite larger uncontrollable variation.

Remark. The proposed robust design method has a well linear structure that would be easy to design using the well-developed robust design approaches. Moreover, since it considers the influence of the nonlinear term, it can design the robustness of the nonlinear system despite large uncontrollable variation. However, the calculation of the upper and lower bounds of the variable sensitivity matrix can increase computational effort, especially for nonconvex, highly nonlinear, and discontinuous system.

4 Case Study

The proposed robust design method will be compared with the traditional Euclidean norm method in two different cases, where the traditional robust design [15,16] is to minimize the singular value of \( J_0 \) defined in Eq. (2).

Since the deviation from its objective can be estimated as \( \| Y(d_0 + \Delta d, p_0 + \Delta p) - Y(d_0, p_0) \|_2 \), the performance index \( E_r \) will be defined as

\[
E_r = \| Y(d_{0, r} + \Delta d, p_0 + \Delta p) - Y(d_{0, r}, p_0) \|_2 - \| Y(d_{0, p} + \Delta d, p_0 + \Delta p) - Y(d_{0, p}, p_0) \|_2
\]

where \( d_{0, r} \) and \( d_{0, p} \) are the design variables gained by the traditional Euclidean norm method and the proposed robust design method, respectively. If the percentage of \( E_r > 0 \) is larger than 50%, when \( \Delta p \) and \( \Delta d \) are randomly sampled from their variation space, then the proposed method is more robust than the traditional Euclidean norm method. Otherwise, the traditional Euclidean norm method is better.

Example 1. Robust design of a belt

Belts are used in the transmission of power between shafts with either parallel or skewed axes. The power transmitted by a belt is

\[
W = f(s)
\]

with

\[
f(s) = (1 - e^{-i\theta})(T - M V^2) V
\]

where \( M \) and \( \theta \) are the mass of the belt per unit length and the contact angle, respectively, \( V \) and \( T \) are the belt speed and the tension in the belt, respectively, \( \xi \) and \( W \) are the coefficient of friction and the transmitted power, respectively. The coefficient of friction \( \xi \), the tension \( T \), the nominal mass \( M \), and the contact angle \( \theta \) are 0.2, 15 N m, 1 kg, and \( \pi/4 \), respectively. The design variable and the performance function are

\[
s = V, \quad Y = W
\]

The design task is to find the design variable \( V_0 \) from the design space \([0.5, 2.5]\) to have a robust performance against the variation \( \Delta s = \Delta V \).

First, the robust design calculated by the traditional Euclidean norm approach is

\[
V_T = 2.5
\]

Then, the proposed robust design is used to design the belt. Let \( \Delta V \in [-R, R] \). From Eq. (28), the performance variation \( \Delta W \) can be expressed as

\[
\Delta W = \Delta V \Delta s = \Delta s
\]

with

\[
J = (1 - e^{-i\theta})(T - M(V^2 + V V_0 + V_0^2))
\]

The maximal \( J_{\text{max}} \) and minimal \( J_{\text{min}} \) values of \( J \) are easily calculated as

\[
J_{\text{max}} = (1 - e^{-i\theta})(T - M((V_0 - R)^2 + (V_0 + R)V_0 + V_0^2)),
\]

\[
J_{\text{min}} = (1 - e^{-i\theta})(T - M((V_0 - R)^2 + (V_0 - R)V_0 + V_0^2))
\]

If \( J_{\text{max}} \) and \( J_{\text{min}} \) have the same sign, then the robust design can be figured out from Eq. (9) with the given bound \( R \). Otherwise, it will be figured out from Eq. (10) with the given bound \( R \).

For comparison verification, let \( \Delta V \) uniformly distributed random variation in \([-R, R] \), and a total of 1000 samples are taken to compare the worst case of the performance variation \( \Delta W \) with respect to the variation \( \Delta V \). Designs from different variation bounds \( R \) are shown in Fig. 8, where it is clear that the worst case of the performance variation obtained by the proposed robust design is smaller than the traditional Euclidean norm approach.
Thus, the proposed robust design has better robust performance than the traditional Euclidean norm approach. This is because the proposed robust design considers the influence from the nonlinear term and the larger uncontrollable variation, but the traditional Euclidean norm approach does not.

Finally, the performance $E_r$ is compared. Let $R=0.1$ and thus $\Delta V$ is bounded in $(-0.1, 0.1)$. Under this bound, the proposed robust design is $V_p=2.24$. A total of 1000 samples are taken to compare the performance $E_r$. From Fig. 9, it is clear that it has about 81.8% chances to have a better design than the traditional one. Thus, the proposed robust design method is more robust than the traditional Euclidean norm method.

Example 2. Robust design of a damper

The damper design example in Fig. 10 is taken from the Ref. [16]. $M$ and $C_d$ are mass and damping coefficients, respectively. The excitation force $F(t)$ is equal to $F \cos(\omega \cdot t)$. The displacement is equal to $X(t)=X \cos(\omega \cdot t + \phi)$, where $\phi$ is the phase.

Moreover, the following relations exist:

$$X = \frac{F}{\omega \sqrt{C_d^2 + \omega^2 M^2}}, \quad \phi = \tan^{-1}\left(\frac{\omega M}{C_d}\right) \tag{32}$$

The design variable $d$, the model parameter $p$, and the performance functions $Y$ are $d = M$, $p = C_d$, $s = [M, C_d]^T$, $Y = \begin{bmatrix} X \\ \phi \end{bmatrix}$.

$F$ and $w$ are set as 200 N and 31.4 rad/s, respectively. The model parameter $C_d$ is equal to $C_{d0}=H$ with the nominal value $C_{d0}=50$, and the design variable $M$ has the variation $\Delta M \in [-R, R]$ around its design point $M_0$, where $H$ and $R$ are variation bound. The objective is to select the design variable $M_0$ from $M_0 \in [2 \text{ kg}, 3 \text{ kg}]$ to have a robust performance against the uncontrollable variation $\Delta s \in [\Delta C_d, \Delta M]^T$ from both the design variable and the model parameter.

For the design problem (Eq. (32)), the robust design calculated by the traditional Euclidean norm method is $M_f=2.5$. The proposed robust design for the damper design can be easily figured out from Eq. (26) with the given bounds $H$ and $R$.

For the comparison verification, let $\Delta M$ and $\Delta C_d$ uniformly distributed random variation in $(-R, R)$ and $(-H, H)$, respectively, and a total of 1000 samples are taken to compare the worst case of the performance variation $\Delta W= [(\Delta X)^2 + (\Delta \phi)^2]$. Define

$$\text{Difference of the worst case} = \max(\Delta W_f) - \max(\Delta W_p) \tag{33}$$

where $\Delta W_f$ and $\Delta W_p$ is the worst case of the performance variation $\Delta W= [(\Delta X)^2 + (\Delta \phi)^2]$ obtained by the traditional and proposed robust designs, respectively. Only if this difference is positive, the proposed robust design has a better robust performance than the traditional one. This difference under the variation bounds $R$ and $H$ is shown in Fig. 11. From Fig. 11, the positive difference shows clearly that the proposed robust design has a better robust performance than the traditional Euclidean norm approach.

For comparison of $E_r$, let $R=0.3$ and $H=15$, and thus, $\Delta M$ and $\Delta C_d$ are bounded in $(-0.3, 0.3)$ and $(-15, 15)$, respectively. Under these bounds, the proposed robust design is $M_f=2.28$. A total of 1000 samples are taken for the comparison of performance $E_r$, as shown in Fig. 12, where it shows about 64.1% (for $E_r>0$)
chances to have a better design than the traditional one. Since the percentage is larger than 50%, the proposed method is more robust than the traditional one.

In conclusion, the proposed robust design method is effective for the design of the nonlinear system despite larger uncontrollable variation.

5 Conclusion

In this paper, the novel robust design method is proposed to design the robustness of the nonlinear system. This new design approach considers not only the nonlinearity of the system, but also the large uncontrollable variation. The variable sensitivity approach can well express the nonlinear system since it considers the influence of the nonlinearity. Moreover, the proposed variable sensitivity-based approach can achieve a good robust design even for the strongly nonlinear system with larger uncontrollable variation.

Simulation examples are used to compare the proposed method with the traditional Euclidean norm method. The comparisons show that the proposed method is more robust than the traditional one when the system is of strong nonlinearity and has larger uncontrollable variation. This is because the proposed robust design considers the influence from both the nonlinear terms and the larger uncontrollable variations, but the traditional deterministic robust design does not.

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